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Critique of “Two-dimensional examples of rank-one convex functions that are not quasiconvex” by M. K. Benaouda and J. J. Telega

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Abstract. It is noted that the examples provided in the paper “Two-dimensional examples of rank-one convex functions that are not quasiconvex” by M. K. Benaouda and J. J. Telega, Ann. Polon. Math. 73 (2000), 291–295, contain unrecoverable errors.

1. Introduction. We consider variational integrals

$$I(\varphi) = \int_{\Omega} W(\nabla\varphi(x))\, dV, \tag{1.1}$$

defined for sufficiently regular deformations $\varphi : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ where Ω is a bounded open subset of $\mathbb{R}^m$. Here, $\nabla\varphi(x)$ denotes the deformation gradient at $x \in \mathbb{R}^m$ and W is a continuous function on the space $\mathbb{M}^{m\times n}$ of all real $m \times n$ matrices. One of the important problems in the calculus of variations is to characterise the integrand W for which the integral I is lower semicontinuous. In this respect the following notions have been introduced (see e.g. [1–3, 5, 6]):

1. W is *rank-one convex* if for each matrix $F \in \mathbb{M}^{m\times n}$ and each rank-one matrix $B \in \mathbb{M}^{m\times n}$, the real-valued function $t \mapsto W(F + t\,B)$ is convex.
2. W is *quasiconvex* if for any matrix $F \in \mathbb{M}^{m\times n}$,

$$\forall\psi : \mathbb{R}^m \to \mathbb{R}^n,\ \psi(x) = F.x,\ x \in \partial\Omega : \quad \int_{\Omega} W(\nabla\psi(x))\, dV \ge W(F)\cdot|\Omega|. \tag{1.2}$$

Quasiconvexity implies that the homogeneous deformation $\varphi(x) = F.x$ is energy optimal for homogeneous boundary conditions.

For “nice” integrands W the quasiconvexity condition is necessary and sufficient for the weak lower semicontinuity of I. However, quasiconvexity is

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a nonlocal condition, difficult to check in practice. The rank-one convexity is in principle easy to verify: for $W \in C^2$ it is the *Legendre–Hadamard ellipticity* condition. Moreover, it can be shown that quasiconvexity implies rank-one convexity. The converse is not true as has been shown by Šverák [8] for the case $m \geq 2$ and $n \geq 3$. It is a long standing open problem [7] whether for $m = n = 2$ rank-one convexity implies quasiconvexity.

The authors of [4] claim to have found a counterexample for this two-dimensional case. I show that their example is not a counterexample.

2. Analysis. In the following, let $\mathbb{M}^{2\times 2}$ denote the set of two times two matrices and define for the deformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ the corresponding deformation gradient

$$\nabla\varphi(x_1, x_2) = \begin{pmatrix} \varphi_{1,x_1}(x_1, x_2) & \varphi_{1,x_2}(x_1, x_2) \\ \varphi_{2,x_1}(x_1, x_2) & \varphi_{2,x_2}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} F_{11}(x_1, x_2) & F_{12}(x_1, x_2) \\ F_{21}(x_1, x_2) & F_{22}(x_1, x_2) \end{pmatrix}.$$

In [4, Lem. 4.1] it is claimed that the quadratic function $W : \mathbb{M}^{2\times 2} \to \mathbb{R}$ (the function g there) given by

$$W(F) = F_{11}\, F_{22} + F_{12}^2 + F_{21}^2 \tag{2.3}$$

is rank-one convex and in [4, Lem. 4.2] it is argued that W is not quasiconvex.

Let us rewrite W in the form

$$\begin{aligned} W(F) &= F_{11}\, F_{22} + F_{12}^2 + F_{21}^2 \\ &= F_{11}\, F_{22} - F_{12}\, F_{21} + F_{12}^2 + F_{21}^2 + F_{12}\, F_{21} \\ &= \det F + (F_{12}^2 + F_{21}^2 + F_{12}\, F_{21}). \end{aligned} \tag{2.4}$$

By Young's inequality it is easy to see that

$$\forall F \in \mathbb{M}^{2\times 2} : \quad F_{12}^2 + F_{21}^2 + F_{12}\, F_{21} > 0. \tag{2.5}$$

Therefore, $F \mapsto F_{12}^2 + F_{21}^2 + F_{12}\, F_{21}$ is a strictly positive quadratic form, hence strictly convex. Altogether, W is the sum of the quasi-affine function $F \mapsto \det F$ and a strictly convex term, hence rank-one convex *and* quasiconvex. The error in [4, Lem. 4.2] stems from the fact that the test-function used is indeed not periodic on the unit cube $[0, 1]^2$.

In [4, Th. 3.1] the same error occurs. The test-function is again not periodic on $[0, 1]^2$, therefore, Theorem 3.1 is wrong. Moreover, the meaning of Theorem 3.1 as far as a counterexample to the above mentioned open question is concerned, is not clear to the author, since "rank-one convexity at 0" does not imply rank one convexity.

The observation that this paper is erroneous is not new; indeed, it is clearly pointed out by Baisheng Yan in Mathematical Reviews [MR1785693 (2001g: 49005)]:

"The authors claim to provide two-dimensional examples of rank-one convex functions that are not quasiconvex. {Reviewer's remarks: Theorem 3.1 and Lemma 4.2 appear to be incorrect because Lemma 2.4 is wrongly quoted and used. The periodic functions in Lemma 2.4 should be of $[0,1]^n$-period, as originally stated in the papers [4, 8] cited [B. Dacorogna, Direct methods in the calculus of variations, Springer, Berlin, 1989; MR0990890 (90e:49001); V. Šverák, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992), no. 1-2, 185–189; MR1149994 (93b:49026)]. This paper had the good intention to solve a very hard open problem, but unfortunately appears not to contain any result that is new and correct.}"

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