ANNALES
POLONICI MATHEMATICI
98.1 (2010)

Frictionless contact problem with adhesion and finite penetration for elastic materials

by Arezki Touzaline (Bab-Ezzouar)

Abstract. The paper deals with the problem of quasistatic frictionless contact between an elastic body and a foundation. The elasticity operator is assumed to vanish for zero strain, to be Lipschitz continuous and strictly monotone with respect to the strain as well as Lebesgue measurable on the domain occupied by the body. The contact is modelled by normal compliance in such a way that the penetration is limited and restricted to unilateral contraints. In this problem we take into account adhesion which is modelled by a surface variable, the bonding field, whose evolution is described by a first-order differential equation. We derive a variational formulation of the mechanical problem and we establish an existence and uniqueness result by using arguments of time-dependent variational inequalities, differential equations and the Banach fixed-point theorem. Moreover, using compactness properties we study a regularized problem which has a unique solution and we obtain the solution of the original model by passing to the limit as the regularization parameter converges to zero.

1. Introduction. Contact mechanics is a branch of mechanics which typically involves two bodies instead of one and focuses on their common interface rather than their interiors. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated to more and more complex behaviour laws leads to the introduction of new and nonstandard models, expressed with the aid of evolution variational inequalities.

An early attempt to study contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [19]. We find there a detailed

2010 *Mathematics Subject Classification*: 74M15, 35J85, 49J40, 47J20.
Key words and phrases: elastic material, normal compliance, adhesion, fixed point, variational inequality, weak solution, frictionless contact, differential equation.

DOI: 10.4064/ap98-1-2

mathematical and numerical analysis of adhesive contact problems. Moreover, existence results for the continuous case were recently established in [1, 6, 8, 16] in the study of unilateral and frictional contact problems for linear elastic materials.

In this paper, we study a mathematical model which describes a frictionless adhesive contact between an elastic body and a foundation. The elasticity operator is assumed to vanish for zero strain, to be Lipschitz continuous and strictly monotone with respect to the strain as well as Lebesgue measurable on the domain occupied by the body. As in [13], the contact is modelled by normal compliance in such a way that the penetration is limited and restricted to unilateral constraints. We recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [4, 5, 9, 18, 19, 20, 21]. As in [10, 11], we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$. When $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [3, 10, 11, 12, 15, 17, 18, 19].

Now, we want to point out the physical interest of the model studied here. Indeed, before the appearance of the reference [13], it was well known that no restriction of the penetration was made in compliance models. However, according to [13], the method presented here concerns a compliance model in which the compliance term does not necessarily represent an important perturbation of the original problem without contact. This will help us to study the models where a strictly limited penetration occurs using the limit procedure for the Signorini contact problem. In this work we extend the result established in [21] to the unilateral contact problem with a modified normal compliance when the penetration is finite and the adhesion between contact surfaces is taken into account. We derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solutions. Moreover, we study a regularized problem which we consider as a frictionless contact problem with adhesion and unlimited penetration. We prove its unique weak solvability and show that the solution of the original model is obtained by passing to the limit as the regularization parameter converges to zero.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 2.2. Finally, in Section 4, we prove a convergence result for a regularized problem, Theorem 4.2.

2. Variational formulation. Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a domain initially occupied by a viscoelastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ partitioned into three measurable parts, $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and meas $\Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in adhesive and frictional contact with a foundation.

Thus, the classical formulation of the mechanical problem is as follows.

PROBLEM P_1. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \le g, \quad \sigma_\nu + p(u_\nu) - c_\nu \beta^2(-R(u_\nu))_+ \le 0 \\ (\sigma_\nu + p(u_\nu) - c_\nu \beta^2(-R(u_\nu))_+)(u_\nu - g) = 0 \end{array}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\sigma_\tau = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$\dot{\beta} = -c_\nu \beta_+ (R(u_\nu))^2 \quad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.8}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) is the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ is the Cauchy stress vector. The condition (2.5) represents the unilateral contact conditions with adhesion in which c_ν is a given adhesion coefficient which may depend on $x \in \Gamma_3$, and $R : \mathbb{R} \to \mathbb{R}$ is a truncation operator defined as

