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On the global attractors for a class of semilinear degenerate parabolic equations

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Abstract. We prove the existence and upper semicontinuity with respect to the nonlinearity and the diffusion coefficient of global attractors for a class of semilinear degenerate parabolic equations in an arbitrary domain.

1. Introduction. The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for dissipative dynamical systems is to consider their global attractor. The first question is to study the existence of a global attractor. Once the global attractor is obtained, the next natural question is to study its most important properties, such as dimension, dependence on parameters, regularity, determining modes, etc. In the last three decades, many authors have obtained relevant results for a large class of PDEs (see e.g. [7, 14, 15] and references therein). However, to the best of our knowledge, little seems to be known about the asymptotic behavior of solutions of degenerate equations.

In this paper we study the following semilinear degenerate parabolic equation with variable, nonnegative coefficients, defined on an arbitrary domain (bounded or unbounded) $\Omega \subset \mathbb{R}^N, N \geq 2$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + f(u) + g(x) &= 0, && x \in \Omega,\ t > 0, \\ u(t,x) &= 0, && x \in \partial\Omega,\ t > 0, \\ u(0,x) &= u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where $u_0 \in L^2(\Omega)$ and $g \in L^2(\Omega)$ are given, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying some conditions specified later.

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Problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of a nuclear reactor) (see [8]). In this case u and σ stand for the neutron flux and neutron diffusion respectively.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes. Motivated by [5], where a degenerate elliptic problem is studied, we assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumptions: when the domain Ω is bounded,

($\mathcal{H}_\alpha$) $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$,

and when Ω is unbounded,

($\mathcal{H}^\infty_{\alpha,\beta}$) σ satisfies condition ($\mathcal{H}_\alpha$) and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$.

For the physical motivation of assumptions ($\mathcal{H}_\alpha$) and ($\mathcal{H}^\infty_{\alpha,\beta}$), we refer the reader to [5, 9, 10, 1].

In order to study problem (1.1) we use the natural energy space $\mathcal{D}^1_0(\Omega,\sigma)$ defined as the closure of $C^\infty_0(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^1_0(\Omega,\sigma)} := \Big(\int_\Omega \sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.$$

The existence and long-time behavior of solutions to problem (1.1) in the case that $f(u) = -\lambda u + |u|^{2\gamma}u$ $(0 \le \gamma < (2-\alpha)/(N-2+\alpha))$, $g(x) = 0$ has been studied in [9, 10] and improved recently in [1]. In [1], the authors considered problem (1.1) with $u_0 \in \mathcal{D}^1_0(\Omega,\sigma)$, $g \in L^2(\Omega)$ given, and $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(u)-f(v)| \le C_0|u-v|(1+|u|^\gamma+|v|^\gamma), \qquad 0 \le \gamma < \frac{4-2\alpha}{N-2+\alpha},$$
$$F(u) \ge -\frac{\mu}{2}u^2 - C_1, \qquad f(u)u \ge -\mu u^2 - C_2,$$

where $C_0, C_1, C_2 \ge 0$, F is the primitive $F(y) = \int_0^y f(s)\,ds$ of f, $\mu < \lambda_1$, and $\lambda_1 > 0$ is the first eigenvalue of the operator $Au := -\operatorname{div}(\sigma(x)\nabla u)$ in Ω with the homogeneous Dirichlet condition. Under the above assumptions on f, the authors proved that problem (1.1) defines a semigroup $S(t) : \mathcal{D}^1_0(\Omega,\sigma) \to \mathcal{D}^1_0(\Omega,\sigma)$, which possesses a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $\mathcal{D}^1_0(\Omega,\sigma)$. Furthermore, for each $u_0 \in \mathcal{D}^1_0(\Omega,\sigma)$, the corresponding solution $u(t)$ tends to the set E of equilibrium points in $\mathcal{D}^1_0(\Omega,\sigma)$ as $t \to +\infty$. The basic tool is the Lyapunov function

$$\Phi(u) = \frac{1}{2}\,\|u\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + \int_\Omega (F(u)+gu)\,dx.$$

Note that the critical exponent of the embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ is $2_\alpha^* = 2N/(N-2+\alpha)$, so the condition $0 \le \gamma < (4-2\alpha)/(N-2+\alpha)$ is necessary to prove the existence of a mild solution by the fixed point method and to ensure the existence of the Lyapunov functional Φ.

In this paper we continue the study of the long-time behavior of solutions to problem (1.1) when the nonlinearity f is supposed to satisfy the polynomial growth condition of arbitrary order. More precisely, we assume that the initial data u_0 and the external force g are in $L^2(\Omega)$, and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$C_1|u|^p - C_0 \le f(u)u \le C_2|u|^p + C_0, \quad p \ge 2, \tag{1.2}$$

$$f'(u) \ge -C_3 \quad \text{for all } u \in \mathbb{R}, \tag{1.3}$$

where C_0, C_1, C_2 and C_3 are positive constants. It is clear that when $|f(u)| \sim |u|^{\gamma+1}$ with $\gamma > (4-2\alpha)/(N-2+\alpha)$, the fixed point method for proving the existence of mild solutions does not work, and the system is no longer a gradient system (because then $\int_\Omega F(u)\,dx$ does not exist when u belongs to the energy space $\mathcal{D}_0^1(\Omega,\sigma)$). However, thanks to the structure of the nonlinearity, we may use the compactness method [11] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of an absorbing set B_0 in the space $\mathcal{D}_0^1(\Omega,\sigma)$ for the semigroup $S(t)$ generated by the solutions of problem (1.1). By the compactness of the embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^2(\Omega)$, the semigroup $S(t)$ is asymptotically compact in $L^2(\Omega)$. This implies the existence of a compact global attractor $\mathcal{A} = \omega(B_0)$ for $S(t)$ in $L^2(\Omega)$.

Besides the problem of existence of the global attractor, its dependence on parameters, such as the shape of the domain, the coefficients, nonlinearities, etc. is also an important object of study. In some recent works [4, 12, 13], the problem of continuity of the global attractor in variations of the domain where the problem is posed has been studied for the reaction-diffusion equation with various boundary conditions. The continuous dependence of the global attractor on the diffusion coefficients is investigated in [2, 3, 6].

In this paper, we study the upper semicontinuity of the global attractor with respect to the nonlinear term and the diffusion coefficient taken as parameters. The more delicate question of the lower semicontinuity of the global attractor is not dealt with.

