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On the local Cauchy problem for first order partial differential functional equations

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Abstract. A theorem on the existence of weak solutions of the Cauchy problem for first order functional differential equations defined on the Haar pyramid is proved. The initial problem is transformed into a system of functional integral equations for the unknown function and for its partial derivatives with respect to spatial variables. The method of bicharacteristics and integral inequalities are applied. Differential equations with deviated variables and differential integral equations can be obtained from the general theory by specializing given operators.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Suppose that $M \in C([0,a],\mathbb{R}^n_+)$, $a>0$, $\mathbb{R}_+=[0,+\infty)$, is nondecreasing and $M(0)=\theta$, $\theta=(0,\dots,0)\in\mathbb{R}^n$. Let E be the *Haar pyramid*

$$E=\{(t,x)\in\mathbb{R}^{1+n} : t\in[0,a],\ -b+M(t)\le x\le b-M(t)\}$$

where $b\in\mathbb{R}^n$ and $b>M(a)$. Write $E_0=[-b_0,0]\times[-b,b]$, where $b_0\in\mathbb{R}_+$ and $B=[-b_0-a,0]\times[-2b,2b]$. For $(t,x)\in E$ define

$$D[t,x]=\{(\tau,y)\in\mathbb{R}^{1+n} : \tau\le 0 \text{ and } (t+\tau,x+y)\in E_0\cup E\}.$$

Then $D[t,x]\subset B$ for $(t,x)\in E_0\cup E$. Given $z\colon E_0\cup E\to\mathbb{R}$ and $(t,x)\in E$, define $z_{(t,x)}\colon D[t,x]\to\mathbb{R}$ by $z_{(t,x)}(\tau,y)=z(t+\tau,x+y)$, $(\tau,y)\in D[t,x]$. Then $z_{(t,x)}$ is the restriction of z to $(E_0\cup E)\cap([-b_0,t]\times\mathbb{R}^n)$, shifted to $D[t,x]$.

Suppose that $\phi_0\colon[0,a]\to\mathbb{R}$ and $\phi=(\phi_1,\dots,\phi_n)\colon E\to\mathbb{R}^n$ are given functions such that $0\le\phi_0(t)\le t$ and $(\phi_0(t),\phi(t,x))\in E_0\cup E$ for $(t,x)\in E$. Write $\varphi(t,x)=(\phi_0(t),\phi(t,x))$ for $(t,x)\in E$. Put $\varOmega=E\times\mathbb{R}\times C(B,\mathbb{R})\times\mathbb{R}^n$ and suppose that $f\colon\varOmega\to\mathbb{R}$ and $\psi\colon E_0\to\mathbb{R}$ are given functions. We propose

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here a new general model of functional dependence in nonlinear differential equations with partial derivatives. We consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \psi(t,x) \quad \text{on } E_0 \tag{2}$$

where $x = (x_1,\dots,x_n)$, $\partial_x z = (\partial_{x_1} z,\dots,\partial_{x_n} z)$.

The aim of the paper is to prove the local existence of weak solutions for the initial value problem (1), (2). We use the method of bicharacteristics. The Cauchy problem is transformed into a system of functional integral equations for the unknown function and for its partial derivatives with respect to spatial variables. We prove the existence of a solution of this system by using the method of successive approximations and theorems on integral inequalities. Classical solutions of the functional integral equations lead to weak solutions to (1), (2).

We give examples of nonlinear equations which can be obtained from (1) by specializing f.

Example 1.1. Suppose that $\tilde{f}\colon E\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is a given function. Set $f(t,x,p,w,q) = \tilde{f}(t,x,p,w(0,\theta),q)$. Then (1) becomes the equation with deviated variables

$$\partial_t z(t,x) = \tilde{f}(t,x,z(t,x),z(\varphi(t,x)),\partial_x z(t,x)). \tag{3}$$

Example 1.2. Suppose that $\varphi(t,x) = (t,x)$ on E. For the above $\tilde{f}$ we put

$$f(t,x,p,w,q) = \tilde{f}\Big(t,x,p,\int_{D[t,x]} w(\tau,y)\,d\tau\,dy,q\Big).$$

Then (1) is equivalent to the integral equation

$$\partial_t z(t,x) = \tilde{f}\Big(t,x,z(t,x),\int_{D[t,x]} z_{(t,x)}(\tau,y)\,d\tau\,dy,\partial_x z(t,x)\Big). \tag{4}$$

It is clear that more complicated examples of differential equations with deviated variables and differential integral equations can be obtained from (1) for suitable f and φ.

Let us give a brief review of existence results concerning local Cauchy problems for first order differential and differential functional equations.

T. Ważewski [11], [12] initiated the theory of classical solutions of the Cauchy problem

$$\partial_t z(t,x) = F(t,x,z(t,x),\partial_x z(t,x)), \tag{5}$$
$$z(t_0,x) = \omega(x). \tag{6}$$

considered on the Haar pyramid. The following existence result can be deduced from [12]. Fix $(t_0,x_0,p_0,q_0) \in \mathbb{R}^{1+n+1+n}$. Write $\chi = (\kappa,\dots,\kappa) \in \mathbb{R}^n$ where $\kappa > 0$, and

$$\Xi = [t_0-\kappa,t_0+\kappa] \times [x_0-\chi,x_0+\chi] \times [p_0-\kappa,p_0+\kappa] \times [q_0-\chi,q_0+\chi].$$

Set $\eta = (x,p,q)$, $\eta = (\eta_1,\dots,\eta_{2n+1})$. Suppose that

1) the function $F\colon \Xi \to \mathbb{R}$ of the variables (t,x,p,q) is continuous and the partial derivatives $\partial_\eta F = (\partial_{\eta_1}F,\dots,\partial_{\eta_{2n+1}}F)$ exist with $\partial_\eta F \in C(\Xi,\mathbb{R}^{2n+1})$,
2) the estimates
$$|F(P)| \le A, \quad |\partial_{\eta_i}F(P)| \le A, \quad i=1,\dots,2n+1,$$
are satisfied for $P=(t,x,p,q) \in \Xi$, and $\partial_\eta F$ satisfies the Lipschitz condition with respect to (x,p,q) with the constant A,
3) $\omega\colon [x_0-\chi,x_0+\chi] \to \mathbb{R}$ is of class C^1 and $|\partial_{x_i}\omega(x)| \le A$, $i=1,\dots,n$, $x \in [x_0-\chi,x_0+\chi]$,
4) the following estimates hold:
$$|\omega(x_0)-p_0| \le \frac{\tilde\kappa}{4}, \quad |\partial_{x_i}\omega(x_0)-q_{0i}| \le \frac{\tilde\kappa}{4}, \quad |q_{0i}| \le A, \quad i=1,\dots,n,$$
where $q_0=(q_{01},\dots,q_{0n})$ and $\tilde\kappa < \kappa$.

Under these assumptions there is a Haar pyramid

$$H = \{(t,x) \in \mathbb{R}^{1+n} : t \in [t_0-a_0,t_0+a_0], \\ x \in [x_0-b+M|t-t_0|, x_0+b-M|t-t_0|]\} \tag{7}$$

such that the initial value problem (5), (6) has a solution defined on H. The formulas for $a_0, b=(b_1,\dots,b_n), M=(M_1,\dots,M_n)$ are given in [12]. The number a_0 and the vectors b, M depend on $\kappa, A, \tilde\kappa$. Hence we have the existence of classical solutions to (5), (6) and the domain of the solution is estimated. The method of characteristics and theorems on differential inequalities are used in [12]. Sufficient conditions for the existence of classical solutions to (5), (6) can also be deduced from [4, Chapter 2].

Initial value problems for nonlinear first order partial differential equations have the following property: the proof of the existence of classical solutions to (5), (6) and the existence results for the Cauchy problem which are global with respect to spatial variables, are based on the same ideas. The following existence result can be deduced from [6]. Let Ω_0 be a bounded domain in $\mathbb{R}^{1+n}$. Assume that $H \subset \Omega_0$ where H is given by (7). Suppose that $F\colon \Omega_0 \times \mathbb{R}^{1+n} \to \mathbb{R}$ is continuous and bounded and

1) $\partial_x F$, $\partial_p F$, $\partial_q F$ exist, are continuous and bounded, and
$$(|\partial_{q_1} F(P)|, \ldots, |\partial_{q_n} F(P)|) \le (M_1, \ldots, M_n) \quad \text{for } P \in \Omega_0 \times \mathbb{R}^{1+n}$$
with $M = (M_1, \ldots, M_n)$ appearing in (7),
2) $\partial_x F$, $\partial_p F$, $\partial_q F$ satisfy the Lipschitz condition with respect to (x, p, q),
3) $\omega \colon [x_0 - b, x_0 + b] \to \mathbb{R}$ is of class C^1 and $\partial_x \omega$ satisfies the Lipschitz condition.

Then there is $0 < c \le a_0$ such that problem (5), (6) has a solution $\tilde{u}$ defined on $H \cap ([-c, c] \times \mathbb{R}^n)$. The constant c can be estimated.

The proof of the above result is based on the method of successive approximations and the theory of characteristics of the second order.

