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Positive solutions for one-dimensional singular p-Laplacian boundary value problems

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Abstract. We consider the existence of positive solutions of the equation

$$\frac{1}{\lambda(t)}(\lambda(t)\varphi_p(x'(t)))' + \mu f(t,x(t),x'(t)) = 0,$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1$, subject to some singular Sturm–Liouville boundary conditions. Using the Krasnosel'skiĭ fixed point theorem for operators on cones, we prove the existence of positive solutions under some structure conditions.

1. Introduction. In this paper, we consider the one-dimensional p-Laplacian equation

$$\frac{1}{\lambda(t)}(\lambda(t)\varphi_p(x'(t)))' + \mu f(t,x(t),x'(t)) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

subject to one of the following three pairs of boundary value conditions:

$$\begin{aligned} &\alpha x(0) - \beta \lim_{t\to 0^+} \lambda(t)^{1/(p-1)}x'(t) = 0,\\ &\gamma \lim_{t\to+\infty} x(t) + \delta \lim_{t\to+\infty} \lambda(t)^{1/(p-1)}x'(t) = 0, \end{aligned} \tag{1.2$_a$}$$

$$\alpha x(0) - \beta \lim_{t\to 0^+} \lambda(t)^{1/(p-1)}x'(t) = 0, \quad \lim_{t\to+\infty} \lambda(t)^{1/(p-1)}x'(t) = 0, \tag{1.2$_b$}$$

$$\begin{aligned} &\lim_{t\to 0^+} \lambda(t)^{1/(p-1)}x'(t) = 0,\\ &\gamma \lim_{t\to+\infty} x(t) + \delta \lim_{t\to+\infty} \lambda(t)^{1/(p-1)}x'(t) = 0, \end{aligned} \tag{1.2$_c$}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\alpha,\beta,\gamma,\delta > 0$, $\mu > 0$ is a parameter, $\lambda(t)$, $f(t,x,y)$ are continuous functions, and $f(t,x,y)$ may be singular at $t = 0$.

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The Sturm–Liouville boundary value problems have been the subject of intensive study during the past years: see for example [1–3, 5–7] and the references therein. In particular, Lian and Ge [2] considered the Sturm–Liouville boundary value problem for the equation

$$(p(t)x'(t))' + \lambda\varphi(t)f(t, x(t)) = 0.$$

By using fixed point theorems in cones, they established the existence criteria. In a recent paper [4], Sun et al. have studied a particular case of (1.1) with $p = 2$, i.e., the nonlinear singular equation

$$\frac{1}{p(t)}(p(t)z'(t))' + \mu f(t, z(t), z'(t)) = 0.$$

They established a relation between the existence of positive solutions and the parameter μ.

In this paper, we investigate the existence of positive solutions to the problems (1.1), (1.2). Our approach is based on the Krasnosel'skiĭ fixed point theorem. Unlike earlier, the equation we consider is quasilinear, so that the theory based on Green's function cannot be applied. In addition, solutions of the problems (1.1), (1.2) may not be concave, and so some efficient methods based on convexity (see for example [1, 6, 7]) could not be available here. In order to overcome these difficulties, a special Banach space and special cones are introduced so that we can establish existence results.

This paper is organized as follows. As preliminaries, in Section 2 we introduce the required Banach space E and suitable cones in E, and the corresponding integral operators defined on the cones; we also give some properties of the functions from the cones. In Section 3, we prove the complete continuity of the operators and finally we apply the Krasnosel'skiĭ fixed point theorem to obtain the existence of positive solutions of the boundary value problem (1.1), (1.2_a). In view of their similarity, for the problems (1.1), (1.2_b) and (1.1), (1.2_c) we only present the results and omit the details of the proof. In Section 4, we give some detailed examples to illustrate our main results.

2. Preliminaries. In this section, we present some necessary definitions and construct some integral operators related to solutions of the problems (1.1), (1.2), which will be used to demonstrate the existence of solutions via the Krasnosel'skiĭ fixed point theorem. Firstly, for the convenience of the readers, we recall the definitions of a cone and a completely continuous operator.