The paper is organized as follows. In Section 2, we recall some results on function spaces which we will use. For clarity, in Sections 3–5, we only consider the case of a bounded domain and the diffusion coefficient σ satisfying condition $(\mathcal{H}_\alpha)$. Section 3 is devoted to the proof of the existence and uniqueness of a weak solution by using the compactness method, and the existence of a compact global attractor $\mathcal{A}$ in $L^2(\Omega)$ for the semigroup

$S(t)$ generated by (1.1). In Section 4, we study the upper semicontinuity of the global attractor with respect to the nonlinearity. The upper semicontinuous dependence of the global attractor on the diffusion coefficients is investigated in Section 5. In the last section, we give some remarks on similar results for an unbounded domain and σ satisfying condition $(\mathcal{H}^{\infty}_{\alpha,\beta})$.

Notations. The $L^2(\Omega)$-norm and the $\mathcal{D}^1_0(\Omega,\sigma)$-norm will be denoted by $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{\mathcal{D}^1_0(\Omega,\sigma)}$, respectively. By $\mathcal{D}^{-1}(\Omega,\sigma)$ we denote the dual space of $\mathcal{D}^1_0(\Omega,\sigma)$. Let (X,d) be a metric space. We use the Hausdorff semidistance $\delta_X(\cdot,\cdot)$ defined on the subsets of X by

$$\delta_X(A,B) := \sup_{a\in A}\inf_{b\in B} d(a,b), \quad \forall A,B\subset X.$$

Let X_1, X_2 be Banach spaces and Z be a topological vector space such that $X_1 \hookrightarrow Z$ and $X_2 \hookrightarrow Z$. Then $X_1\cap X_2$ and X_1+X_2 are Banach spaces equipped with the norms

$$\begin{aligned}\|u\|_{X_1\cap X_2} &= \|u\|_{X_1} + \|u\|_{X_2},\\ \|u\|_{X_1+X_2} &= \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2} : u = u_1+u_2\}.\end{aligned}$$

It is known that if $X_1\cap X_2$ is dense both in X_1 and X_2, then $(X_1\cap X_2)^* = X_1^* + X_2^*$.

2. Preliminaries. We recall some basic results on function spaces from [5]. Let $N\ge 2$, $\alpha\in(0,2)$, and

$$2^*_\alpha = \begin{cases} \dfrac{4}{\alpha}\in(2,\infty) & \text{if } N=2,\\ \dfrac{2N}{N-2+\alpha}\in\left(2,\dfrac{2N}{N-2}\right) & \text{if } N\ge 3.\end{cases}$$

The number 2^*_α has the role of the critical exponent in the classical Sobolev embedding.

The natural energy space for problem (1.1) involves the space $\mathcal{D}^1_0(\Omega,\sigma)$, defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}^1_0(\Omega,\sigma)} := \Big(\int_\Omega \sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.$$

It is a Hilbert space with respect to the scalar product

$$(u,v) := \int_\Omega \sigma(x)\nabla u\nabla v\,dx.$$

The following lemmas come from [5, Propositions 3.3–3.5].

Lemma 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$, $N\ge 2$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^{2^*_\alpha}(\Omega)$ *continuously;*
(ii) $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in [1, 2^*_\alpha)$.

LEMMA 2.2. *Assume that Ω is an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, and σ satisfies $(\mathcal{H}^\infty_{\alpha,\beta})$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *continuously for every* $p \in [2^*_\beta, 2^*_\alpha]$*;*
(ii) $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in (2^*_\beta, 2^*_\alpha)$.

We now consider the case where Ω is a bounded domain (the unbounded case is considered similarly with $(\mathcal{H}^\infty_{\alpha,\beta})$ instead of $(\mathcal{H}_\alpha)$).

We consider the boundary value problem

$$-\operatorname{div}(\sigma(x)\nabla u) = h(x) \in L^2(\Omega), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.1}$$

Set

$$X = L^2(\Omega), \quad D(\tilde{A}) = C_0^\infty(\Omega), \quad \tilde{A}u = -\operatorname{div}(\sigma(x)\nabla u).$$

Then problem (2.1) corresponds to the operator equation

$$\tilde{A}u = h, \quad u \in C_0^\infty(\Omega),\ h \in X.$$

For every $u, v \in C_0^\infty(\Omega)$, we have

$$(\tilde{A}u, v) = \int_\Omega \sigma(x)\nabla u \nabla v\, dx = (u, \tilde{A}v).$$

It follows from Lemma 2.1 that there is a constant $C > 0$ such that

$$(\tilde{A}u, u) \geq C\|u\|_X^2 \quad \text{for any } u \in C_0^\infty(\Omega).$$

Hence, $\tilde{A}$ is symmetric and strongly monotone. Applying the Friedrichs extension theorem [16, Vol. IIA, pp. 126–135], we find that the energy space X_E equals $\mathcal{D}_0^1(\Omega,\sigma)$ since X_E is the completion of $D(\tilde{A}) = C_0^\infty(\Omega)$ with respect to the scalar product $(u,v) = \int_\Omega \sigma(x)\nabla u\nabla v\, dx$, and the extensions satisfy

$$\tilde{A} \subset A \subset A_E,$$

where $A_E : \mathcal{D}_0^1(\Omega,\sigma) \to \mathcal{D}^{-1}(\Omega,\sigma)$ is the energetic extension, and $A = -\operatorname{div}(\sigma(x)\nabla)$ is the Friedrichs extension of $\tilde{A}$ with the domain of definition

$$D(A) = \{u \in \mathcal{D}_0^1(\Omega,\sigma) : Au \in X\}.$$

Noticing that $2^*_\alpha > 2$, we have an evolution triple

$$\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{D}^{-1}(\Omega,\sigma)$$

with compact and dense embbedings. Hence, there exists a complete orthonormal system of eigenvectors (e_j, λ_j) such that

$$(e_j, e_k) = \delta_{jk} \quad \text{and} \quad -\operatorname{div}(\sigma(x)\nabla e_j) = \lambda_j e_j, \quad j,k = 1,2,\ldots,$$
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to +\infty \text{ as } j \to \infty.$$

3. Existence of global attractors. Denote

$$\Omega_T = \Omega \times (0,T), \qquad A = -\operatorname{div}(\sigma(x)\nabla),$$
$$V = L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma)) \cap L^p(\Omega_T),$$
$$V^* = L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma)) + L^{p'}(\Omega_T).$$

In what follows, we assume that $g \in L^2(\Omega)$ and $u_0 \in L^2(\Omega)$ are given.