$$R(s) = \begin{cases} -L & \text{if } s \le -L, \\ s & \text{if } |s| < L, \\ L & \text{if } s \ge L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see [20]) and p is a normal compliance function which satisfies the assumption (2.16) below. We denote by g the maximum value of the penetration. When $u_\nu < 0$, i.e. when there is separation between the body and the foundation, then the condition (2.5) combined with hypotheses (2.16) on the function p shows that the reaction of the foundation vanishes ($\sigma_\nu = 0$). When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited, that is, $u_\nu \le g$. In the case of penetration (i.e. $u_\nu \ge 0$), if $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$, which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \le 0$. Since p is an increasing function, the reaction is increasing with the penetration. If $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, condition (2.5) becomes the Signorini contact condition with adhesion with a gap,

$$u_\nu \le g, \quad \sigma_\nu - c_\nu\beta^2(-R(u_\nu))_+ \le 0, \quad (\sigma_\nu - c_\nu\beta^2(-R(u_\nu))_+)(u_\nu - g) = 0.$$

When $g = 0$, the condition (2.5) combined with hypothesis (2.16) results in the Signorini contact condition with adhesion, given by

$$u_\nu \le 0, \quad \sigma_\nu - c_\nu\beta^2(-R(u_\nu))_+ \le 0, \quad (\sigma_\nu - c_\nu\beta^2(-R(u_\nu))_+)u_\nu = 0.$$

This contact condition was used in [20, 21]. We also note that when $g = 0$, the condition (2.5) without adhesion becomes the classical Signorini contact condition without a gap,

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0.$$

Equation (2.6) represents a frictionless contact condition and shows that the tangential stress vanishes on the contact surface during the process. Also it means that the glue does not provide any resistance to the tangential motion of the body on the foundation. Equation (2.7) is an ordinary differential equation which describes the evolution of the bonding field, in which $r_+ = \max\{r, 0\}$, and it was already used in [4]. Since $\dot\beta \le 0$ on $\Gamma_3 \times (0,T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [14] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2,3)$; and $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $\|v\| = (v.v)^{1/2}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau.\tau)^{1/2}$. Here and below, the indices i and j run between 1 and d, and the summation convention over repeated indices is adopted.

Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d,$$
$$Q = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad Q_1 = \{\sigma \in Q;\ \operatorname{div}\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma,\tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} \, dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \left(\frac{1}{2}\,(u_{i,j} + u_{j,i})\right);$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu.$$

Also, we denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and if σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

and the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu.v \, da \qquad \forall v \in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements be given by

$$K = \{v \in V;\ v_\nu \le g \text{ on } \Gamma_3\},$$

where $g \ge 0$. Since $\operatorname{meas}\Gamma_1 > 0$, the following Korn's inequality holds [7]:

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \qquad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \le d_\Omega \|v\|_V \qquad \forall v \in V. \tag{2.10}$$

For $p \in [1,\infty]$, we use the standard norm of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0,T;H), \qquad \varphi_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)^d) \tag{2.11}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t),v)_V = \int_\Omega \varphi_1(t).v\,dx + \int_{\Gamma_2} \varphi_2(t).v\,da \qquad \forall v\in V,\ t\in[0,T]. \tag{2.12}$$

Using (2.11) and (2.12) yields

$$f\in W^{1,\infty}(0,T;V).$$

In the study of the mechanical problem P_1 we assume that the elasticity operator $F:\Omega\times S_d\to S_d$ satisfies

$$\begin{cases} \text{(a) there exists } M>0 \text{ such that} \\ \qquad \|F(x,\varepsilon_1)-F(x,\varepsilon_2)\| \le M\|\varepsilon_1-\varepsilon_2\| \\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d \text{ and a.e. } x \text{ in } \Omega; \\ \text{(b) there exists } m>0 \text{ such that} \\ \qquad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m\|\varepsilon_1-\varepsilon_2\|^2 \\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d \text{ and a.e. } x \text{ in } \Omega; \\ \text{(c) } x\mapsto F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \text{ in } S_d; \\ \text{(d) } F(x,0)=0 \text{ for a.e. } x \text{ in } \Omega. \end{cases} \tag{2.13}$$