Weak solutions to (5), (6) were considered in [2]. Existence results are based on a method of quasilinearization which was introduced and widely studied in nonfunctional setting by M. Cinquini Cibrario [2]. The method consists in linearization of the right-hand side of (5) with respect to the last variable. In the second step, a quasilinear system is constructed for the unknown function and for its spatial derivatives. The system thus obtained is equivalent to a system of integral equations of the Volterra type. Classical solutions of the integral equations lead to weak solutions of (5), (6).

Various models of functional dependence in partial differential equations are used in the literature. Several papers deal with an initial value problem for the equation

$$\partial_t z(t, x) = G(t, x, z(\cdot), \partial_x z(t, x)) \tag{8}$$

or a weakly coupled system ([9], [10]). The variable $z(\cdot)$ represents the functional argument. Sufficient conditions for the existence of classical solutions defined on the Haar pyramid can be deduced from [9]. The above existence results can be characterized as follows: theorems have simple assumptions and their proofs are very natural. Unfortunately, only a small class of functional differential equations is covered by this theory. The results given in [9], [10] are not applicable to (3) and (4).

An extension of this result to functional differential equations of the Volterra type was given in [5, Theorem 2.4]. Classical solutions are obtained by using the method of successive approximations.

Numerous papers concern initial value problems for equations

$$\partial_t z(t, x) = \tilde{F}(t, x, (Wz)(t, x), \partial_x z(t, x)) \tag{9}$$

where W is an operator of the Volterra type and $\tilde{F}$ is defined on finite-dimensional Euclidean space. The main assumptions in existence theorems for (9) concern the operator W. They are formulated in the form of norm inequalities in appropriate function spaces ([1], [3], [8]). These inequalities are linear, which is the main shortcoming of this theory.

Uniqueness criteria for initial value problems are obtained by using differential or differential functional methods and they can be found in [5], [9].

2. Bicharacteristics. The maximum norm in the space $C(B,\mathbb{R})$ will be denoted by $\|\cdot\|_B$. For a function $w\in C(B,\mathbb{R})$ and for a point $(t,x)\in E$ we put $E_t=(E_0\cup E)\cap([-b_0,t]\times\mathbb{R}^n)$, $0\le t\le a$ and

$$\|w\|_{D[t,x]}=\max\{|w(\tau,y)|:(\tau,y)\in D[t,x]\}.$$

CONDITION (V). Suppose that $f\colon \Omega\to\mathbb{R}$ is a given function of the variables (t,x,p,w,q), $q=(q_1,\dots,q_n)$. We will say that f satisfies *condition* (V) if for each $(t,x,p,q)\in E\times\mathbb{R}^{1+n}$ and for $w,\bar w\in C(B,\mathbb{R})$ such that $w(\tau,y)=\bar w(\tau,y)$ for $(\tau,y)\in D[\varphi(t,x)]$ we have $f(t,x,p,w,q)=f(t,x,p,\bar w,q)$.

Note that condition (V) means that the value of f at the point $(t,x,p,w,q)\in\Omega$ depends on (t,x,p,q) and on the restriction of w to the set $D[\varphi(t,x)]$ only.

Given $\psi\colon E_0\to\mathbb{R}$, put $E_t=(E_0\cup E)\cap([-b_0,t]\times\mathbb{R}^n)$, $0\le t\le a$, and

$$H_t=E\cap([0,t]\times\mathbb{R}^n),\quad S_t=[-b+M(t),b-M(t)],\qquad t\in[0,a],$$
$$I[x]=\{t\in[0,a]:-b+M(t)\le x\le b-M(t)\},\qquad x\in[-b,b].$$

We consider weak solutions of initial problems. A function $\tilde z\colon E_c\to\mathbb{R}$, where $0<c\le a$, is a solution of (1), (2) provided

(i) $\tilde z\in C(E_c,\mathbb{R})$ and $\partial_x z$ exists on $E_c\setminus E_0$,
(ii) $\tilde z(\cdot,x)\colon I[x]\to\mathbb{R}$ is absolutely continuous for each $x\in[-b,b]$,
(iii) for each $x\in[-b,b]$ equation (1) is satisfied for almost all $t\in I[x]$ and condition (2) holds.

This class of solutions lies between classical solutions and solutions in the Carathéodory sense, and both inclusions are strict.

We will denote by $\mathbb{L}([\tau,t],\mathbb{R}^k_+)$, $c>0$, $[\tau,t]\subset\mathbb{R}$, the class of all integrable functions $\gamma\colon[\tau,t]\to\mathbb{R}^k_+$. The maximum norms in $C(E_t,\mathbb{R})$ and $C(E_t,\mathbb{R}^n)$ are denoted by $\|\cdot\|_{(t,\mathbb{R})}$ and $\|\cdot\|_{(t,\mathbb{R}^n)}$, respectively. Given $(a_1,a_2)\in\mathbb{R}^2_+$, we denote by $\mathbb{K}$ the set of all $\psi\in C(E_0,\mathbb{R})$ such that

(i) $(\partial_{x_1}\psi,\dots,\partial_{x_n}\psi)=\partial_x\psi$ exists on E_0 and $\partial_x\psi\in C(E_0,\mathbb{R}^n)$,
(ii) for $(t,x),(t,\bar x)\in E$ we have

$$\|\partial_x\psi(t,x)\|\le a_1\quad\text{and}\quad\|\partial_x\psi(t,x)-\partial_x\psi(t,\bar x)\|\le a_2\|x-\bar x\|.$$

Let $\psi\in\mathbb{K}$ be given and let $0<c\le a$, $d\in\mathbb{R}_+$, $s=(s_1,s_2)\in\mathbb{R}^2_+$ and $s_1>a_1$, $d>2a_1$, $s_2>2a_2$. We denote by $C_{\psi.c}[d]$ the set of all $z\in C(E_c,\mathbb{R})$ such that $z(t,x)=\psi(t,x)$ on E_0 and

$$|z(t,x)-z(t,\bar x)|\le d\|x-\bar x\|,\qquad (t,x),(t,\bar x)\in E_c.$$

Let $C_{\partial\psi.c}[s]$ be the class of all $u \in C(E_c, \mathbb{R}^n)$ such that $u(t,x) = \partial_x \psi(t,x)$ on E_0 and

$$\|u(t,x)\| \le s_1 \quad \text{and} \quad \|u(t,x) - u(t,\bar{x})\| \le s_2 \|x - \bar{x}\| \quad \text{on } E_c.$$

Assumption H_1. The function $f\colon \Omega \to \mathbb{R}$ of the variables (t,x,p,w,q) satisfies condition (V), the derivatives $(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$ exist on Ω and the following conditions hold:

1) $\partial_q f(\cdot, x, p, w, q)\colon I[x] \to \mathbb{R}^n$ is measurable for $(x,p,w,q) \in [-b,b] \times \mathbb{R} \times C(B,\mathbb{R}) \times \mathbb{R}^n$ and $\partial_q f(t,\cdot)\colon S_t \times \mathbb{R} \times C(B,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for almost all $t \in [0,a]$,
2) there are $L \in \mathbb{L}([0,a], \mathbb{R}_+)$ and $\alpha \in \mathbb{L}([0,a], \mathbb{R}^n_+)$ such that

$$(|\partial_{q_1} f(t,x,p,w,q)|, \dots, |\partial_{q_n} f(t,x,p,w,q)|) \le (\alpha_1(t), \dots, \alpha_n(t))$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and

$$\|\partial_q f(t,x,p,w,q) - \partial_q f(t,\bar{x},\bar{p},\bar{w},\bar{q})\| \le L(t)\big[\|x-\bar{x}\| + |p-\bar{p}| + \|w-\bar{w}\|_B + \|q-\bar{q}\|\big] \tag{10}$$

on Ω,
3) for $t \in [0,a]$ we have $M(t) = \int_0^t \alpha(\tau)\,d\tau$,
4) $\phi_0 \in C([0,a],\mathbb{R})$, $\phi \in C(E,\mathbb{R}^n)$ and
   (i) for $(t,x) \in E$ we have $\varphi(t,x) = (\phi_0(t), \phi(t,x)) \in E$ and $\phi_0(t) \le t$,
   (ii) the partial derivatives $\partial_x \phi = [\partial_{x_j} \phi_i]_{i,j=1,\dots,n}$ exist on E and $\|\partial_x \phi(t,x)\| \le Q$ on E.

Lemma 2.1. *Suppose that $\Lambda\colon \Omega \to \mathbb{R}$ is continuous and*

1) *$\varphi = (\phi_0, \phi)$ satisfies conditions* 4) *of Assumption H_1,*
2) *there is $\tilde{\lambda}\colon [0,a] \to \mathbb{R}_+$ such that*

$$\|\Lambda(t,x,p,w,q) - \Lambda(t,\bar{x},\bar{p},\bar{w},\bar{q})\| \le \tilde{\lambda}(t)[\|x-\bar{x}\| + |p-\bar{p}| + \|w-\bar{w}\|_B + \|q-\bar{q}\|] \quad \text{on } \Omega$$

and Λ satisfies condition (V),
3) *$\psi \in \mathbb{K}$ and $z \in C_{\psi.c}[d]$, $u \in C_{\partial\psi.c}[s]$.*

Then

$$\|\Lambda(t,x,z(t,x),z_{\varphi(t,x)},u(t,x)) - \Lambda(t,\bar{x},z(t,\bar{x}),z_{\varphi(t,\bar{x})},u(t,\bar{x}))\| \le \tilde{\lambda}(t)(1 + d(1+Q) + s_2)\|x-\bar{x}\| \quad \text{on } H_c.$$

Proof. Note that the functions $z_{\varphi(t,x)}$ and $z_{\varphi(t,\bar{x})}$ have different domains. Hence we need the following construction. Write $Y = [-b_0, a] \times [-4b, 4b]$. There is $\tilde{z}\colon Y \to \mathbb{R}$ such that

(i) $\tilde{z} \in C(Y,\mathbb{R})$ and $\tilde{z}|_{E_0 \cup E} = z$,
(ii) for $(t,x), (t,\bar{x}) \in Y$ we have $|\tilde{z}(t,x) - \tilde{z}(t,\bar{x})| \le d\|x - \bar{x}\|$.