Definition 3.1. A function $u(x,t)$ is called a *weak solution* of (1.1) on $(0,T)$ iff

$$u \in V, \qquad \frac{\partial u}{\partial t} \in V^*, \qquad u|_{t=0} = u_0 \quad \text{a.e. in } \Omega$$

and

$$\int_{\Omega_T} \left(\frac{\partial u}{\partial t}\varphi + \sigma \nabla u \nabla \varphi + f(u)\varphi + g\varphi \right) dx\, dt = 0 \tag{3.1}$$

for all test functions $\varphi \in V$.

The following proposition shows the continuity of weak solutions with respect to time t, which makes the initial condition meaningful.

Proposition 3.1. *If $u \in V$ and $\partial u/\partial t \in V^*$, then $u \in C([0,T];L^2(\Omega))$.*

Proof. We select a sequence $u_n \in C^1([0,T];\mathcal{D}_0^1(\Omega,\sigma) \cap L^p(\Omega))$ such that

$$u_n \to u \quad \text{in } V, \qquad \frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{in } V^*.$$

Then, for all $t, t_0 \in [0,T]$, we have

$$\|u_n(t) - u_m(t)\|^2_{L^2(\Omega)} = \|u_n(t_0) - u_m(t_0)\|^2_{L^2(\Omega)} + 2\int_{t_0}^{t} \langle u_n'(s) - u_m'(s), u_n(s) - u_m(s)\rangle\, ds.$$

We choose t_0 so that

$$\|u_n(t_0) - u_m(t_0)\|^2_{L^2(\Omega)} = \frac{1}{T}\int_0^T \|u_n(t) - u_m(t)\|^2_{L^2(\Omega)}\, dt.$$

We have

$$\int_\Omega |u_n(t) - u_m(t)|^2\, dt$$
$$= \frac{1}{T}\int_\Omega \int_0^T |u_n(t) - u_m(t)|^2\, dt\, dx + 2\int_\Omega \int_{t_0}^{t} (u_n'(s) - u_m'(s))(u_n(s) - u_m(s))\, ds\, dx$$
$$\le \frac{1}{T}\int_\Omega \int_0^T |u_n(t) - u_m(t)|^2\, dx\, dt + 2\|u_n' - u_m'\|_{V^*}\|u_n - u_m\|_V.$$

Hence, $\{u_n\}$ is a Cauchy sequence in $C([0,T];L^2(\Omega))$. Thus it converges in $C([0,T];L^2(\Omega))$ to a function $v\in C([0,T];L^2(\Omega))$. On the other hand, since $u_n\to u$ in V, $u_n(t)\to u(t)$ in $L^2(\Omega)$ for a.e. $t\in[0,T]$. Therefore, we deduce that $u=v$ a.e. This implies that $u\in C([0,T];L^2(\Omega))$ (after possibly redefining it on a set of zero measure). ∎

THEOREM 3.2. *Under conditions* (1.2)–1.3), *problem* (1.1) *has a unique weak solution $u(t)$ satisfying*

$$u\in C([0,\infty);L^2(\Omega))\cap L^2_{\mathrm{loc}}(0,\infty;\mathcal{D}^1_0(\Omega,\sigma))\cap L^p_{\mathrm{loc}}(0,\infty;L^p(\Omega))$$

and

$$\frac{\partial u}{\partial t}\in L^2_{\mathrm{loc}}(0,\infty;\mathcal{D}^{-1}(\Omega,\sigma))+L^{p'}_{\mathrm{loc}}(0,\infty;L^{p'}(\Omega)),$$

where p' is the conjugate of p. Moreover, the mapping $u_0\mapsto u(t)$ is continuous on $L^2(\Omega)$.

Proof. Existence. We look for an approximate solution $u_n(t)$ that belongs to the finite-dimensional space spanned by the first n eigenfunctions of the operator A, so that $u_n(t)=\sum_{j=1}^n u_{nj}(t)e_j$, and solves the following problem:

$$\begin{cases}\left\langle\dfrac{\partial u}{\partial t},e_j\right\rangle+\langle Au_n,e_j\rangle+\langle f(u_n),e_j\rangle+(g,e_j)=0, & 1\le j\le n,\\ (u_n(0),e_j)=(u_0,e_j).\end{cases}\tag{3.2}$$

The existence of $u_n(t)$ follows from the Peano theorem. We now establish some *a priori* estimates for u_n. We have

$$\frac12\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}+\|u_n\|^2_{\mathcal{D}^1_0(\Omega,\sigma)}+\int_\Omega f(u_n)u_n\,dx+\int_\Omega gu_n\,dx=0.$$

Using condition (1.2) and the Cauchy inequality, we get

$$\begin{aligned}\frac12\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}+\|u_n\|^2_{\mathcal{D}^1_0(\Omega,\sigma)}-C_0|\Omega|+C_1\int_\Omega|u_n|^p\,dx&\\ \le\frac{1}{2\lambda_1}\,\|g\|^2_{L^2(\Omega)}+\frac{\lambda_1}{2}\,\|u_n\|^2_{L^2(\Omega)},&\end{aligned}\tag{3.3}$$

where λ_1 is the first eigenvalue of A in Ω with the homogeneous Dirichlet condition (note that $\|u\|^2_{\mathcal{D}^1_0(\Omega,\sigma)}\ge\lambda_1\|u\|^2_{L^2(\Omega)}$). Hence

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}\le-\lambda_1\|u_n\|^2_{L^2(\Omega)}+C_4,$$

where $C_4=(1/\lambda_1)\|g\|^2+2C_0|\Omega|$. Using the Gronwall inequality, we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)}\le e^{-\lambda_1 t}\|u_n(0)\|^2_{L^2(\Omega)}+\frac{C_4}{\lambda_1}(1-e^{-\lambda_1 t}).\tag{3.4}$$

This estimate ensures that the solution $u_n(t)$ of (3.2) can be extended to $+\infty$.

From (3.3), we have

$$\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + 2C_1 \int_\Omega |u_n(t)|^p\,dx \le C_4.$$

Let T be an arbitrary positive number. Integrating both sides of the above inequality from 0 to T, we get

$$\|u_n(T)\|^2_{L^2(\Omega)} + \int_0^T \|u_n(t)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)}\,dt + 2C_1 \int_0^T |u_n|^p\,dx\,dt \le \|u_n(0)\|^2_{L^2(\Omega)} + C_4T.$$

This inequality shows that

- $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
- $\{u_n\}$ is bounded in $L^2(0,T;\mathcal{D}^1_0(\Omega,\sigma))$;
- $\{u_n\}$ is bounded in $L^p(\Omega_T)$.