Definition 2.1. A nonempty, convex and closed subset P of a Banach space E is called a *cone* if

(i) $P \neq \{0\}$,
(ii) if $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, and $x, y \in P$, then $\alpha x + \beta y \in P$,
(iii) if $x \in P$ and $-x \in P$, then $x = 0$.

Definition 2.2. An operator $T : E \to E$ is said to be *completely continuous* if T is continuous and maps bounded sets into precompact sets.

The following is the well-known Krasnosel'skiĭ fixed point theorem (see for example [5]).

Proposition 2.3. *Let E be a Banach space and $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are two bounded open sets in E such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Throughout this paper, we need the following assumptions:

(H1) The function $f : (0, +\infty) \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous and singular at the point $t = 0$, with $0 \leq f(t, x, y) \leq a(t)g(t, x)$, where $a : (0, +\infty) \to [0, +\infty)$ is continuous and singular at $t = 0$; $g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous and $g(t, x)$ is bounded for x in any bounded set and for all $t \in [0, +\infty)$;
(H2) $\lambda \in C[0, +\infty) \cap C^1(0, +\infty)$ with $\lambda(t) > 0$ on $(0, +\infty)$ and
$$0 < \int_0^{+\infty} \varphi_q\left(\frac{1}{\lambda(t)}\right) dt < +\infty,$$
where $1/p + 1/q = 1$;
(H3) $0 < \int_0^{+\infty} \lambda(t) a(t)\, dt < \infty$.

In Section 3, we prove the existence of positive solutions of the boundary value problems (1.1), (1.2) under the assumptions (H1)–(H3). In addition, in Section 4, we give detailed examples to show that all of the assumptions (H1)–(H3) can be satisfied.

Because of the possible singularity, we give the exact meaning of solutions to the problems (1.1), (1.2). By a *positive solution* of the boundary value problem (1.1), (1.2), we mean a function $x(t)$ satisfying the following conditions:

(i) $x \in C[0, +\infty) \cap C^1(0, +\infty)$ and the following three limits exist:
$$\lim_{t\to\infty} x(t), \qquad \lim_{t\to 0^+} \lambda(t)^{1/(p-1)} x'(t), \qquad \lim_{t\to\infty} \lambda(t)^{1/(p-1)} x'(t);$$
(ii) $x(t) > 0$ for all $t \in (0, +\infty)$ and satisfies (a), (b) or (c) of (1.2);

(iii) $\lambda(t)\varphi_p(x'(t))$ is locally absolutely continuous in $(0,+\infty)$ and

$$\frac{1}{\lambda(t)}(\lambda(t)\varphi_p(x'(t)))' + \mu f(t,x(t),x'(t)) = 0$$

almost everywhere in $(0,+\infty)$.

Before proving the main results, we make some preparations. Let $k>1$ be a constant and

$$y(t) = \int_{1/k}^{t} \varphi_q\left(\frac{1}{\lambda(s)}\int_s^t \lambda(\tau)\,d\tau\right) ds + \int_t^k \varphi_q\left(\frac{1}{\lambda(s)}\int_t^s \lambda(\tau)\,d\tau\right) ds, \quad t \in \left[\frac{1}{k},k\right].$$

From the above definition, we find that $y(t)$ is continuous and positive on $[1/k,k]$. For notational convenience, we set

$$M_1 = \int_0^{+\infty} \varphi_q(1/\lambda(t))\,dt, \quad M = \max\{\beta/\alpha,1\}, \quad \tilde{M} = \max\{\delta/\gamma,1\},$$

$$m = \max\left\{\frac{\alpha}{\beta},\frac{\gamma}{\delta}\right\},\ h = \frac{\beta\delta}{M(\alpha\delta+\beta\gamma+\alpha\gamma M_1)},\ \mathit{\Lambda} = \min\left\{y(t) : t \in \left[\frac{1}{k},k\right]\right\}.$$

We consider the Banach space E defined by

$$E = \left\{x \in C[0,+\infty)\cap C^1(0,+\infty) : \begin{array}{c}\lim\limits_{t\to+\infty} x(t),\ \lim\limits_{t\to 0^+} \lambda(t)^{1/(p-1)}x'(t) \\ \text{and } \lim\limits_{t\to+\infty} \lambda(t)^{1/(p-1)}x'(t) \text{ exist}\end{array}\right\}$$

with the norm

$$\|x\| = \max\{\|x\|_1,\|x\|_2\},$$

where

$$\|x\|_1 = \frac{1}{1+M_1}\sup_{0\le t<+\infty}|x(t)|, \quad \|x\|_2 = \sup_{0<t<+\infty}|\lambda(t)^{1/(p-1)}x'(t)|.$$