For given (t,x) define $w,\bar{w}\colon B\to\mathbb{R}$ by

$$w(\tau,y)=\tilde{z}_{\varphi(t,x)}(\tau,y)=\tilde{z}(\psi(t,x)+(\tau,y)),$$
$$\bar{w}(\tau,y)=\tilde{z}_{\varphi(t,\bar{x})}(\tau,y)=\tilde{z}(\psi(t,\bar{x})+(\tau,y)),$$

where $(\tau,y)\in B$. Then

$$\begin{aligned}\|\varLambda(t,x,z(t,x),z_{\varphi(t,x)},u(t,x))-\varLambda(t,\bar{x},z(t,\bar{x}),z_{\varphi(t,\bar{x})},u(t,\bar{x}))\|\\ &=\|\varLambda(t,x,z(t,x),w,u(t,x))-\varLambda(t,\bar{x},z(t,\bar{x}),\bar{w},u(t,\bar{x}))\|\\ &\le\tilde{\lambda}(t)[\|x-\bar{x}\|+|z(t,x)-z(t,\bar{x})|+\|w-\bar{w}\|_B+\|u(t,x)-u(t,\bar{x})\|]\\ &=\tilde{\lambda}(t)[\|x-\bar{x}\|+|z(t,x)-z(t,\bar{x})|\\ &\quad+\|\tilde{z}_{\varphi(t,x)}-\tilde{z}_{\varphi(t,\bar{x})}\|_B+\|u(t,x)-u(t,\bar{x})\|]\\ &\le\tilde{\lambda}(t)(1+d(1+Q)+s_2)\|x-\bar{x}\|.\end{aligned}$$

This completes the proof.

Suppose that Assumption H_1 is satisfied and $\psi\in\mathbb{K}$, $z\in C_{\psi.c}[d]$, $u\in C_{\partial\psi.c}[s]$. Write $T[z,u](t,x)=(t,x,z(t,x),z_{\varphi(t,x)},u(t,x))$. Let $g[z,u](\cdot,t,x)$ denote the solution of the Cauchy problem

$$\eta'(\tau)=-\partial_q f(T[z,u](\tau,\eta(\tau))),\qquad \eta(t)=x, \tag{11}$$

where $(t,x)\in H_c$. The function $g[z,u](\cdot,t,x)$ is the bicharacteristic of (1) corresponding to (z,u). The main properties of the bicharacteristic are given in the following lemma.

LEMMA 2.2. *Suppose that Assumption H_1 is satisfied and*

$$\psi,\tilde{\psi}\in\mathbb{K},\quad z\in C_{\psi.c}[d],\quad \tilde{z}\in C_{\tilde{\psi}.c}[d],\quad u\in C_{\partial\psi.c}[s],\quad \tilde{u}\in C_{\partial\tilde{\psi}.c}[s]$$

where $0<c\le a$. Then the bicharacteristics $g[z,u](\cdot,t,x)$ and $g[\tilde{z},\tilde{u}](\cdot,t,x)$ exist on intervals $[0,\kappa[z,u](t,x)]$ and $[0,\kappa[\tilde{z},\tilde{u}](t,x)]$ such that

$$(\kappa[z,u](t,x),g[z,u]](\kappa[z,u](t,x),t,x))\in\partial H_c$$

and

$$(\kappa[\tilde{z},\tilde{u}](t,x),g[z](\kappa[\tilde{z},\tilde{u}](t,x),t,x))\in\partial H_c$$

where ∂H_c is the boundary of H_c. The solution of (11) *is unique and we have the estimates*

$$\|g[z,u](\tau,t,x)-g[z,u](\tau,t,\bar{x})\|\le\Theta(c)\|x-\bar{x}\| \tag{12}$$

where $\tau\in[0,\min\{\kappa[z,u](t,x),\kappa[z,u](\bar{t},\bar{x})\}]$ and

$$\begin{aligned}\|g[z,u](\tau,t,x)-g[\tilde{z},\tilde{u}](\tau,t,x)\|\\ \le\Theta(c)\Big|\int_\tau^t L(\zeta)[\|z-\tilde{z}\|_{(\zeta,\mathbb{R})}+\|u-\tilde{u}\|_{(\zeta,\mathbb{R}^n)}]\,d\zeta\Big|\end{aligned} \tag{13}$$

where $\tau \in [0, \min\{\kappa[z,u](t,x), \kappa[\tilde{z},\tilde{u}](t,x)\}]$ *and*

$$\Theta(\tau) = 2\exp\Big[(1 + d(1+Q) + s_2)\int_0^\tau L(\zeta)\,d\zeta\Big].$$

Proof. The existence and uniqueness of the solution of (11) follows from classical theorems. Note that the right-hand side of the differential system satisfies the Carathéodory conditions, and the following Lipschitz condition holds:

$$\|\partial_q f(T[z,u](\tau,y)) - \partial_q f(T[z,u](\tau,\bar{y}))\| \le L(\tau)(1 + d(1+Q) + s_2)\|y - \bar{y}\|.$$

We prove that the bicharacteristic $g[z,u](\cdot,t,x)$ exists on $[0, \kappa[z,u](t,x)]$. Suppose that $[t_0, t]$ is the interval on which the bicharacteristic is defined. Then

$$-\alpha(\tau) \le \frac{d}{d\tau} g[z,u](\tau,t,x) \le \alpha(\tau) \quad \text{for } \tau \in [t_0,t],$$

and consequently

$$-b + M(\tau) \le g[z,u](\tau,t,x) \le b - M(\tau) \quad \text{for } \tau \in [t_0,t].$$

This yields $(\tau, g[z,u](\tau,t,x)) \in E$ for $\tau \in [t_0,t]$ and the assertion follows. It follows from Assumption H_1 and Lemma 2.1 that the bicharacteritics satisfy the integral inequality

$$\|g[z,u](\tau,t,x) - g[z,u](\tau,t,\bar{x})\| \\ \le \|x - \bar{x}\| + (1 + d(1+Q) + s_2)\Big|\int_\tau^t L(\zeta)\|g[z,u](\zeta,t,x) - g[z,u](\zeta,t,\bar{x})\|d\zeta\Big|,$$

where $\tau \in [0, \min\{\kappa[z,u](t,x), \kappa[z,u](t,\bar{x})\}]$. It follows from the Gronwall inequality that estimate (12) is satisfied. Now we prove inequality (13). For $z \in C_{\psi.c}[d]$, $\tilde{z} \in C_{\tilde{\psi}.c}[d]$, $u \in C_{\partial\psi.c}[s]$, $\tilde{u} \in C_{\partial\tilde{\psi}.c}[s]$ we have

$$\|g[z,u](\tau,t,x) - g[\tilde{z},\tilde{u}](\tau,t,x)\| \\ \le 2\Big|\int_\tau^t L(\zeta)[\|z - \tilde{z}\|_{(\zeta,\mathbb{R})} + \|u - \tilde{u}\|_{(\zeta,\mathbb{R}^n)}]\,d\zeta\Big| \\ + (1 + d(1+Q) + s_2)\Big|\int_\tau^t L(\zeta)\|g[z,u](\zeta,t,x) - g[\tilde{z},\tilde{u}](\zeta,t,x)\|\,d\zeta\Big|$$

where $\tau \in [0, \min\{\kappa[z,u](t,x), \kappa[\tilde{z},\tilde{u}](t,x)\}]$. From the Gronwall inequality we deduce (13).

3. Integral equations. Let us denote by $CL(B,\mathbb{R})$ the class of all continuous linear operators from $C(B,\mathbb{R})$ to $\mathbb{R}$. The norm in $CL(B,\mathbb{R})$ generated by the maximum norm in $C(B,\mathbb{R})$ will be denoted by $\|\cdot\|_\star$. The scalar product in $\mathbb{R}^n$ is denoted by $\circ$.