We first use the boundedness of $\{u_n\}$ in $L^p(\Omega_T)$ to prove the boundedness of $\{f(u_n)\}$ in $L^{p'}(\Omega_T)$, where p' is the conjugate of p. Indeed, the condition (1.2) implies that

$$|f(u)| \le C_5(1+|u|^{p-1}).$$

Therefore,

$$\begin{aligned}\|f(u_n)\|^{p'}_{L^{p'}(\Omega)} &= \int_0^T\int_\Omega |f(u_n)|^{p'}\,dx\,dt\\ &\le C\int_0^T\int_\Omega (1+|u_n|^{p-1})^{p'}\,dx\,dt \le C\int_0^T\int_\Omega (1+|u_n|^p)\,dx\,dt.\end{aligned}$$

Hence $\{f(u_n)\}$ is bounded in $L^{p'}(\Omega_T)$.

Next, we show that $\{\partial u_n/\partial t\}$ is bounded in $L^{p'}(0,T;\mathcal{D}^{-1}(\Omega,\sigma)+L^{p'}(\Omega))$. Indeed, since

$$\frac{\partial u_n}{\partial t} = -Au_n - f(u_n) - g,$$

we conclude that $\{\partial u_n/\partial t\}$ is bounded in V^*. Combining this with the fact that $L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma))$ and $L^{p'}(\Omega_T)$ are continuously embedded into $L^{p'}(0,T;\mathcal{D}^{-1}(\Omega,\sigma)+L^{p'}(\Omega))$, we obtain the desired result.

From the above results, we can assume that

- $u_n \rightharpoonup u$ in $L^2(0,T;\mathcal{D}^1_0(\Omega,\sigma))$;
- $u_n \rightharpoonup u$ in $L^p(\Omega_T)$;
- $f(u_n) \rightharpoonup \chi$ in $L^{p'}(\Omega_T)$;
- $\partial u_n/\partial t \rightharpoonup \partial u/\partial t$ in V^*.

Since $u \in V$ and $u_t \in V^*$, we conclude that $u \in C([0,T]; L^2(\Omega))$ thanks to Proposition 3.1.

It remains to show that $\chi = f(u)$ and $u(0) = u_0$. Since $\{u_n\}$ is bounded in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$ and $\{\partial u_n/\partial t\}$ is bounded in $L^{p'}(0,T;\mathcal{D}^{-1}(\Omega,\sigma) + L^{p'}(\Omega))$, it follows from the Aubin–Lions Lemma [11, p. 58] that

$$u_n \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Hence, we can choose a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \to u \qquad \text{a.e. in } \Omega_T.$$

It follows from the continuity of the function f that

$$f(u_{n_k}) \to f(u) \qquad \text{a.e. in } \Omega_T.$$

In view of the boundedness of $\{f(u_{n_k})\}$ in $L^{p'}(\Omega_T)$, by Lemma 1.3 in [11, Chapter 1], we have

$$f(u_{n_k}) \rightharpoonup f(u) \quad \text{in } L^{p'}(\Omega_T),$$

and taking into account the uniqueness of a weak limit, we get $\chi = f(u)$.

To prove $u(0) = u_0$, choosing some test function $\varphi \in C^1([0,T];\mathcal{D}_0^1(\Omega,\sigma) \cap L^p(\Omega))$ with $\varphi(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\int_0^T -\langle u_n, \varphi' \rangle \, dt + \int_{\Omega_T} (\sigma \nabla u_n \nabla \varphi + f(u_n)\varphi + g\varphi) \, dx \, dt = (u_n(0), \varphi(0)).$$

Taking limits as $n \to \infty$ we obtain

$$\int_0^T -\langle u, \varphi' \rangle \, dt + \int_{\Omega_T} (\sigma(x) \nabla u \nabla \varphi + f(u)\varphi + g\varphi) \, dx \, dt = (u_0, \varphi(0)), \tag{3.5}$$

since $u_n(0) \to u_0$. On the other hand, for the "limiting equation", we have

$$\int_0^T -\langle u, \varphi' \rangle \, dt + \int_{\Omega_T} (\sigma(x) \nabla u \nabla \varphi + f(u)\varphi + g\varphi) \, dx \, dt = (u(0), \varphi(0)). \tag{3.6}$$

Comparing (3.5) with (3.6) we get $u(0) = u_0$. Thus, u is a weak solution to (1.1). The global existence of the solution u follows from the following inequality, which is proved similarly to (3.4):

$$\|u(t)\|^2_{L^2(\Omega)} \le e^{-\lambda_1 t} \|u(0)\|^2_{L^2(\Omega)} + \frac{C_4}{\lambda_1} (1 - e^{-\lambda_1 t}). \tag{3.7}$$

Uniqueness and continuous dependence. Let u, v be two solutions of problem (1.1) with initial data $u_0, v_0 \in L^2(\Omega)$. Then $w = u - v$ satisfies

$$\begin{cases} w_t + Aw + f(u) - f(v) = 0, & x \in \Omega,\ t > 0, \\ w|_{\partial\Omega} = 0, \quad w(0) = u_0 - v_0. \end{cases}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \|w\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + \int_\Omega (u-v)(f(u)-f(v))\,dx = 0.$$

Using condition (1.3) we have

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + 2\|w\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} \le 2C_3\|w\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we obtain

$$\|w(t)\|_{L^2(\Omega)} \le \|w(0)\|_{L^2(\Omega)} e^{2C_3 t}.$$

This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of the solution on the initial data. ∎

Theorem 3.2 allows us to define a continuous (nonlinear) semigroup $S(t): L^2(\Omega) \to L^2(\Omega)$ (in the sense of Definition 2.1 in [14]) associated to problem (1.1) as follows:

$$S(t)u_0 := u(t),$$

where $u(t)$ is the unique weak solution of problem (1.1) with the initial datum u_0. We will prove that the semigroup $S(t)$ possesses a compact connected global attractor $\mathcal{A}$ in $L^2(\Omega)$.