The adhesion coefficient satisfies

$$c_\nu \in L^\infty(\Gamma_3) \quad \text{and} \quad c_\nu \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{2.14}$$

Finally, we assume that the initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3), \qquad 0\le\beta_0\le 1 \quad \text{a.e. on } \Gamma_3. \tag{2.15}$$

We define the functional $j: L^2(\Gamma_3)\times V\times V\to\mathbb{R}$ by

$$j(\beta,u,v) = \int_{\Gamma_3} (p(u_\nu) - c_\nu\beta^2(-R(u_\nu))_+)v_\nu\,da \qquad \forall\beta\in L^2(\Gamma_3),\ u,v\in V.$$

As in [13], we assume that the normal compliance function p satisfies

$$\begin{cases} \text{(a) } p:(-\infty,g]\to\mathbb{R}; \\ \text{(b) there exists } L_p>0 \text{ such that} \\ \qquad |p(r_1)-p(r_2)| \le L_p|r_1-r_2| \text{ for all } r_1,r_2\le g; \\ \text{(c) } (p(r_1)-p(r_2))(r_1-r_2)\ge 0 \text{ for all } r_1,r_2\le g; \\ \text{(d) } p(r)=0 \text{ for all } r<0. \end{cases} \tag{2.16}$$

As in [19], the functional j has the following properties:

$$j(\beta_1,u_1,u_2-u_1)+j(\beta_2,u_2,u_1-u_2)\le C\|\beta_1-\beta_2\|_{L^2(\Gamma_2)}\|u_1-u_2\|_V, \tag{2.17}$$

$$j(\beta,u_1,u_2-u_1)+j(\beta,u_2,u_1-u_2)\le 0, \tag{2.18}$$

$$j(\beta,u_1,v)-j(\beta,u_2,v)\le C\|u_1-u_2\|_V\|v\|_V, \tag{2.19}$$

where C is a positive constant, and

$$j(\beta,v,v)\ge 0. \tag{2.20}$$

Finally, we need to introduce the set

$$B=\{\theta:[0,T]\to L^2(\Gamma_3);\ 0\le\theta(t)\le 1,\ \forall t\in[0,T],\ \text{a.e. on } \Gamma_3\}.$$

Now by assuming the solution to be sufficiently regular, we deduce by using Green's formula and techniques similar to those presented in [20] that Problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u\in W^{1,\infty}(0,T;V)$ and a bonding field $\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap B$ such that

$$\begin{aligned} u(t)\in K,\quad &(F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t)))_Q+j(\beta(t),u(t),v-u(t))\\ &\ge (f(t),v-u(t))_V\quad \forall v\in K,\ t\in[0,T], \end{aligned} \tag{2.21}$$

$$\dot\beta(t)=-c_\nu(\beta(t))_+(R(u_\nu(t)))^2\quad \text{a.e. } t\in(0,T), \tag{2.22}$$

$$\beta(0)=\beta_0. \tag{2.23}$$

As in [20, 21], our main result, which will be established in the next section, is the following theorem.

Theorem 2.2. *Let* (2.11) *and* (2.13)–(2.16) *hold. Then Problem P_2 has a unique solution.*

3. Existence and uniqueness result. The proof of Theorem 2.2 is carried out in several steps. In the first step, let $k>0$ and consider the space

$$X=\{\beta\in C([0,T];L^2(\Gamma_3));\ \sup_{t\in[0,T]}[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}]<+\infty\}.$$

It is well known that X is a Banach space for the norm

$$\|\beta\|_X=\sup_{t\in[0,T]}[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}].$$

Next for a given $\beta\in X$, we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta\in C([0,T];V)$ such that

$$\begin{aligned} u_\beta(t)\in K,\quad &(F\varepsilon(u_\beta(t)),\varepsilon(v-u_\beta(t)))_Q+j(\beta(t),u_\beta(t),v-u_\beta(t))\\ &\ge (f(t),v-u_\beta(t))_V\quad \forall v\in K,\ t\in[0,T]. \end{aligned} \tag{3.1}$$

We have the following result.