Assumption H_2. Assumption H_1 is satisfied and

1) $f(\cdot,x,p,w,q): I[x] \to \mathbb{R}$ is measurable for $(x,p,w,q) \in [-b,b]\times\mathbb{R}\times C(B,\mathbb{R})\times\mathbb{R}^n$,
2) $(\partial_{x_1}f,\dots,\partial_{x_n}f) = \partial_x f$, $\partial_p f$ exist on Ω and
$$\partial_x f(\cdot,x,p,w,q): I[x]\to\mathbb{R}^n, \qquad \partial_p f(\cdot,x,p,w,q): I[x]\to\mathbb{R}$$
are measurable for $(x,p,w,q)\in[-b,b]\times\mathbb{R}\times C(B,\mathbb{R})\times\mathbb{R}^n$,
3) the Fréchet derivative $\partial_w f$ exists and $\partial_w f(t,x,p,w,q)\in CL(B,\mathbb{R})$ on Ω, and $\partial_w f(\cdot,x,p,w,q)\tilde{w} : I[x]\to\mathbb{R}$ is measurable for $(x,p,w,q)\in[-b,b]\times\mathbb{R}\times C(B,\mathbb{R})\times\mathbb{R}^n)$, $\tilde{w}\in C(B,\mathbb{R})$,
4) there are $\alpha_0,\beta\in\mathbb{L}([0,a],\mathbb{R}_+)$ such that $|f(t,x,p,w,q)|\le\alpha_0(t)$ and
$$\|\partial_x f(t,x,p,w,q)\|, |\partial_p f(t,x,p,w,q)|, \|\partial_w f(t,x,p,w,q)\|_\star \le \beta(t)$$
on Ω and the expressions
$$\|\partial_x f(t,x,p,w,q)-\partial_x f(t,\bar{x},\bar{p},\bar{w},\bar{q})\|,$$
$$|\partial_p f(t,x,p,w,q)-\partial_p f(t,\bar{x},\bar{p},\bar{w},\bar{q})|,$$
$$\|\partial_w f(t,x,p,w,q)-\partial_w f(t,\bar{x},\bar{p},\bar{w},\bar{q})\|_\star$$
are bounded from above by $L(t)\big[\|x-\bar{x}\|+|p-\bar{p}|+\|w-\bar{w}\|_D+\|q-\bar{q}\|\big]$,
5) $Q\in\mathbb{R}_+$ is such that $\|\partial_x\phi(t,x)-\partial_x\phi(t,\bar{x})\|\le Q\|x-\bar{x}\|$ on E.

Write $P[z,u](\tau,t,x)=T[z,u](\tau,g[z,u](\tau,t,x)))$ and
$$\partial_w f(P)\star\tilde{w} = (\partial_w f(P)\tilde{w}_1,\dots,\partial_w f(P)\tilde{w}_n),$$
$$u_{\varphi(t,x)} = ((u_1)_{\varphi(t,x)},\dots,(u_n)_{\varphi(t,x)})$$
where $P=(t,x,p,w,q)\in\Omega$, $\tilde{w}=(\tilde{w}_1,\dots,\tilde{w}_n)\in C(B,\mathbb{R}^n)$ and $u=(u_1,\dots,u_n)\in C(E_0\cup E,\mathbb{R}^n)$. Let $(u_{\varphi(t,x)})\partial_x\phi(t,x)\colon D[\psi(t,x)]\to\mathbb{R}^n$ be defined by
$$(u_{\varphi(t,x)})\partial_x\phi(t,x) = \Big(\sum_{\nu=1}^n \partial_{x_1}\phi_\nu(t,x)(u_\nu)_{\varphi(t,x)},\dots,\sum_{\nu=1}^n \partial_{x_n}\phi_\nu(t,x)(u_\nu)_{\varphi(t,x)}\Big).$$

Define $F[z,u]$ and $G[z,u]=(G_1[z,u],\dots,G_n[z,u])$ by
$$F[z,u](t,x) = \psi(0,g[z,u](0,t,x)) + \int_0^t \big[f(P[z,u](\zeta,t,x)) - \partial_q f(P[z,u](\zeta,t,x))\circ u(\zeta,g[z,u](\zeta,t,x))\big]\,d\zeta$$

and

$$G[z,u](t,x) = \partial_x\psi(0, g[z,u](0,t,x)) + \int_0^t \partial_x f(P[z,u](\zeta,t,x))\,d\zeta$$
$$+ \int_0^t \partial_p f(P[z,u](\zeta,t,x)) u(\zeta, g[z,u](\zeta,t,x))\,d\zeta$$
$$+ \int_0^t [\partial_w f(P[z,u](\zeta,t,x)) \star u_{\varphi(\zeta, g[z,u](\zeta,t,x))}]$$
$$\cdot\, \partial_x\phi(\zeta, g[z,u](\zeta,t,x))\,d\zeta.$$

We shall consider the following system of functional integral equations:

$$z(t,x) = F[z,u](t,x), \quad u(t,x) = G[z,u](t,x), \tag{14}$$

$$g[z,u](\tau,t,x) = x + \int_\tau^t \partial_q f(P[z,u](\zeta,t,x))\,d\zeta, \tag{15}$$

$$z(t,x) = \psi(t,x), \qquad u(t,x) = \partial_x\psi(t,x) \qquad \text{on } E_0. \tag{16}$$

The proof of the existence of a solution of the above problem is based on the following method of successive approximations. Suppose that Assumption H_2 is satisfied and $\psi \in \mathbb{K}$. We define sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ in the following way. We first put

$$z^{(0)}(t,x) = \begin{cases} \psi(t,x) & \text{on } E_0, \\ \psi(0,x) & \text{on } H_c, \end{cases} \qquad u^{(0)}(t,x) = \begin{cases} \partial_x\psi(t,x) & \text{on } E_0, \\ \partial_x\psi(0,x) & \text{on } H_c. \end{cases}$$

If $z^{(m)} : E_c \to \mathbb{R}$, $u^{(m)} : E_c \to \mathbb{R}^n$ are already defined then $u^{(m+1)} = \partial_x\psi(t,x)$ on E_0 and $u^{(m+1)}$ is a solution of the equation

$$u(t,x) = G^{(m)}[u](t,x), \qquad (t,x) \in E \cap ([0,c] \times \mathbb{R}^n), \tag{17}$$

where $G^{(m)} = (G_1^{(m)}, \ldots, G_n^{(m)})$ is defined by

$$G^{(m)}[u](t,x) = \partial_x\psi(0, g[z^{(m)},u](0,t,x)) + \int_0^t \partial_x f(P[z^{(m)},u](\zeta,t,x))\,d\zeta$$
$$+ \int_0^t \partial_p f(P[z^{(m)},u](\zeta,t,x)) u^{(m)}(\zeta, g[z^{(m)},u](\zeta,t,x))\,d\zeta$$
$$+ \int_0^t [\partial_w f(P[z^{(m)},u](\zeta,t,x)) \star (u^{(m)})_{\varphi(\zeta, g[z^{(m)},u](\zeta,t,x))}]$$
$$\cdot\, \partial_x\phi(\zeta, g[z^{(m)},u](\zeta,t,x))\,d\zeta.$$

The function $z^{(m+1)}$ is given by

(18) $$z^{(m+1)}(t,x) = F[z^{(m)}, u^{(m+1)}](t,x) \quad \text{on } H_c,$$

(19) $$z^{(m+1)}(t,x) = \psi(t,x) \quad \text{on } E_0.$$

Remark 3.1. Observe that the equations

$$u(t,x) = G[z^{(m)}, u](t,x)$$

and (17) are not identical. Equation (17) is obtained in the following way. Suppose that $z^{(m)} \colon E_c \to \mathbb{R}$ and $u^{(m)} \colon E_c \to \mathbb{R}^n$ are known functions. Consider the differential equation

(20) $$\partial_t z(t,x) = f(t,x,z^{(m)}(t,x), z^{(m)}_{\varphi(t,x)}, \partial_x z(t,x)).$$

We put $u = \partial_x z$ in (20). Then we obtain the differential equations for u:

(21) $$\begin{aligned}\partial_t u_i(t,x) &= \partial_{x_i} f(S[z^{(m)},u](t,x)) + \partial_p f(S[z^{(m)},u](t,x))\partial_{x_i} z^{(m)}(t,x)\\ &\quad + [\partial_w f(S[z^{(m)},u](t,x)) \star (\partial_x z^{(m)})_{\varphi(t,x)}] \circ \partial_{x_i}\phi(t,x)\\ &\quad + \partial_q f(S[z^{(m)},u](t,x)) \circ \partial_x u_i(t,x), \qquad i = 1,\dots,n,\end{aligned}$$

where $S[z^{(m)},u](t,x) = (t,x,z^{(m)}(t,x), z^{(m)}_{\varphi(t,x)}, u(t,x))$. If we assume that $\partial_x z^{(m)} = u^{(m)}$ (see Theorem 3.1), then by integrating (21) along the bicharacteristic $g[z^{(m)},u](\cdot,t,x)$ we obtain (17).

We prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ exist on E_c provided $c \in (0,a]$ is sufficiently small. Write

$$\tilde{A}(\tau) = \Theta(c)\Big(\int_0^\tau \beta(\zeta)\,d\zeta + s_1 \int_0^\tau L(\zeta)\,d\zeta\Big),$$

$$A(\tau) = \Theta(c)[1 + d(1+Q) + s_2]\int_0^\tau L(\zeta)\,d\zeta,$$

$$B(\tau) = \Theta(c)\int_0^\tau \beta(\zeta)\,d\zeta, \qquad \tau \in (0,c].$$

Assumption H_3. The constant $c \in (0,a]$ is so small that

$$d \ge a_1\Theta(c) + \tilde{A}(c)(1 + d(Q+1) + s_2) + \|M(c)\|\Theta(c)s_2,$$

$$s_1 \ge a_1 + (1 + s_1(1+Q))\int_0^c \beta(\zeta)\,d\zeta,$$

$$s_2 \ge a_2\Theta(c) + A(c)(1 + s_1(1+Q)) + B(c)(s_2(1+Q^2) + s_1 Q).$$

Remark 3.2. Since $d > 2a_1$, $s_1 > a_1$, $s_2 > 2a_2$, there is $c \in (0,a]$ such that Assumption H_3 is satisfied.