First, from (3.7) we deduce the existence of an absorbing set in $L^2(\Omega)$: There are a constant R and a time $t_0(\|u_0\|_{L^2(\Omega)})$ such that for the solution $u(t) = S(t)u_0$,

$$\|u(t)\|_{L^2(\Omega)} \le R \quad \text{for all } t \ge t_0(\|u_0\|_{L^2(\Omega)}).$$

Multiplying the equation in (1.1) by u and using (1.2), we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + C_1\int_\Omega |u(t)|^p\,dx - C_0|\Omega| + \int_\Omega gu\,dx \le 0.$$

Integrating between t and $t+1$, we obtain

$$\int_t^{t+1}\Big[\|u(s)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + C_1\int_\Omega |u(s)|^p\,dx + \int_\Omega gu\,dx\Big]\,ds \le C_0|\Omega| + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}.$$

This shows that

$$\int_t^{t+1}\Big[\|u(s)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + C_1\int_\Omega |u(s)|^p\,dx + \int_\Omega gu\,dx\Big]\,ds \le C_0|\Omega| + \frac{1}{2}R^2 \tag{3.8}$$

for all $t \ge t_0(\|u_0\|_{L^2(\Omega)})$. Noting that

$$C_6(|u|^p - 1) \le F(u) \le C_7(|u|^p + 1), \tag{3.9}$$

where $F(u) = \int_0^u f(\xi)\,d\xi$, we obtain

$$\int_t^{t+1}\left[\frac{1}{2}\|u(s)\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + \int_\Omega F(u)\,dx + \int_\Omega gu\,dx\right] ds \le C_8 \tag{3.10}$$

for all $t \ge t_0(\|u_0\|_{L^2(\Omega)})$.

In what follows, we shall derive an *a priori* estimate in $\mathcal{D}_0^1(\Omega,\sigma)\cap L^p(\Omega)$ for the solutions, which holds for smooth functions and will become rigorous by using a Galerkin truncation and a limiting process. Taking the inner product of (1.1) with u_t, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + \int_\Omega (F(u)+gu)\,dx\right] = -\|u_t\|^2_{L^2(\Omega)} \le 0. \tag{3.11}$$

Using the uniform Gronwall inequality, from (3.10), (3.11) and (3.9) we conclude that

$$\|u(t)\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + \int_\Omega |u(t)|^p\,dx \le C_9$$

provided that $t \ge t_0(\|u_0\|)+1$. It follows that the ball B_0 centered at 0 with radius C_9 is an absorbing set for $S(t)$ in $\mathcal{D}_0^1(\Omega,\sigma)\cap L^p(\Omega)$.

Using the absorbing set B_0 in $\mathcal{D}_0^1(\Omega,\sigma)$ and noting that $\mathcal{D}_0^1(\Omega,\sigma)\hookrightarrow L^2(\Omega)$ compactly and $L^2(\Omega)$ is connected, we obtain the following theorem.

THEOREM 3.3. *Under conditions* (1.2)–(1.3), *the semigroup* $S(t)$ *associated to problem* (1.1) *possesses a compact connected global attractor* $\mathcal{A} = \omega(B_0)$ *in* $L^2(\Omega)$.

REMARK 3.1. In fact, if we are only concerned with the existence of the global attractor in $L^2(\Omega)$ for the semigroup $S(t)$, then the assumption (1.3) can be replaced by a weaker assumption

$$(f(u)-f(v))(u-v) \ge -C_3|u-v|^2 \quad \text{for any } u,v\in\mathbb{R}.$$

However, we need to use the stronger assumptions, namely $f\in C^1(\mathbb{R})$ and (1.3), in the next section (to prove Lemma 4.1 and (4.5)).

4. Continuous dependence of the attractor on the nonlinearity. In this section we consider a family of C^1 functions f_λ, $\lambda\in\Lambda$, such that for each $\lambda\in\Lambda$, f_λ satisfies conditions (1.2)–(1.3) with the constants independent of λ. The family Λ is endowed with a topology $\mathcal{T}$ such that the convergence $\lambda_j\to\lambda$ with respect to $\mathcal{T}$ implies that

$$f_{\lambda_j}(u)\to f_\lambda(u) \quad \text{for any } u.$$

Let $S_t(\lambda, u_0)$ be the semigroup generated by the problem

$$(4.1)\qquad \begin{aligned} u_t - \operatorname{div}(\sigma(x)\nabla u) + f_\lambda(u) + g(x) &= 0, && x \in \Omega,\ t > 0,\\ u(t,x) &= 0, && x \in \partial\Omega,\ t > 0,\\ u(0,x) &= u_0(x), && x \in \Omega. \end{aligned}$$

From the results in Section 3, this semigroup has a compact absorbing set

$$B_\lambda = \{u \in L^2(\Omega) : \|u\|_{\mathcal{D}_0^1(\Omega,\sigma)} \le R_\lambda\},$$

and a compact global attractor $\mathcal{A}_\lambda = \omega(B_\lambda)$ in $X = L^2(\Omega)$.

Lemma 4.1. *Let u be the weak solution of problem* (4.1) *with the initial data $u_0 \in L^2(\Omega)$, $\|u_0\|_{L^2(\Omega)} \le R$. Then for any $\tau > 0$, $u(\tau) \in \mathcal{D}_0^1(\Omega,\sigma)$ and*

$$\|u(\tau)\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} \le C(R)/\tau.$$

Proof. Let $\{u_m\}$ be a sequence of approximate solutions. As in the proof of Theorem 3.2, we have

$$(4.2)\qquad \|u_m\|_{L^2(0,\tau;\mathcal{D}_0^1(\Omega,\sigma))} \le \widetilde{C}(R).$$

On the other hand,

$$t\left(\frac{d}{dt}u_m(t), Au_m\right) + t(Au_m, Au_m) + t(f(u_m), Au_m) + t(g, Au_m) = 0.$$

Hence

$$\frac{1}{2}\,t\,\frac{d}{dt}\|u_m\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + t\|Au_m\|^2_{L^2(\Omega)} - t\int_\Omega f'(u_m)\sigma(x)|\nabla u_m|^2\,dx + t(g, Au_m) = 0.$$

Using the Cauchy inequality and noting that $f'(u) \ge -C$, we obtain

$$(4.3)\qquad \frac{1}{2}\,t\,\frac{d}{dt}\|u_m\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} \le \frac{1}{2}\,t\|g\|^2_{L^2(\Omega)}.$$

Integrating (4.3) with respect to t on $(0,\tau)$ we get

$$\tau\|u_m(\tau)\|^2_{\mathcal{D}_0^1(\Omega)} \le \int_0^\tau \|u_m(t)\|^2_{\mathcal{D}_0^1(\Omega)}\,dt + \int_0^\tau t\|g\|^2_{L^2(\Omega)}\,dt.$$