Define the following subsets of E:

$$P_a = \left\{x \in E : \begin{array}{l} x(t)\ge 0, t\in[0,+\infty), \alpha x(0) - \beta\lim\limits_{t\to 0^+}\lambda(t)^{1/(p-1)}x'(t) = 0, \\ \gamma\lim\limits_{t\to+\infty}x(t) + \delta\lim\limits_{t\to+\infty}\lambda(t)^{1/(p-1)}x'(t) = 0, \\ \lambda(t)^{1/(p-1)}x'(t) \text{ is nonincreasing on } (0,\infty)\end{array}\right\},$$

$$P_b = \left\{x \in E : \begin{array}{l} x(t)\ge 0, t\in[0,+\infty), \alpha x(0) - \beta\lim\limits_{t\to 0^+}\lambda(t)^{1/(p-1)}x'(t) = 0, \\ \lim\limits_{t\to+\infty}\lambda(t)^{1/(p-1)}x'(t) = 0, \\ \lambda(t)^{1/(p-1)}x'(t) \text{ is nonincreasing on } (0,\infty)\end{array}\right\},$$

$$P_c = \left\{ x \in E: \begin{array}{l} x(t) \geq 0,\ t \in [0,+\infty),\ \lim\limits_{t\to 0^+} \lambda(t)^{1/(p-1)} x'(t) = 0, \\ \gamma \lim\limits_{t\to+\infty} x(t) + \delta \lim\limits_{t\to+\infty} \lambda(t)^{1/(p-1)} x'(t) = 0, \\ \lambda(t)^{1/(p-1)} x'(t) \text{ is nonincreasing on } (0,\infty) \end{array} \right\}.$$

It is easy to check that P_a, P_b and P_c are all cones in E. Define the corresponding operators T_a, T_b, T_c by

$$(T_a x)(t) = \begin{cases} \dfrac{\beta}{\alpha}\varphi_q\Big(\mu \displaystyle\int_0^A \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big) \\ \qquad + \displaystyle\int_0^t \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^A \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\bigg) ds, \quad 0 \leq t < A, \\ \dfrac{\delta}{\gamma}\varphi_q\Big(\mu \displaystyle\int_A^\infty \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big) \\ \qquad + \displaystyle\int_t^\infty \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_A^s \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\bigg) ds, \quad A \leq t < \infty, \end{cases}$$

for $x \in P_a$,

$$(T_b x)(t) = \frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^\infty \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big) + \int_0^t \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^\infty \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\bigg)\, ds, \quad t \in [0,\infty),$$

for $x \in P_b$, and

$$(T_c x)(t) = \frac{\delta}{\gamma}\varphi_q\Big(\mu \int_0^\infty \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big) + \int_t^\infty \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_0^s \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\bigg)\, ds, \quad t \in [0,\infty),$$

for $x \in P_c$, where A is a solution of the equation

$$z_0(t) = z_1(t),$$

with

$$z_0(t) := \frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^t \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big) + \int_0^t \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^t \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\bigg)\, ds, \quad 0 \leq t < \infty,$$

$$z_1(t) := \frac{\delta}{\gamma}\varphi_q\Big(\mu\int_t^{\infty}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\Big)$$
$$+\int_t^{\infty}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int_t^{s}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\bigg)\,ds,\quad 0\le t<\infty.$$

Because z_0 is a nondecreasing continuous function on $[0,+\infty)$ with $z_0(0) = 0$, and z_1 is a nonincreasing continuous function on $[0,+\infty)$ with $z_1(\infty) = 0$, there exists $A \in (0,+\infty)$ such that $z_0(A) = z_1(A)$. Moreover, if $A_1, A_2 \in (0,+\infty), A_1 < A_2$ and $z_0(A_i) = z_1(A_i) (i = 1,2)$, then we have $\lambda(t)f(t,x(t),x'(t)) \equiv 0$ on $[A_1,A_2]$. Therefore, the mapping T_a is well defined.