LEMMA 3.1. *Problem $P_{1\beta}$ has a unique solution.*

Proof. Let $t \in [0,T]$ and let $A_t : V \to V$ be the operator defined by

$$(A_t u, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + j(\beta(t), u, v) \quad \forall u, v \in V.$$

Using the hypotheses on F and the properties of j we see that A_t is strongly monotone and Lipschitz continuous. Then from [2], since K is a closed convex subset of V, using the standard results for elliptic variational inequalities, we deduce that there exists a unique element $u_\beta(t) \in K$ which satisfies (3.1). As in [19], to show that $u_\beta \in C([0,T];V)$, it suffices to see from (3.1) that there exists a constant $C > 0$ such that

$$\|u_\beta(t_1) - u_\beta(t_2)\|_V \\ \le C(\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f(t_1) - f(t_2)\|_V) \quad \forall t_1, t_2 \in [0,T],$$

and conclude the proof by making use of the fact that $f \in C([0,T];V)$ and $\beta \in C([0,T];L^2(\Gamma_3))$. ∎

In the second step we consider the following problem.

PROBLEM $P_{2\beta}$. Find $\beta^* : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot\beta^*(t) = -c_\nu(\beta^*(t))_+(R(u_{\beta^*\nu}(t)))^2 \quad \text{a.e. } t \in (0,T), \tag{3.2}$$

$$\beta^*(0) = \beta_0. \tag{3.3}$$

We obtain the following result.

LEMMA 3.2. *Problem $P_{2\beta}$ has a unique solution β^* which satisfies*

$$\beta^* \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B.$$

Proof. Consider the mapping $T : X \to X$ defined as

$$T\beta(t) = \beta_0 - \int_0^t c_\nu(\beta(s))_+(R(u_{\beta\nu}(s)))^2\, ds,$$

where u_β is the solution of Problem $P_{1\beta}$. Then for $\beta_1, \beta_2 \in X$, we have

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \\ \le c\int_0^t \|(\beta_1(s))_+(R(u_{\beta_1\nu}(s)))^2 - (\beta_2(s))_+(R(u_{\beta_2\nu}(s)))^2\|_{L^2(\Gamma_3)}\, ds,$$

where $c > 0$. Using the definition of R and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \\ \le c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds + c\int_0^t \|u_{\beta_1\nu}(s) - u_{\beta_2\nu}(s)\|_{L^2(\Gamma_3)}\, ds.$$

Moreover, from (2.10), we obtain

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,ds + cd_\Omega\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V\,ds.$$

Now for $t \in [0,T]$, we use the inequality (3.1), the assumption (2.13)(b) on F and the property (2.19) of j to find that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \le c_1\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \tag{3.4}$$

for some constant $c_1 > 0$. Hence, we deduce that there exists a constant $d_1 > 0$ such that

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le d_1\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,ds.$$

On the other hand, we have

$$\begin{aligned}\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} &= \int_0^t \exp(ks)[\exp(-ks)\|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}]\,ds\\ &\le \|\beta_1 - \beta_2\|_X\int_0^t \exp(ks)\,ds.\end{aligned}$$

Since

$$\int_0^t \exp(ks)\,ds = \frac{\exp(kt)-1}{k} \le \frac{\exp(kt)}{k},$$

therefore

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le d_1\|\beta_1 - \beta_2\|_X\,\frac{\exp(kt)}{k} \qquad \forall t \in [0,T],$$

which implies that

$$\exp(-kt)\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le \frac{d_1}{k}\,\|\beta_1 - \beta_2\|_X \qquad \forall t \in [0,T].$$

So we obtain

$$\|T\beta_1 - T\beta_2\|_X \le \frac{d_1}{k}\,\|\beta_1 - \beta_2\|_X. \tag{3.5}$$

The inequality (3.5) shows that for k sufficiently large, T is a contraction. Thus it has a unique fixed point β^* which satisfies (3.2) and (3.3). To prove that $\beta^* \in B$, it suffices to invoke [20, Remark 3.1]. ∎

Finally, as in [20, 21] we conclude that

$$(u_{\beta^*}, \beta^*) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$$

such that $u_{\beta^*}(t) \in K$ for all $t \in [0,T]$ is the unique solution to Problem P_2.