Combining this with (4.2) we deduce that

$$(4.4)\qquad \|u_m(\tau)\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} \le C(R)/\tau.$$

As (4.4) holds for all m, this completes the proof. ∎

Lemma 4.2. *$S_t(\cdot,\cdot)$ is continuous in $\Lambda \times X$ for any fixed $t > 0$.*

Proof. Let $(\lambda_0, u_0) \in \Lambda \times X$ and $(\lambda_j, u_{j0}) \in \Lambda \times X$ be such that $\lambda_j \to \lambda_0$ and $u_{j0} \to u_0$. Let $u_j(t) = S_t(\lambda_j, u_{j0})$ be the solution of problem (1.1) with the nonlinearity f_{λ_j} and the initial data u_{j0}. Since f_{λ_j} satisfies (1.2)–(1.3) with the same constants and $\{u_{j0}\}$ is bounded, by using arguments as in the proof of Theorem 3.2, we find that

- $\{u_j\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
- $\{u_j\}$ is bounded in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$;
- $\{f_{\lambda_j}(u_j)\}$ is bounded in $L^{p'}(\Omega_T)$;
- $\{\partial_t u_j\}$ is bounded in $L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma)) + L^{p'}(\Omega_T)$.

We can apply the Aubin–Lions Lemma [11, p. 58] to conclude that $\{u_j\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. Hence, there exists a subsequence (still denoted by) u_j such that

- $u_j \rightharpoonup^* u$ in $L^\infty(0,T;L^2(\Omega))$;
- $u_j \rightharpoonup u$ in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$;
- $u_j \to u$ strongly in $L^2(0,T;L^2(\Omega))$;
- $u_j \to u$ a.e. in $\Omega \times (0,T)$;
- $f_{\lambda_j}(u_j) \rightharpoonup \omega$ in $L^{p'}(\Omega_T)$;
- $\partial_t u_j \rightharpoonup \partial_t u$ in $L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma)) + L^{p'}(\Omega_T)$.

Combining these with the hypotheses imposed on f_λ and the fact that f_{λ_j} converges almost everywhere to f_{λ_0} we have

$$f_{\lambda_j}(u_j) \to f_{\lambda_0}(u) \quad \text{almost everywhere in } \Omega \times (0,T). \tag{4.5}$$

From Lemma 1.3 in [11, Chapter 1], we have $\omega = f_{\lambda_0}(u)$. By passing to the weak limit, we find that u is the solution of problem (4.1) with the initial datum u_0 and the nonlinearity f_{λ_0}, that is, $u(t) = S_t(\lambda_0, u_0)$.

Since $u_j \to u$ in $L^2(0,T;L^2(\Omega))$, $u_j(t) \to u(t)$ in $L^2(\Omega)$ for all $t \in (0,T) \setminus E$, with $\mu(E) = 0$. Denote $M = \mathcal{D}^{-1}(\Omega,\sigma) + L^{p'}(\Omega)$. For any fixed $t > 0$, we choose $t_j \notin E$ such that $t_j \to t$ and

$$\|u_j(t_j) - u(t_j)\|_M \to 0 \quad \text{as } j \to \infty.$$

We have

$$\begin{aligned}
&\|u_j(t) - u(t)\|_M \\
&\qquad \le \|u_j(t) - u_j(t_j)\|_M + \|u_j(t_j) - u(t_j)\|_M + \|u(t_j) - u(t)\|_M \\
&\qquad = \Big\| \int_{t_j}^{t} u_j'(s)\,ds \Big\|_M + \|u_j(t_j) - u(t_j)\|_M + \Big\| \int_{t_j}^{t} u'(s)\,ds \Big\|_M \\
&\qquad \le \|\partial_t u_j\|_{L^{p'}(0,T;M)} |t - t_j|^{1/p} + \|u_j(t_j) - u(t_j)\|_M \\
&\qquad\qquad + \|\partial_t u\|_{L^{p'}(0,T;M)} |t - t_j|^{1/p}.
\end{aligned}$$

From the boundedness of $\{\partial_t u_j\}$ and $\partial_t u$ in $L^{p'}(0,T;M)$ we conclude that $u_j(t) \to u(t)$ in M.

On the other hand, using Lemma 4.1 we can prove that $\{u_j(t)\}$ is bounded in $\mathcal{D}_0^1(\Omega,\sigma)$ for any fixed $t>0$. As $\mathcal{D}_0^1(\Omega,\sigma)\hookrightarrow L^2(\Omega)$ compactly, there is a subsequence, still denoted by u_j, such that $u_j(t)\to v(t)$ strongly in $L^2(\Omega)$ and thus in M. By the uniqueness of the limit in M, we have $v(t)=u(t)$.

We have proved that for any sequences $(\lambda_j,u_{j0})\to(\lambda_0,u_0)$, there exists a subsequence of $S_t(\lambda_j,u_{j0})$ which converges in $L^2(\Omega)$, the limit is independent of the subsequence, and it equals $S_t(\lambda_0,u_0)$, so the whole sequence $S_t(\lambda_j,u_{j0})$ converges to $S_t(\lambda_0,u_0)$. Hence, $S_t(\cdot,\cdot)$ is continuous at (λ_0,u_0). ∎

THEOREM 4.1. *The family $\{\mathcal{A}_\lambda:\lambda\in\Lambda\}$ depends upper semicontinuously on the parameter λ, i.e.,*

$$\limsup_{\lambda\to\lambda_0}\delta_X(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})=0.$$

Proof. For any $\lambda_j\in\Lambda$, the semigroup $S_t(\lambda_j,u)$ has a compact absorbing set

$$B_{\lambda_j}=\{u\in L^2(\Omega):\|u\|_{\mathcal{D}_0^1(\Omega,\sigma)}\le R\},$$

where R is a sufficiently large constant depending only on the constants in (1.2)–(1.3). Hence, we can choose R independent of λ_j. Hence, there exists

$$B_0=\{u\in L^2(\Omega):\|u\|_{\mathcal{D}_0^1(\Omega,\sigma)}\le R\}$$

such that for any bounded set $B\subset L^2(\Omega)$ and for any λ, there is $\tau=\tau(\lambda,B)$ with $S_t(\lambda,B)\subset B_0$ for $t\ge\tau$. Let $\epsilon>0$. There exists $T=T(\epsilon)>0$ such that

$$\delta_X(S_T(\lambda_0,B),\mathcal{A}_{\lambda_0})<\epsilon.$$

By Lemma 4.2, for any $x\in B_0$, there are open neighborhoods $V(x)$ and $W(\lambda_0)$ in X and Λ such that

$$\delta_X(S_T(\lambda,V(x)),\mathcal{A}_{\lambda_0})<\epsilon\quad\text{for any }\lambda\in W(\lambda_0).$$