THEOREM 3.1. *If $\psi \in \mathbb{K}$ and Assumptions H_2, H_3 are satisfied then for any $m \in \mathbb{N}$:*

(I_m) *the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are defined on E_c and $z^{(m)} \in C_{\psi.c}[d]$, $u^{(m)} \in C_{\partial\psi.c}[s]$,*

(II_m) *$\partial_x z^{(m)} = u^{(m)}$ on E_c.*

Proof. We argue by induction. It follows from the definitions of $z^{(0)}$ and $u^{(0)}$ that conditions (I_0) and (II_0) are satisfied. Suppose that (I_m) and (II_m) hold for a given $m \geq 0$. We first prove that there is $u^{(m+1)} \in C_{\partial\psi.c}[s]$. We claim that

$$G^{(m)} \colon C_{\partial\psi.c}[s] \to C_{\partial\psi.c}[s]. \tag{22}$$

Suppose that $u \in C_{\partial\psi.c}[s]$ and $(t,x), (t,\bar{x}) \in H_c$. It follows from Assumptions H_2 and H_3 that

$$\|G^{(m)}[u](t,x)\| \leq a_1 + (1 + s_1(1+Q)) \int_0^c \beta(\zeta)\, d\zeta \leq s_1$$

and

$$\|G^{(m)}[u](t,x) - G^{(m)}[u](t,\bar{x})\|$$
$$\leq a_2\Theta(c) + A(c)[1+s_1(1+Q)] + B(c)(s_2(1+Q^2)+s_1Q)\|x-\bar{x}\| \leq s_2\|x-\bar{x}\|.$$

This proves (22). It follows from Assumption H_2 and from (I_m) that there is $\Gamma \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that for $u, \bar{u} \in C_{\partial_\psi.c}[s]$ we have

$$\|G^{(m)}[u](t,x) - G^{(m)}[\bar{u}](t,x)\| \leq \int_0^t \Gamma(\zeta)\|u-\bar{u}\|_{(\zeta,\mathbb{R}^n)}\, d\zeta, \quad (t,x) \in H_c.$$

Write

$$[\![u-\bar{u}]\!] = \max\Big\{\|u-\bar{u}\|_{(\tau,\mathbb{R}^n)} \exp\Big\{-2\int_0^\tau \Gamma(s)\,ds\Big\} : \tau \in [0,c]\Big\}.$$

Then we have

$$\|G^{(m)}[u](t,x) - G^{(m)}[\bar{u}](t,x)\|$$
$$\leq [\![u^{(m+1)} - \bar{u}^{(m+1)}]\!] \Big|\int_0^t \Gamma(\tau) \exp\Big\{2\int_0^\tau \Gamma(s)\,ds\Big\}\,d\tau\Big|$$
$$\leq \frac{1}{2}[\![u^{(m+1)} - \bar{u}^{(m+1)}]\!] \exp\Big\{2\int_0^t \Gamma(s)\,ds\Big\}$$

and consequently

$$[\![G^{(m)}[u] - G^{(m)}[\bar{u}]]\!] \leq \frac{1}{2}|u-\bar{u}|.$$

From the Banach fixed point theorem we find that $u^{(m+1)} \in C_{\partial\psi.c}[s]$ exists and is unique. It follows easily that $z^{(m+1)}$ given by (18), (19) satisfies the condition $z^{(m+1)} \in C_{\psi.c}[d]$. Now we will show (II_{m+1}). Write

$$U(t,x,\bar{x}) = z^{(m+1)}(t,\bar{x}) - z^{(m+1)}(t,x) - u^{(m+1)}(t,x) \circ (\bar{x} - x).$$

We will prove that there is $K > 0$ such that

$$|U(t,x,\bar{x})| \le K\|x - \bar{x}\|^2 \quad \text{for } (t,x),(t,\bar{x}) \in E_c. \tag{23}$$

It follows that

$$U(t,x,\bar{x}) = F[z^{(m)}, u^{(m+1)}](t,\bar{x}) - F[z^{(m)}, u^{(m+1)}](t,x) \\ - G^{(m)}[u^{(m+1)}](t,x) \circ (\bar{x} - x).$$

For $m \in \mathbb{N}$ write

$$g^{(m)}(\tau,t,x) = g[z^{(m)}, u^{(m+1)}](\tau,t,x),$$
$$P^{(m)}(\tau,t,x) = T[z^{(m)}, u^{(m+1)}](\tau, g^{(m)}(\tau,t,x)).$$

Note that $(z^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,\bar{x}))}$ and $z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,x))}$ have different domains. Hence we need the following construction. Put $\Delta = [-b_0 - a, a] \times [-4b, 4b]$. There are $Z^{(m)} \in C(\Delta, \mathbb{R})$ and $U^{(m)} \in C(\Delta, \mathbb{R}^n)$ such that

(i) $Z^{(m)}(t,x) = z^{(m)}(t,x)$ and $U^{(m)}(t,x) = u^{(m)}(t,x)$ on $E_0 \cup E_c$, where $U^{(m)} = (U_1^{(m)}, \ldots, U_n^{(m)})$,
(ii) $\partial_x Z^{(m)}(t,x) = U^{(m)}(t,x)$ on Δ.

Then the functions

$$(Z^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,\bar{x}))}, (Z^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))}, (U^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))},$$

for $(t,x),(t,\bar{x}) \in E_c$, $\xi \in [0,c]$, are defined on B. Write

$$T^{(m)}(\tau,\xi,t,x,\bar{x}) = \tau T[Z^{(m)}, u^{(m+1)}](\xi, g^{(m)}(\xi,t,\bar{x})) \\ + (1-\tau) T[Z^{(m)}, u^{(m+1)}](\xi, g^{(m)}(\xi,t,x)), \quad 0 \le \tau \le 1.$$

We apply the Hadamard mean value theorem to the difference

$$f(P[z^{(m)}, u^{(m+1)}](\xi,t,\bar{x})) - f(P[z^{(m)}, u^{(m+1)}](\xi,t,x)).$$

Then we have

$$
\begin{aligned}
(24)\qquad U(t,x,\bar{x}) &= \psi(0,g^{(m)}(0,t,\bar{x}))\\
&- \psi(0,g^{(m)}(0,t,x)) - \partial_x\psi(0,g^{(m)}(0,t,\bar{x}))\circ(\bar{x}-x)\\
&+ \int_0^t\int_0^1 \partial_x f(T^{(m)}(\tau,\xi,t,x,\bar{x}))\circ[g^{(m)}(\xi,t,\bar{x})-g^{(m)}(\xi,t,x)]\,d\xi\\
&+ \int_0^t\int_0^1 \partial_p f(T^{(m)}(\tau,\xi,t,x,\bar{x}))\\
&\qquad\cdot[z^{(m)}(\xi,g^{(m)}(\xi,t,\bar{x}))-z^{(m)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi\\
&+ \int_0^t\int_0^1 \partial_w f(T^{(m)}(\tau,\xi,t,x,\bar{x}))\star[Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,\bar{x}))}-Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,x))}]\,d\xi\\
&+ \int_0^t\int_0^1 \partial_q f(T^{(m)}(\tau,\xi,t,x,\bar{x}))\\
&\qquad\circ[u^{(m+1)}(\xi,g^{(m)}(\xi,t,\bar{x}))-u^{(m+1)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi\\
&- \int_0^t [\partial_q f(P^{(m)}(\xi,t,\bar{x}))\circ u^{(m+1)}(\xi,g^{(m)}(\xi,t,\bar{x}))\\
&- \partial_q f(P^{(m)}(\xi,t,x))\circ u^{(m+1)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi\\
&- \int_0^t \partial_x f(P^{(m)}(\xi,t,x))\,d\xi\circ(\bar{x}-x)\\
&- \int_0^t \partial_p f(P^{(m)}(\xi,t,x))u^{(m)}(\xi,g^{(m)}(\xi,t,x))\,d\xi\circ(\bar{x}-x)\\
&- \int_0^t [\partial_w f(P^{(m)}(\xi,t,x))\star(U^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))}]\\
&\qquad\cdot\partial_x\phi(\xi,g^{(m)}(\xi,t,x))\,d\xi\circ(\bar{x}-x).
\end{aligned}
$$