4. The regularized problem. In this section we consider a frictionless contact problem with normal compliance and adhesion with unlimited penetration. The contact condition (2.5) is replaced by the contact condition

$$-\sigma_\nu = p_\delta(u_{\delta\nu}) - c_\nu\beta^2(-R(u_{\delta\nu}))_+ \quad \text{on } \Gamma_3 \times (0,T),$$

where as in [13] the regularized functional $p_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$p_\delta(r) = \begin{cases} p(r) & \text{if } r \le g, \\ \dfrac{r-g}{\delta} + p(g) & \text{if } r > g. \end{cases} \tag{4.1}$$

We recall that $\delta > 0$ is the regularization parameter and $1/\delta$ is interpreted as the stiffness coefficient of the foundation. We understand that when δ is small, the reaction of the foundation to the penetration is important; also when δ is large then the reaction of the foundation to the penetration is smaller. We study the behavior of the solution as $\delta \to 0$ and prove that in the limit we obtain the solution of the adhesive frictionless contact problem with normal compliance and finite penetration. We define the functional $j_\delta : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_\delta(\beta, u, v) = \int_{\Gamma_3} (p_\delta(u_\nu) - c_\nu\beta^2(-R(u_\nu))_+)v_\nu \, da \quad \forall\beta \in L^2(\Gamma_3),\ u, v \in V.$$

With these notations, the formulation of the regularized problem with frictionless contact and adhesion is the following.

Problem $P_{1\delta}$. Find a displacement field $u_\delta : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta_\delta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$\operatorname{div}\sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$\sigma = F\varepsilon(u_\delta) \quad \text{in } \Omega \times (0,T), \tag{4.3}$$

$$u_\delta = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{4.4}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{4.5}$$

$$-\sigma_\nu = p_\delta(u_{\delta\nu}) - c_\nu\beta^2(-R(u_{\delta\nu}))_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{4.6}$$

$$\dot\beta_\delta = -c_\nu(\beta_\delta)_+(R(u_{\delta\nu}))^2 \quad \text{on } \Gamma_3 \times (0,T), \tag{4.7}$$

$$\beta_\delta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{4.8}$$

Problem $P_{1\delta}$ has the following variational formulation.

Problem $P_{2\delta}$. Find $(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$ such that

$$(F\varepsilon(u_\delta(t)), \varepsilon(v))_Q + j_\delta(\beta_\delta(t), u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T], \tag{4.9}$$

$$\dot\beta_\delta(t) = -c_\nu(\beta_\delta(t))_+(R(u_{\delta\nu}(t)))^2 \quad \text{a.e. } t \in (0,T), \tag{4.10}$$

$$\beta_\delta(0) = \beta_0. \tag{4.11}$$

We have the following result.

THEOREM 4.1. *Problem $P_{2\delta}$ has a unique solution.*

Proof. As in [20], the proof of Theorem 4.1 is similar to the proof of Theorem 2.1 and it is carried out in several steps. We omit the details and just recall the main steps:

(i) For any $\beta \in X$, we prove that there exists a unique $u_\delta \in C([0,T];V)$ such that

$$(4.12)\qquad (F\varepsilon(u_\delta(t)),\varepsilon(v))_Q + j_\delta(\beta(t),u_\delta(t),v) = (f(t),v)_V \qquad \forall v \in V,\ t\in[0,T].$$

To make this step for all $t\in[0,T]$ we consider the operator $T_t : V \to V$ defined by

$$(T_t u, v)_V = (F\varepsilon(u),\varepsilon(v))_Q + j_\delta(\beta(t),u,v) \quad \forall u,v\in V.$$

We use the properties (2.17)–(2.20) satisfied by the functional j and (4.1) to see that the operator T_t is strongly monotone and Lipschitz continuous, and therefore invertible.