Since B_0 is compact in X, there exists a neighborhood W of λ_0 such that

$$\delta_X(S_T(\lambda,B_0),\mathcal{A}_{\lambda_0})<\epsilon\quad\text{for any }\lambda\in W.$$

Therefore

$$\delta_X(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})<\epsilon\quad\text{for any }\lambda\in W. \ ∎$$

5. Continuous dependence of the attractor on the diffusion. We consider a family of diffusion coefficients $\{\sigma_\epsilon\}_{0\le\epsilon<\epsilon_0}$ such that for each $0\le\epsilon<\epsilon_0$, σ_ϵ satisfies the following conditions:

($\mathcal{H}_{\alpha_\epsilon}$) $\sigma_\epsilon\in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha_\epsilon\in(0,2)$, $\liminf_{x\to z}|x-z|^{-\alpha_\epsilon}\sigma_\epsilon(x)>0$ for every $z\in\overline{\Omega}$.

Moreover, we assume that $\sigma_\epsilon \to \sigma_0$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0$, and that the embeddings $\mathcal{D}^1_0(\Omega,\sigma_\epsilon) \hookrightarrow \mathcal{D}^1_0(\Omega,\sigma_0)$ are continuous uniformly with respect to $\epsilon \in (0,\epsilon_0)$, that is, there exists $C > 0$ independent of ϵ such that

$$\|u\|_{\mathcal{D}^1_0(\Omega,\sigma_0)} \le C\|u\|_{\mathcal{D}^1_0(\Omega,\sigma_\epsilon)} \quad \text{for all } u \in \mathcal{D}^1_0(\Omega,\sigma_\epsilon) \ \ (0<\epsilon<\epsilon_0).$$

A typical example which satisfies the above assumptions is $\Omega = B_{\mathbb{R}^N}(0,1)$ and $\sigma_\epsilon(x) \sim |x|^{\alpha_\epsilon}$, where $\alpha \in (0,2)$, $\alpha_\epsilon := \alpha - \epsilon \in (0,2)$, $\sigma_0(x) = |x|^\alpha$.

Let $S_\epsilon(t)$ $(0 \le \epsilon < \epsilon_0)$ be the semigroup generated by the problem

$$\begin{aligned} u_t - \operatorname{div}(\sigma_\epsilon(x)\nabla u) + f(u) + g(x) &= 0, && x \in \Omega,\ t>0,\\ u(t,x) &= 0, && x \in \partial\Omega,\ t>0,\\ u(0,x) &= u_0, && x\in\Omega. \end{aligned} \tag{5.1}$$

From the results in Section 3, for each $0 \le \epsilon < \epsilon_0$ the corresponding semigroup $S_\epsilon(t)$ has a compact absorbing set

$$B_\epsilon = \{u \in L^2(\Omega) : \|u\|_{\mathcal{D}^1_0(\Omega,\sigma_\epsilon)} \le R\},$$

and has a compact global attractor $\mathcal{A}_\epsilon = \omega(B_\epsilon)$ in the space $X = L^2(\Omega)$.

Because R depends only on $\|u_0\|_{L^2(\Omega)}$ and on the constants in (1.2)–(1.3), we can choose an absorbing set B_0 for all semigroups $S_\epsilon(t)$,

$$B_0 = \{u \in L^2(\Omega) : \|u\|_{\mathcal{D}^1_0(\Omega,\sigma_\epsilon)} \le R,\ \forall 0 \le \epsilon < \epsilon_0\}.$$

Since B_0 is the absorbing set, we have

$$\mathcal{A}_\epsilon = S_\epsilon(t)\mathcal{A}_\epsilon \subset B_0$$

for all $t > t_0(\|u_0\|_{L^2(\Omega)}) + 1$ and $0 \le \epsilon < \epsilon_0$. Thus

$$\bigcup_{0\le\epsilon<\epsilon_0} \mathcal{A}_\epsilon \subset B_0.$$

On the other hand, by using arguments as in Theorem 3.2, one can easily prove that there is a constant C such that for all weak solutions u^ϵ on $(0,T)$, $\epsilon \in [0,\epsilon_0)$, of problems (5.1) with the same initial data $u_0 \in L^2(\Omega)$, we have

$$\|u^\epsilon(t)\|^2_{L^2(0,T;\mathcal{D}^1_0(\Omega,\sigma_\epsilon))} \le C \quad \text{for all } \epsilon \in [0,\epsilon_0),\ t>0, \tag{5.2}$$

where $C = C(\|u_0\|_{L^2(\Omega)}, T, C_0, |\Omega|, \lambda_1)$.

We now prove the following lemma.

Lemma 5.1. *For any fixed $t \ge 0$, $S_\epsilon(t)u_0 \to S_0(t)u_0$ uniformly on bounded subsets B of $L^2(\Omega)$ as $\epsilon \to 0$, that is,*

$$\sup_{u_0\in B} \|S_\epsilon(t)u_0 - S_0(t)u_0\|_{L^2(\Omega)} \to 0 \quad \textit{as } \epsilon \to 0.$$

Proof. Let $u^\epsilon(t) := S_\epsilon(t)u_0$ $(0 \le \epsilon < \epsilon_0)$ be the solution of problem (5.1) with the initial data $u_0 \in B$. Then, by the definition of weak solutions, we have

$$\int_0^T \langle u_t^\epsilon, \varphi\rangle\, dt + \int_{\Omega_T} \sigma_\epsilon(x)\nabla u^\epsilon \nabla\varphi\, dx\, dt + \int_{\Omega_T} f(u^\epsilon)\varphi\, dx\, dt + \int_{\Omega_T} g(x)\varphi\, dx\, dt = 0, \tag{5.3}$$

and

$$\int_0^T \langle u_t^0, \varphi\rangle\, dt + \int_{\Omega_T} \sigma(x)\nabla u^0 \nabla\varphi\, dx\, dt + \int_{\Omega_T} f(u^0)\varphi\, dx\, dt + \int_{\Omega_T} g(x)\varphi\, dx\, dt = 0, \tag{5.4}$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$.