For simplicity of formulation of the next properties of $U(t,x,\bar{x})$ we write

$$
\begin{aligned}
\bar{U}(t,x,\bar{x}) = &\int_0^t\int_0^1 [\partial_x f(T^{(m)}(\tau,\xi,t,x,\bar{x}))-\partial_x f(P^{(m)}(\xi,t,x))]\,d\tau\\
&\qquad\cdot[g^{(m)}(\xi,t,\bar{x})-g^{(m)}(\xi,t,x)]\,d\xi\\
&+\int_0^t\int_0^1 [\partial_p f(T^{(m)}(\tau,\xi,t,x,\bar{x}))-\partial_p f(P^{(m)}(\xi,t,x))]\,d\tau\\
&\qquad\cdot[z^{(m)}(\xi,g^{(m)}(\xi,t,\bar{x}))-z^{(m)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi
\end{aligned}
$$

$$+\int_0^t\int_0^1 [\partial_w f(T^{(m)}(\tau,\xi,t,x,\bar{x})) - \partial_w f(P^{(m)}(\xi,t,x))]\,d\tau$$

$$\star\,[Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,\bar{x}))} - Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,x))}]\,d\xi$$

$$+\int_0^t\int_0^1 [\partial_q f(T^{(m)}(\tau,\xi,t,x,\bar{x})) - \partial_q f(P^{(m)}(\xi,t,x))]\,d\tau$$

$$\circ\,[u^{(m+1)}(\xi,g^{(m)}(\xi,t,\bar{x})) - u^{(m+1)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi,$$

$$U_\psi(t,x,\bar{x}) = \psi(0,g^{(m)}(0,t,\bar{x})) - \psi(0,g^{(m)}(0,t,x)) - \partial_x\psi(0,g^{(m)}(0,t,x))\circ[g^{(m)}(0,t,\bar{x}) - g^{(m)}(0,t,x)],$$

and

$$\tilde{U}(t,x,\bar{x}) = \int_0^t \partial_p f(P^{(m)}(\xi,t,x))[z^{(m)}(\xi,g^{(m)}(\xi,t,\bar{x})) - z^{(m)}(\xi,g^{(m)}(\xi,t,x))$$

$$- u^{(m)}(\xi,g^{(m)}(\xi,t,x))\circ(g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x))]\,d\xi$$

$$+\int_0^t \partial_w f(P^{(m)}(\xi,t,x))\star[Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,\bar{x}))} - Z^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,x))}$$

$$- (U^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))}\partial_x\phi(\xi,g^{(m)}(\xi,t,x))\circ(g^{(m)}(\xi,t,\bar{x}) - g^{(m)}(\xi,t,x))]\,d\xi,$$

$$\Gamma(t,x,\bar{x}) = \partial_x\psi(0,g^{(m)}(0,t,x))\circ\int_0^t [\partial_q f(P^{(m)}(\tau,t,\bar{x})) - \partial_q f(P^{(m)}(\tau,t,x))]\,d\tau$$

$$+\int_0^t \partial_x f(P^{(m)}(\xi,t,x))\circ\int_\xi^t [\partial_q f(P^{(m)}(\tau,t,\bar{x})) - \partial_q f(P^{(m)}(\tau,t,x))]\,d\tau\,d\xi$$

$$+\int_0^t \partial_p f(P^{(m)}(\xi,t,x))u^{(m)}(\xi,g^{(m)}(\xi,t,x))$$

$$\circ\int_\xi^t [\partial_q f(P^{(m)}(\tau,t,\bar{x})) - \partial_q f(P^{(m)}(\tau,t,x))]\,d\tau\,d\xi$$

$$+\int_0^t \partial_w f(P^{(m)}(\xi,t,x))\star(u^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))}\partial_x\phi(\xi,g^{(m)}(\xi,t,x))$$

$$\circ\int_\xi^t [\partial_q f(P^{(m)}(\tau,t,\bar{x})) - \partial_q f(P^{(m)}(\tau,t,x))]\,d\tau\,d\xi.$$

We conclude from Assumptions H_1, H_2 that there is $\tilde{C} > 0$ such that

$$|\bar{U}(t,x,\bar{x})| + |U_\psi(t,x,\bar{x})| + |\tilde{U}(t,x,\bar{x})| \le \tilde{C}\|x-\bar{x}\|^2 \tag{25}$$

for $(t,x),(t,\bar{x})\in E_c$. It follows from (11) that

$$g^{(m)}(\xi,t,\bar{x})-g^{(m)}(\xi,t,x)-(\bar{x}-x)=\int_{\xi}^{t}[\partial_q f(P^{(m)}(\tau,t,\bar{x}))-\partial_q f(P^{(m)}(\tau,t,x))]\,d\tau.$$

The above relation and (24) imply

$$\begin{aligned}(26)\quad U(t,x,\bar{x})&=\bar{U}(t,x,\bar{x})+U_{\psi}(t,x,\bar{x})+\tilde{U}(t,x,\bar{x})+\mathit{\Gamma}(t,x,\bar{x})\\&-\int_0^t\{\partial_q f(P^{(m)}(\xi,t,\bar{x}))\circ u^{(m+1)}(\xi,g^{(m)}(\xi,t,\bar{x}))\\&-\partial_q f(P^{(m)}(\xi,t,x))\circ u^{(m+1)}(\xi,g^{(m)}(\xi,t,x))\}\,d\xi\\&+\int_0^t\partial_q f(P^{(m)}(\xi,t,x))\circ[u^{(m+1)}(\xi,g^{(m)}(\xi,t,\bar{x}))-u^{(m+1)}(\xi,g^{(m)}(\xi,t,x))]\,d\xi.\end{aligned}$$

It follows easily that

$$\begin{aligned}\mathit{\Gamma}(t,x,\bar{x})=\int_0^t[\partial_q f(P^{(m)}(\tau,t,\bar{x}))-\partial_q f(P^{(m)}(\tau,t,x))]\circ\Big\{\partial_x\psi(0,g^{(m)}(0,t,x))\\+\int_0^{\tau}\partial_x f(P^{(m)}(\xi,t,x))\,d\xi+\int_0^{\tau}\partial_p f(P^{(m)}(\xi,t,x))u^{(m)}(\xi,g^{(m)}(\xi,t,x))\,d\xi\\+\int_0^{\tau}\partial_w f(P^{(m)}(\xi,t,x))\star(u^{(m)})_{\varphi(\xi,g^{(m)}(\xi,t,x))}\partial_x\phi(\xi,g^{(m)}(\xi,t,x))\,d\xi\Big\}d\tau.\end{aligned}$$

The bicharacteristics satisfy the following group property:

$$g^{(m)}(\xi,\tau,g^{(m)}(\tau,x,y))=g^{(m)}(\xi,x,y).$$

Therefore we get

$$\begin{aligned}u^{(m+1)}(\tau,g^{(m)}(\tau,t,x))&=\partial_x\psi(0,g^{(m)}(0,t,x))\\&+\int_0^{\tau}\partial_x f(P^{(m)}(\xi,t,x))\,d\xi+\int_0^{\tau}\partial_p f(P^{(m)}(\xi,t,x))u^{(m)}(\xi,g^{(m)}(\xi,t,x))\,d\xi\\&+\int_0^{\tau}[\partial_w f(P^{(m)}(\xi,t,x))\star u^{(m)}_{\varphi(\xi,g^{(m)}(\xi,t,x))}]\partial_x\phi(\xi,g^{(m)}(\xi,t,x))\,d\xi\end{aligned}$$

and consequently

$$\mathit{\Gamma}(t,x,\bar{x})=\int_0^t[\partial_q f(P^{(m)}(\tau,t,\bar{x}))-\partial_q f(P^{(m)}(\tau,t,x))]\circ u^{(m+1)}(\tau,g^{(m)}(\tau,t,x))\,d\tau.$$

Write

$$U_\star(t,x,\bar{x}) = -\int_0^t [\partial_q f(P^{(m)}(\xi,t,\bar{x})) - \partial_q f(P^{(m)}(\xi,t,x))] \circ [u^{(m+1)}(\tau,g^{(m)}(\tau,t,\bar{x})) - u^{(m+1)}(\tau,g^{(m)}(\tau,t,x))]\,d\tau.$$

It is clear that there is $C_\star > 0$ such that

$$|U_\star(t,x,\bar{x})| \le C_\star \|x-\bar{x}\|^2, \qquad (t,x),(t,\bar{x}) \in E_c. \tag{27}$$

We conclude from (26) and from the above relation that

$$U(t,x,\bar{x}) = U_\psi(t,x,\bar{x}) + \bar{U}(t,x,\bar{x}) + \tilde{U}(t,x,\bar{x}) + U_\star(t,x,\bar{x})$$

for $(t,x),(t,\bar{x}) \in E_c$. The above relation and (25), (27) imply (23). This completes the proof of the theorem.