(ii) There exists a unique β_δ such that

$$(4.13)\qquad \beta_\delta \in W^{1,\infty}(0,T;L^2(\Gamma_3)),$$

$$(4.14)\qquad \dot\beta_\delta(t) = -c_\nu(\beta_\delta(t))_+(R(u_{\delta\nu}(t)))^2 \quad \text{a.e. } t\in(0,T),$$

$$(4.15)\qquad \beta_\delta(0)=\beta_0.$$

(iii) Let β_δ be as in (ii) and denote again by u_δ the function obtained in step (i) for $\beta=\beta_\delta$. Then, by using (4.13)–(4.15), it is easy to see that (u_δ,β_δ) is the unique solution to Problem $P_{2\delta}$ and it satisfies

$$(u_\delta,\beta_\delta)\in W^{1,\infty}(0,T;V)\times W^{1,\infty}(0,T;L^2(\Gamma_3))\cap B.$$

Now, as in [20, 21], we specify the convergence of the solution (u_δ,β_δ) as $\delta\to 0$ in the following theorem.

THEOREM 4.2. *Assume that* (2.13)–(2.15) *hold. Then we have the following convergences for all $t\in[0,T]$:*

$$(4.16)\qquad \lim_{\delta\to0}\|u_\delta(t)-u(t)\|_V = 0,$$

$$(4.17)\qquad \lim_{\delta\to0}\|\beta_\delta(t)-\beta(t)\|_{L^2(\Gamma_3)} = 0.$$

The proof is carried out in several steps. In the first step, we show the following lemma.

LEMMA 4.3. *For each $t\in[0,T]$, there exists $\bar u(t)\in K$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$(4.18)\qquad u_\delta(t)\to\bar u(t) \quad \textit{weakly in } V.$$

Proof. Let $t\in[0,T]$. Take $v=u_\delta(t)$ in (4.9) to find

$$(4.19)\qquad (F\varepsilon(u_\delta(t)),\varepsilon(u_\delta(t)))_Q + j(\beta_\delta(t),u_\delta(t),u_\delta(t)) = (f(t),u_\delta(t))_V.$$

Since $j_\delta(\beta_\delta(t), u_\delta(t), u_\delta(t)) \geq 0$, it follows from (4.19) that

$$(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t)))_Q \leq (f(t), u_\delta(t))_V.$$

Now, keeping in mind the assumption (2.13)(b) on F, we deduce that there exists a constant $C > 0$ such that

$$\|u_\delta(t)\|_V \leq C\|f(t)\|_V.$$

The sequence $(u_\delta(t))$ is bounded in V. Hence there exists $\bar{u}(t) \in V$ and a subsequence again denoted $(u_\delta(t))$ such that (4.18) holds. Also from (4.19) we have

$$j_\delta(\beta_\delta(t), u_\delta(t), u_\delta(t)) \leq (f(t), u_\delta(t))_V.$$

Using the definition of j_δ we see that

$$j_\delta(\beta_\delta(t), u_\delta(t), u_\delta(t)) = \int_{\Gamma_3} (p_\delta(u_{\delta\nu}(t)) - c_\nu \beta^2(-R(u_{\delta\nu}(t)))_+) u_{\delta\nu}(t)\, da,$$

and since

$$\int_{\Gamma_3} c_\nu \beta^2(-R(u_{\delta\nu}(t)))_+ u_{\delta\nu}(t)\, da \leq 0,$$

it follows that

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da \leq (f(t), u_\delta(t))_V.$$

Now according to [13], we have

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da = \int_{\Gamma_3\cap\{u_{\delta\nu}(t)\leq g\}} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da + \int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da.$$

As

$$\int_{\Gamma_3\cap\{u_{\delta\nu}(t)\leq g\}} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da = \int_{\Gamma_3\cap\{u_{\delta\nu}(t)\leq g\}} p(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da \geq 0,$$

we find that

$$\int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da \leq (f(t), u_\delta(t))_V.$$