Repeating the arguments in the proof of Theorem 3.2, and since $\mathcal{D}_0^1(\Omega,\sigma_\epsilon) \hookrightarrow \mathcal{D}_0^1(\Omega,\sigma_0)$ continuously uniformly with respect to $\epsilon \in (0,\epsilon_0)$, we deduce that

- $\{u^\epsilon\}$ is bounded in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma))$;
- $\{u^\epsilon\}$ is bounded in $L^p(\Omega_T)$;
- $\{f(u^\epsilon)\}$ is bounded in $L^{p'}(\Omega_T)$;
- $\{u_t^\epsilon\}$ is bounded in $L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma_0)) + L^{p'}(\Omega_T)$.

Hence there is a subsequence (still denoted by) $\{u^\epsilon\}$ such that

- $u^\epsilon \rightharpoonup v$ in $L^2(0,T;\mathcal{D}_0^1(\Omega,\sigma_0))$;
- $f(u^\epsilon) \rightharpoonup f(v)$ in $L^{p'}(\Omega_T)$;
- $u_t^\epsilon \rightharpoonup v_t$ in $L^2(0,T;\mathcal{D}^{-1}(\Omega,\sigma_0)) + L^{p'}(\Omega_T)$;
- $u^\epsilon \to v$ in $L^2(0,T;L^2(\Omega))$.

Since

$$\int_{\Omega_T} \sigma_\epsilon(x)\nabla u^\epsilon \nabla\varphi = \int_{\Omega_T} \sigma_0(x)\nabla u^\epsilon \nabla\varphi + \int_{\Omega_T} (\sigma_\epsilon(x) - \sigma_0(x))\nabla u^\epsilon \nabla\varphi,$$

and

$$\begin{aligned}\Big|\int_{\Omega_T} (\sigma_\epsilon(x) - \sigma_0(x))\nabla u^\epsilon \nabla\varphi\Big| &\le \Big(\int_{\Omega_T} |\sigma_\epsilon(x) - \sigma_0(x)|\,|\nabla u^\epsilon|^2\Big)^{1/2}\Big(\int_{\Omega_T} |\sigma_\epsilon(x) - \sigma_0(x)|\,|\nabla\varphi|^2\Big)^{1/2} \\ &\le \Big(\int_{\Omega_T} (\sigma_\epsilon(x) + \sigma_0(x))|\nabla u^\epsilon|^2\Big)^{1/2}\Big(\int_{\Omega_T} |\sigma_\epsilon(x) - \sigma_0(x)||\nabla\varphi|^2\Big)^{1/2} \to 0,\end{aligned}$$

as $\epsilon \to 0$, where we have used (5.2) and the fact that $\sigma_\epsilon \to \sigma_0$ in $L^1_{\mathrm{loc}}(\Omega)$, we deduce that

$$\int_{\Omega_T} \sigma_\epsilon(x)\nabla u^\epsilon \nabla\varphi \to \int_{\Omega_T} \sigma_0(x)\nabla v\nabla\varphi \quad \text{ as } \epsilon \to 0.$$

By passing to the weak limit in (5.3), we see that v is a weak solution to problem (5.1) with $\epsilon = 0$. By the uniqueness of the weak solution, we conclude that $u^0 = v$. Hence, using arguments as in the proof of Proposition 3.1, one can prove that for any bounded subset B of $L^2(\Omega)$, $u_0 \in B$, and for any $\delta > 0$ given, there exists an $\epsilon(\delta) > 0$ such that $\|u^\epsilon(t) - u^0(t)\|_{L^2(\Omega)} < \delta$ for all $0 < \epsilon \le \epsilon(\delta)$. The proof is complete. ∎

We are now in a position to prove the following

THEOREM 5.1. *The family* $\{\mathcal{A}_\epsilon : 0 \le \epsilon < \epsilon_0\}$ *depends upper semicontinuously on the diffusion coefficients* σ_ϵ, *i.e.*,

$$\limsup_{\epsilon\to 0} \delta_X(\mathcal{A}_\epsilon, \mathcal{A}_0) = 0.$$

Proof. Given $\delta > 0$, we first show that $S_\epsilon(t)B_0 \subset N(\mathcal{A}_0, \delta)$ for some $t > 0$ and $\epsilon \le \epsilon(\delta)$. Now, since $\mathcal{A}_0$ attracts B_0 there exists a time t such that

$$S_0(t)B_0 \subset N(\mathcal{A}_0, \delta/2).$$

Then, for ϵ sufficiently small, we can ensure that

$$\sup_{u\in B_0} \|S_\epsilon(t)u - S_0(t)u\|_{L^2(\Omega)} < \delta/2.$$

Thus in fact, for $\epsilon \le \epsilon(\delta)$, since $\mathcal{A}_\epsilon \subset B_0$,

$$\mathcal{A}_\epsilon = S_\epsilon(t)A_\epsilon \subset S_\epsilon(t)B_0 \subset N(\mathcal{A}_0, \delta),$$

and it follows that $\delta_X(\mathcal{A}_\epsilon, \mathcal{A}_0) \le \delta$. This completes the proof. ∎

6. Some remarks on the case of an unbounded domain. In this section we discuss the case of an unbounded domain $\Omega \subset \mathbb{R}^N, N \ge 2$, and we assume that the weight function $\sigma(x)$ satisfies the condition $(\mathcal{H}^\infty_{\alpha,\beta})$. Then the operator $A = -\operatorname{div}(\sigma(x)\nabla)$ has the same properties as in the case of a bounded domain. On the other hand, we still have the continuous embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^{2^*_\alpha}(\Omega)$, and in particular the embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact. Therefore, we may apply the methods used for a bounded domain to this case with some small changes in the conditions imposed on the nonlinearity f.

More precisely, we assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,u)| \le C_1|u|^{p-1} + h_1(x), \tag{6.1}$$
$$f(x,u)u \ge C_2|u|^p - h_2(x), \tag{6.2}$$
$$f'_u(x,u) \ge -C_3, \tag{6.3}$$

where $h_1 \in L^{p'}(\Omega), h_2 \in L^1(\Omega)$ are nonnegative real-valued functions. We can now repeat the arguments used in Section 3 to obtain

THEOREM 6.1. *Under conditions* $(\mathcal{H}^\infty_{\alpha,\beta})$ *and* (6.1)–(6.3), *problem* (1.1) *defines a semigroup* $S(t) : L^2(\Omega) \to L^2(\Omega)$, *which has a compact connected global attractor* $\mathcal{A}$ *in* $L^2(\Omega)$.

We can also prove the upper semicontinuous dependence of the global attractor on the nonlinearity and on the diffusion coefficient by using arguments as in Sections 4 and 5.

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