4. Existence of solutions to initial value problems. We formulate the main result of the paper.

THEOREM 4.1. *If Assumptions H_1, H_2, H_3 are satisfied then there is a solution $\bar{z}\colon E_c \to \mathbb{R}$ of* (1), (2). *If $\tilde{\psi} \in \mathbb{K}$ and $\tilde{z}\colon E_c \to \mathbb{R}$ is a solution of* (1) *with the initial condition $\tilde{z}(t,x) = \tilde{\psi}(t,x)$ on E_0 then there is $\tilde{K} \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that*

$$\|\bar{z}-\tilde{z}\|_{(t,\mathbb{R})} + \|\partial_x\bar{z} - \partial_x\tilde{z}\|_{(t,\mathbb{R}^n)} \le \exp\Big\{\int_0^t \tilde{K}(\tau)\,d\tau\Big\}[\|\psi-\tilde{\psi}\|_{(0,\mathbb{R})} + \|\partial_x\psi - \partial_x\tilde{\psi}\|_{(0,\mathbb{R}^n)}], \qquad t \in [0,c]. \tag{28}$$

Proof. We first prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on E_c. It follows from (17), (18) that there are $K_0, K_1, K_2 \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that

$$\|z^{(m+1)} - z^{(m)}\|_{(t,\mathbb{R})} + \|u^{(m+1)} - u^{(m)}\|_{(t,\mathbb{R}^n)} \le \int_0^t K_0(\tau)[\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R})} + \|u^{(m+1)} - u^{(m)}\|_{(\tau,\mathbb{R}^n)}]\,d\tau \tag{29}$$

and

$$\|u^{(m+1)} - u^{(m)}\|_{(t,\mathbb{R}^n)} \le \int_0^t K_1(s)\|u^{(m+1)} - u^{(m)}\|_{(\tau,\mathbb{R}^n)}\,d\tau + \int_0^t K_2(\tau)[\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R})} + \|u^{(m)} - u^{(m-1)}\|_{(\tau,\mathbb{R}^n)}]\,d\tau \tag{30}$$

for $t \in [0,c]$, $m \geq 0$. From (30) we deduce that

$$\text{(31)} \quad \|u^{(m+1)} - u^{(m)}\|_{(t,\mathbb{R}^n)} \leq \exp\Big[\int_0^t K_1(\tau)\,d\tau\Big] \int_0^t [\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R})} + \|u^{(m)} - u^{(m-1)}\|_{(\tau,\mathbb{R}^n)}]\,d\tau$$

for $t \in [0,c]$. We conclude from (29) and (31) that there is $K \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that

$$\text{(32)} \quad \|z^{(m+1)} - z^{(m)}\|_{(t,\mathbb{R})} + \|u^{(m+1)} - u^{(m)}\|_{(t,\mathbb{R}^n)} \leq \int_0^t K(\tau)[\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R})} + \|u^{(m)} - u^{(m-1)}\|_{(\tau,\mathbb{R}^n)}]\,d\tau,$$

where $t \in [0,c]$ and $m \geq 0$. Write

$$W^{(m)}(t) = \max\Big\{[\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R})} + \|u^{(m)} - u^{(m-1)}\|_{(\tau,\mathbb{R}^n)}] \cdot \exp\Big[-2\int_0^\tau K(\xi)\,d\xi\Big] : \tau \in [0,t]\Big\}.$$

We conclude from (32) that

$$\begin{aligned} \|z^{(m+1)} - z^{(m)}\|_{(t,\mathbb{R})} + \|u^{(m+1)} - u^{(m)}\|_{(t,\mathbb{R}^n)} &\leq W^{(m)}(t) \int_0^t K(s) \exp\Big[2\int_0^s K(\xi)\,d\xi\Big]\,ds \\ &\leq \frac{1}{2}\,W^{(m)}(t) \exp\Big[2\int_0^t K(\xi)\,d\xi\Big], \qquad t \in [0,c], \end{aligned}$$

and consequently

$$W^{(m+1)}(t) \leq \frac{1}{2}\,W^{(m)}(t), \qquad t \in [0,c].$$

There is $C_0 \in \mathbb{R}_+$ such that $W^{(1)}(t) \leq C_0$ for $t \in [0,c]$. We thus get

$$\lim_{m\to\infty} W^{(m)}(t) = 0 \quad \text{uniformly on } [0,c]$$

and there are $\bar{z} \in C_{\psi.c}[d]$ and $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_n) \in C_{\partial\psi.c}[s]$ such that

$$\bar{z}(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \qquad \bar{u}(t,x) = \lim_{m\to\infty} u^{(m)}(t,x) \qquad \text{uniformly on} E_c.$$

It follows from Theorem 3.1 and from the definition of the sequence that $\partial_t\bar{z}$ and $\partial_x\bar{z}$ exist on E_c and $\partial_x\bar{z} = \bar{u}$. Furthermore,

$$\text{(33)} \qquad \bar{z}(t,x) = F[\bar{z},\bar{u}](t,x), \qquad \bar{u}(t,x) = G[\bar{z},\bar{u}](t,x), \qquad (t,x) \in E_c.$$

For a given $(t,x) \in E_c$ set $y = g[\bar{z}, \partial_x \bar{z}](0,t,x)$. Then $g[\bar{z}, \partial_x \bar{z}](\tau,t,x) = g[\bar{z}, \partial_x \bar{z}](\tau,0,y)$ for $\tau \in [0, \kappa(t,x)]$ where the interval $[0,\kappa(t,x)]$ is the domain of $g[\bar{z}, \partial_x \bar{z}](\cdot,t,x)$. Then the relations (33) imply

$$(34)\qquad \bar{z}(t, g[\bar{z}, \partial_x \bar{z}](t,0,y)) = \psi(0,y)$$
$$+\int_0^t [f(P[\bar{z}, \partial_x \bar{z}](\tau,0,y)) - \partial_q f(P[\bar{z}, \partial_x \bar{z}](\tau,0,y)) \circ \partial_x \bar{z}(\tau, g[\bar{z}, \partial_x \bar{z}](\tau,0,y))]\, d\tau.$$

The relations $y = g[\bar{z}, \partial_x \bar{z}](0,t,x)$ and $x = g[\bar{z}, \partial_x \bar{z}](t,0,y)$ are equivalent. By differentiating (34) with respect to t and by putting again $x = g[\bar{z}, \partial_x \bar{z}](t,0,y)$ we find that $\bar{z}$ satisfies (1) on E_c. Now we prove (32). There is $\tilde{K} \in \mathbb{L}([0,a], \mathbb{R}_+)$ such that

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R})} + \|\partial_x \bar{z} - \partial_x \tilde{z}\|_{(t,\mathbb{R}^n)} \le \|\psi - \tilde{\psi}\|_{(0,\mathbb{R})} + \|\partial_x \psi - \partial_x \tilde{\psi}\|_{(0,\mathbb{R}^n)}$$
$$+ \int_0^t \tilde{K}(\tau)[\|\bar{z} - \tilde{z}\|_{(\tau,\mathbb{R})} + \|\partial_x \bar{z} - \partial_x \tilde{z}\|_{(\tau,\mathbb{R}^n)}]\, d\tau, \qquad t \in (0,c].$$

Then we obtain (32) from the Gronwall inequality. This completes the proof.

Remark 4.1. Let z and $\bar{z}$ be solutions of the Cauchy problem (1), (2). Then Theorem 4.1 shows that z and $\bar{z}$ coincide on the whole domain.

Remark 4.2. Suppose that $\phi_0^{(i)} \colon [0,a] \to \mathbb{R}$, $(\phi_1^{(i)}, \ldots, \phi_n^{(i)}) \colon E \to \mathbb{R}^n$, $i = 1, \ldots, k$, are given functions and $0 \le \phi_0^{(i)}(t) \le t$,

$$(\phi_0^{(i)}(t), \phi_1^{(i)}(t,x), \ldots, \phi_n^{(i)}(t,x)) \in E_0 \cup E$$

for $(t,x) \in E$, $i = 1, \ldots, k$. Write

$$\varphi^{(i)}(t,x) = (\phi_0^{(i)}(t), \phi_1^{(i)}(t,x), \ldots, \phi_n^{(i)}(t,x)), \qquad i = 1, \ldots, k.$$

Put $\Xi = E \times \mathbb{R} \times (C(B,\mathbb{R}))^k \times \mathbb{R}^n$ and suppose that $G \colon \Xi \to \mathbb{R}$ is a given function of the variables $(t,x,p,w_1,\ldots,w_k,q)$. We will say that G satisfies condition (V) if for each $(t,x,p,q) \in E \times \mathbb{R}^{1+n}$ and for $w_i, \bar{w}_i \in C(B,\mathbb{R})$, $i = 1, \ldots, k$, such that $w_i(\tau,y) = \bar{w}_i(\tau,y)$, $(\tau,y) \in D[\varphi^{(i)}(t,x)]$, $i = 1, \ldots, k$, we have

$$G(t,x,p,w_1,\ldots,w_k,q) = G(t,x,p,\bar{w}_1,\ldots,\bar{w}_k,q).$$

Given $\psi \colon E_0 \to \mathbb{R}$, we consider the functional differential equation

$$(35)\qquad \partial_t z(t,x) = G(t,x,z(t,x), z_{\varphi^{(1)}(t,x)}, \ldots, z_{\varphi^{(k)}(t,x)}, \partial_x z(t,x))$$

with initial condition (2). It is clear that Theorem 4.1 can be extended to the Cauchy problem (35), (2).

5. Examples and comments. Now we consider two examples of functional differential equations. Suppose that $F \colon E \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\phi_0, \tilde{\phi}_0$:

$[0,a] \to \mathbb{R}$, $\phi, \tilde{\phi}\colon E \to \mathbb{R}^n$ are given functions. Assume that $0 \le \phi_0(t), \tilde{\phi}_0(t) \le t$ and $\varphi(t,x) = (\phi_0(t), \phi(t,x)) \in E_0 \cup E$, $\tilde{\varphi}(t,x) = (\tilde{\phi}_0(t), \tilde{\phi}(t,x)) \in E_0 \cup E$ for $(t,x) \in E$. Consider the differential equation with deviated variables

$$\partial_t z(t,x) = F(t,x,z(\varphi(t,x))z(\tilde{\varphi}(t,x)), \partial_x z(t,x)) \tag{36}$$

and the differential integral equation

$$\partial_t z(t,x) = F\Big(t,x, \int_{D[t,x]} z^2(\tau,y)\,dy\,d\tau, \partial_x z(t,x)\Big). \tag{37}$$

We formulate sufficient conditions for the existence of solutions to (36), (2) and (37), (2).