The left hand side can be written as

$$\int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} p_\delta(u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da = \int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} \frac{u_{\delta\nu}(t) - g}{\delta}\,(u_{\delta\nu}(t) - g)\, da$$
$$+ \int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} \frac{g}{\delta}\,(u_{\delta\nu}(t) - g)\, da + \int_{\Gamma_3\cap\{u_{\delta\nu}(t)> g\}} p(g) u_{\delta\nu}(t)\, da,$$

from which we deduce

$$\int\limits_{\Gamma_3\cap\{u_{\delta\nu}(t)>g\}} \frac{u_{\delta\nu}(t)-g}{\delta}\,(u_{\delta\nu}(t)-g)\,da \le (f(t), u_\delta(t))_V.$$

This inequality implies that

$$\|(u_{\delta\nu}(t)-g)_+\|^2_{L^2(\Gamma_3)} \le \delta C$$

for some constant $C > 0$. Hence, using (4.18), we deduce that

$$\|(\bar{u}_\nu(t)-g)_+\|_{L^2(\Gamma_3)} \le \liminf_{\delta\to 0} \|(u_{\delta\nu}(t)-g)_+\|_{L^2(\Gamma_3)} = 0. \tag{4.20}$$

Hence $(\bar{u}_\nu(t)-g)_+ = 0$, i.e. $\bar{u}_\nu(t) \le g$ a.e. on Γ_3, which shows that $\bar{u}(t) \in K$. ∎

Now we state the following problem.

Problem P_a. Find $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\beta}(t) = -c_\nu(\beta(t))_+(R(\bar{u}_\nu(t)))^2 \quad \text{a.e. } t \in (0,T),$$
$$\beta(0) = \beta_0.$$

As in [21, Lemma 3.2] we have the following result.

Lemma 4.4. *Problem P_a has a unique solution $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$.*

Also as in [21], we show the following convergence result.

Lemma 4.5. *Let β be the solution to Problem P_a. Then*

$$\lim_{\delta\to 0} \|\beta_\delta(t)-\beta(t)\|_{L^2(\Gamma_3)} = 0 \quad \textit{for all } t \in [0,T]. \tag{4.21}$$

Proof. As in [21, Lemma 3.2], we have

$$\|\beta_\delta(t)-\beta(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|u_{\delta\nu}(s)-\bar{u}_\nu(s)\|_{L^2(\Gamma_3)}\,ds \tag{4.22}$$

for some constant $C > 0$. Using (4.18) we deduce that $u_{\delta\nu}(t) \to \bar{u}_\nu(t)$ strongly in $L^2(\Gamma_3)$ as $\delta \to 0$. On the other hand,

$$\|u_{\delta\nu}(t)-\bar{u}_\nu(t)\|_{L^2(\Gamma_3)} \le d_\Omega\|u_\delta(t)-\bar{u}(t)\|_V \le d_\Omega\bigg(\frac{\|f(t)\|_V}{m} + \|\bar{u}(t)\|_V\bigg),$$

which implies that there exists a constant $C > 0$ such that

$$\|u_{\delta\nu}(t)-\bar{u}_\nu(t)\|_{L^2(\Gamma_3)} \le C.$$

Then it follows from Lebesgue's dominated convergence theorem that

$$\lim_{\delta\to 0}\int_0^t \|u_{\delta\nu}(s)-\bar{u}_\nu(s)\|_{L^2(\Gamma_3)}\,ds = 0. \tag{4.23}$$

The convergence result is now a consequence of (4.22) and (4.23). ∎

Next, we prove the following lemma.

Lemma 4.6. *We have $\bar{u}(t) = u(t)$ for all $t \in [0,T]$.*

Proof. Let $v \in K$ and take $v - u_\delta(t)$ in (4.12) to obtain

$$(4.24)\qquad (F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t)))_Q + j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \geq (f(t), v - u_\delta(t))_V \quad \forall v \in K.$$

We have

$$j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) = \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) \leq g\}} (p(u_{\delta\nu}(t)) - c_\nu \beta_\delta^2 (-R(u_{\delta\nu}(t)))_+)(v_\nu - u_{\delta\nu}(t))\, da + \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) > g\}} (p_\delta(u_{\delta\nu}(t)) - c_\nu \beta^2 (-R(u_{\delta\nu}(t)))_+)(v_\nu - u_{\delta\nu}(t))\, da.$$

Since

$$\int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) > g\}} (p_\delta(u_{\delta\nu}(t)) - c_\nu \beta_\delta^2 (-R(u_{\delta\nu}(t)))_+)(v_\nu - u_{\delta\nu}(t))\, da = \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) > g\}} \frac{u_{\delta\nu}(t) - g}{\delta}((v_\nu - g) - (u_{\delta\nu}(t) - g))\, da + \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) > g\}} p(g)((v_\nu - g) - (u_{\delta\nu}(t) - g))\, da - \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) > g\}} c_\nu \beta_\delta^2 (-R(u_{\delta\nu}(t)))_+((v_\nu - g) - (u_{\delta\nu}(t) - g))\, da \leq 0,$$

we deduce that

$$j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \leq \int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) \leq g\}} (p(u_{\delta\nu}(t)) - c_\nu \beta_\delta^2 (-R(u_{\delta\nu}(t)))_+)(v_\nu - u_{\delta\nu}(t))\, da.$$

We now use (2.17), (4.1), (4.16), (4.17) and the properties of R to see that

$$\int\limits_{\Gamma_3 \cap \{u_{\delta\nu}(t) \leq g\}} (p(u_{\delta\nu}(t)) - c_\nu \beta_\delta^2 (-R(u_{\delta\nu}(t)))_+)(v_\nu - u_{\delta\nu}(t))\, da \to j(\beta(t), \bar{u}(t), v - \bar{u}(t)) \quad \text{as } \delta \to 0.$$

Therefore, passing to the limit in (4.24) as $\delta \to 0$, we obtain

$$(4.25)\qquad \bar{u}(t) \in K, \quad (F\varepsilon(\bar{u}(t)), \varepsilon(v - \bar{u}(t)))_Q + j(\beta(t), \bar{u}(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \quad \forall v \in K.$$

Now, taking $v = u(t)$ in (4.25) and $v = \bar{u}(t)$ in (2.21) and adding the resulting inequalities, we find by using the assumption (2.13)(b) on F that

$$m\|\bar{u}(t) - u(t)\|_V^2 \leq j(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + j(\beta(t), u(t), \bar{u}(t) - u(t)).$$

Moreover, using (2.18), we see that

$$j(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + j(\beta(t), u(t), \bar{u}(t) - u(t)) \leq 0,$$

and therefore

$$\bar{u}(t) = u(t). \tag{4.26}$$

Now, we have all the ingredients to prove Theorem 4.2. Indeed, from (4.26), we immediately deduce (4.17). To prove (4.16), we take $v = u(t)$ in (4.24) to obtain, by using the assumption (2.13)(b) on F,

$$\begin{aligned} m\|u_\delta(t) - u(t)\|_V^2 \\ &\leq j(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) - j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \\ &\quad + j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) + (F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t)))_Q \\ &\quad + (f(t), u_\delta(t) - u(t))_V. \end{aligned}$$

Letting $\delta \to 0$ and using the convergences

$$\begin{aligned} &j(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) - j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0, \\ &j(\beta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0, \\ &(F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t)))_Q + (f(t), u_\delta(t) - u(t))_V \to 0, \end{aligned}$$

we obtain

$$\|u_\delta(t) - u(t)\|_V \to 0 \quad \text{for all } t \in [0, T].$$

Acknowledgments. I want to express my gratitude to the anonymous referee for his/her helpful suggestions and comments.

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Arezki Touzaline
Laboratoire de Systèmes Dynamiques
Faculté de Mathématiques, USTHB
BP 32 El Alia
Bab-Ezzouar, 16111, Algeria
E-mail: ttouzaline@yahoo.fr

Received 6.3.2009
and in final form 4.11.2009 (1986)