Assumption H_4. The function $F\colon E \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of the variables (t,x,r,q) satisfies the conditions:

1) $F(\cdot,x,r,q)\colon I[x] \to \mathbb{R}$ is measurable for $(x,r,q) \in [-b,b] \times \mathbb{R} \times \mathbb{R}^n$, the derivatives $\partial_x F$, $\partial_r F$, $\partial_q F$ exist and
$$\partial_x F(\cdot,x,r,q), \partial_q F(\cdot,x,r,q)\colon I[x] \to \mathbb{R}^n, \qquad \partial_r F(\cdot,x,r,q)\colon I[x] \to \mathbb{R}$$
are measurable for $(x,r,q) \in [-b,b] \times \mathbb{R} \times \mathbb{R}^n$,
2) $\partial_x F(t,\cdot), \partial_q F(t,\cdot)\colon S_t \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\partial_r F(t,\cdot)\colon S_t \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are continuous and there are $\alpha \in \mathbb{L}([0,a],\mathbb{R}^n_+)$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, and $\beta, L \in \mathbb{L}([0,a],\mathbb{R}^n_+)$ such that
$$(|\partial_{q_1} F(P)|, \ldots, |\partial_{q_n} F(P)|) \le (\alpha_1(t),\ldots,\alpha_n(t)),$$
$$\|\partial_x F(t,x,r,q)\| \le \beta(t),$$
where $P = (t,x,r,q) \in E \times \mathbb{R} \times \mathbb{R}^n$ and the expressions
$$\|\partial_x F(t,x,r,q) - \partial_x F(t,\bar{x},\bar{r},\bar{q})\|, \qquad \|\partial_q F(t,x,r,q) - \partial_q F(t,\bar{x},\bar{r},\bar{q})\|$$
are bounded from above by $L(t)[\|x-\bar{x}\| + |r-\bar{r}| + \|q-\bar{q}\|]$,
3) there is $\gamma \in \mathbb{L}([0,a],\mathbb{R}_+)$ such that $\tilde{f}(t,x,r,q) = r\partial_r F(t,x,r,q)$ satisfies the conditions
$$|\tilde{f}(t,x,r,q)| \le \gamma(t),$$
$$|\tilde{f}(t,x,r,q) - \tilde{f}(t,\bar{x},\bar{r},\bar{q})| \le \gamma(t)[\|x-\bar{x}\| + |r-\bar{r}| + \|q-\bar{q}\|],$$
on $E \times \mathbb{R} \times \mathbb{R}^n$.

Lemma 5.1. *If Assumption H_4 is satisfied and $\psi \in \mathbb{K}$ then there are $c \in (0,a]$ and $u\colon E_c \to \mathbb{R}$ such that u is a solution of* (37), (2) *on E_c.*

The above lemma is a consequence of Theorem 4.1.

Note that the result given in [3] is not applicable to (37), (2). The initial value problem (9), (2) is investigated in [3]. We consider equation (9) with

$$(Wz)(t,x) = \int_{D[t,x]} z^2(\tau,y)\,dy\,d\tau. \tag{38}$$

Let us denote by $\|\cdot\|_{E_t}$ the maximum norm in the space $C(E_t,\mathbb{R})$. The following assumption on W is needed in [3]: there are $A_0, B_0, L \in \mathbb{R}_+$ such that

$$\|W[z]\|_{E_t} \le A_0 + B_0\|z\|_{E_t}, \tag{39}$$

$$\|W[z] - W[\bar{z}]\|_{E_t} \le L\|z-\bar{z}\|_{E_t} \tag{40}$$

for $t \in [0,a]$, $z, \bar{z} \in C(E_0 \cup E, \mathbb{R})$. It is clear that the above conditions are not satisfied for W given by (38).

The initial value problem (8), (2) is investigated in [5] and a theorem on the existence of classical solutions is obtained.

Let us consider equation (8) with

$$G(t,x,z(\cdot),q) = F\Big(t,x,\int\limits_{D[t,x]} z^2(\tau,y)\,dy\,d\tau, q\Big). \tag{41}$$

The following assumption on G is needed in [5]: there are A_0, A_1 such that

$$\|\partial_x G(t,x,z,q)\| \le A_0 + A_1\Big[\|z\|_{E_t} + \|\partial_t z\|_{E_t} + \sum_{i=1}^{n} \|\partial_{x_i} z\|_{E_t}\Big], \tag{42}$$

where $z\colon E_0 \cup E \to \mathbb{R}$ is of class C^1. It is clear that the above condition is not satisfied for G given by (41).

Consider the equation (36) with the initial condition (2).

Assumption H_5. The functions $\phi_0, \tilde{\phi}_0\colon [0,a] \to \mathbb{R}$, $\phi, \tilde{\phi}\colon E \to \mathbb{R}^n$ satisfy:

(i) for $t \in [0,a]$ we have $0 \le \phi_0(t), \tilde{\phi}_0(t) \le t$,
(ii) $\varphi(t,x) = (\phi_0(t), \phi(t,x)) \in E_0 \cup E$, $\tilde{\varphi}(t,x) = (\tilde{\phi}_0(t), \tilde{\phi}(t,x)) \in E_0 \cup E$ for $(t,x) \in E$,
(iii) the partial derivatives

$$\partial_x \phi = [\partial_{x_j}\phi_i]_{i,j=1,\dots,n}, \qquad \partial_x \tilde{\phi} = [\partial_{x_j}\tilde{\phi}_i]_{i,j=1,\dots,n},$$

exist on E, the functions $\partial_x\phi$ and $\partial_x\tilde{\phi}$ are continuous and there is $Q \in \mathbb{R}_+$ such that

$$\|\partial_x\phi(t,x) - \partial_x\phi(t,\bar{x})\| \le Q\|x-\bar{x}\|,$$
$$\|\partial_x\tilde{\phi}(t,x) - \partial_x\tilde{\phi}(t,\bar{x})\| \le Q\|x-\bar{x}\|$$

for $(t,x), (t,\bar{x}) \in E$.

Lemma 5.2. *If Assumptions H_4, H_5 are satisfied and $\psi \in \mathbb{K}$ then there are $c \in (0,a]$ and $u\colon E_c \to \mathbb{R}$ such that u is a solution of* (36), (2) *on E_c.*

The above lemma is a consequence of Theorem 4.1 (see also Remark 4.2). Note that the result given in [3] is not applicable to (36), (2). We consider equation (9) with

$$(Wz)(t,x) = z(\varphi(t,x))z(\tilde{\varphi}(t,x)). \tag{43}$$

It is clear that the conditions (39), (40) are not satisfied for W given by (43).

Let us consider equation (8) with

$$G(t,x,z(\cdot),q) = F(t,x,z(\varphi(t,x))z(\tilde{\varphi}(t,x)),q). \tag{44}$$

The condition (42) which is needed in [5] is not satisfied for G given by (44).

Remark 5.1. The result of this paper can be extended to functional differential systems

$$\partial_t z_i(t,x) = f_i(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z_i(t,x)), \qquad i=1,\ldots,k,$$

with the initial condition $z(t,x)=\psi(t,x)$ on E_0 where $z=(z_1,\ldots,z_k)$ and $f=(f_1,\ldots,f_k)\colon E\times\mathbb{R}^k\times C(B,\mathbb{R}^k)\times\mathbb{R}^n$, $\psi\colon E_0\to\mathbb{R}^k$.

Remark 5.2. Suppose that f does not depend on the functional variable and $d_0=0$. Then (1), (2) reduces to the classical Cauchy problem

$$\partial_t z(t,x) = f(t,x,z(t,x),\partial_x z(t,x)), \tag{45}$$

$$z(0,x) = \psi(x) \quad \text{for } x\in[-b,b], \tag{46}$$

where $\psi\colon[-b,b]\to\mathbb{R}$ is a given function. Then Theorem 4.1 gives sufficient conditions for the existence of weak solutions to (45), (46).

There are the following relations between our results and known theorems for (45), (46).

The assumptions of Theorem 4.1 for (45), (46) and the assumptions in the existence theorem given in [2] are the same. Hence Theorem 4.1 is a generalization of the existence result of [2].

Now we adopt additional assumptions on f and we consider classical solutions to (45), (46). Suppose that the functions

$$f(\cdot,x,p,q)\colon I[x]\to\mathbb{R}, \qquad \partial_p f(\cdot,x,p,q)\colon I[x]\to\mathbb{R},$$
$$\partial_x f(\cdot,x,p,q)\colon I[x]\to\mathbb{R}^n, \qquad \partial_q f(\cdot,x,p,q)\colon I[x]\to\mathbb{R}^k$$

are continuous. In this case we can assume that the functions α_0,α,β,L are constant on $[0,a]$. Then Theorem 4.1 gives sufficient conditions for the existence of classical solutions to (45), (46).

Note that our assumptions on regularity of given functions in (45), (46) and assumptions in the existence theorem presented in [6] are the same. Estimates of the existence domain which can be deduced from [6] and our estimates are not the same. This is due to the fact that the method of successive approximations is used in [6]. Our results are obtained by using the method of quasilinearization.

It follows that Theorem 4.1 is an extension of classical theorems for first order partial differential equations. It is clear that the above observations can be extended to the Cauchy problem for functional differential systems.

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