From the definition of T_a, we deduce that for each $x \in P_a$, $T_a x$ satisfies (1.2_a) and $(T_a x)(A)$ is the maximum value of $(T_a x)(t)$ on $[0,+\infty)$, since

$$(T_a x)'(t) = \begin{cases} \varphi_q\bigg(\dfrac{\mu}{\lambda(t)}\displaystyle\int_t^{A}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\bigg), & 0<t\le A,\\ -\varphi_q\bigg(\dfrac{\mu}{\lambda(t)}\displaystyle\int_A^{t}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\bigg), & A\le t<\infty,\end{cases}$$

and $(T_a x)'(A) = 0$. Moreover,

$$\lambda(t)^{1/(p-1)}(T_a x)'(t) = \begin{cases} \varphi_q\Big(\mu\displaystyle\int_t^{A}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\Big), & 0<t\le A,\\ -\varphi_q\Big(\mu\displaystyle\int_A^{t}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\Big), & A\le t<\infty,\end{cases}$$

$$\varphi_p(\lambda(t)^{1/(p-1)}(T_a x)'(t)) = \begin{cases} \mu\displaystyle\int_t^{A}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau, & 0<t\le A,\\ -\mu\displaystyle\int_A^{t}\lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau, & A\le t<\infty,\end{cases}$$

$$(\lambda(t)\varphi_p((T_a x)'(t)))' + \mu\lambda(t)f(t,x(t),x'(t)) = 0,\qquad 0<t<+\infty.$$

This shows that $T_a(P_a) \subset P_a$ and each fixed point of T_a in P_a is a solution of $(1.1), (1.2_a)$. In the same way, we can deduce that $T_i(P_i) \subset P_i$ and each fixed point of T_i in P_i is a solution of $(1.1), (1.2_i)$ $(i = b, c)$.

Now we state some properties of the functions in P_a, P_b, P_c. By the definitions of the norms in the Banach space E, we can deduce

LEMMA 2.4.

(1) *For each* $x \in P_a$, $\|x\|_2 = \max\{\alpha x(0)/\beta, \gamma x(\infty)/\delta\} \le m \sup_{0 \le t < \infty} x(t)$.
(2) *For each* $x \in P_b$, $\|x\| = \max\{x(\infty)/(1+M_1), \alpha x(0)/\beta\}$.
(3) *For each* $x \in P_c$, $\|x\| = \max\{x(0)/(1+M_1), \gamma x(\infty)/\delta\}$.

LEMMA 2.5. *For each* $x \in P_a \cup P_b$,

$$\|x\|_1 \le M\|x\|_2,$$

and for each $x \in P_c$,

$$\|x\|_1 \le \tilde{M}\|x\|_2.$$

Proof. For each $x \in P_a \cup P_b$, we have

$$\begin{aligned}
\frac{x(t)}{1+M_1} &= \frac{x(0) + \int_0^t x'(s)\,ds}{1+M_1} \\
&= \frac{1}{1+M_1}\left(\frac{\beta}{\alpha}\lim_{t\to 0^+}\lambda(t)^{1/(p-1)}x'(t) + \int_0^t \lambda(s)^{1/(p-1)}x'(s)\varphi_q\left(\frac{1}{\lambda(s)}\right)ds\right) \\
&\le \frac{1}{1+M_1}\,\frac{\beta}{\alpha}\|x\|_2 + \frac{M_1}{1+M_1}\|x\|_2 \\
&\le M\|x\|_2.
\end{aligned}$$

In a similar way we can show that $x(t)/(1+M_1) \le \tilde{M}\|x\|_2$ for all $x \in P_c$ and $t \in [0,\infty)$. ∎

LEMMA 2.6.

(1) *For each* $x \in P_a$, $x(t) \ge h\|x\|$ *for all* $t \in [0,+\infty)$.
(2) *For each* $x \in P_b$, $x(t) \ge \frac{\beta}{\alpha M}\|x\|$ *for all* $t \in [0,+\infty)$.
(3) *For each* $x \in P_c$, $x(t) \ge \frac{\delta}{\gamma \tilde{M}}\|x\|$ *for all* $t \in [0,+\infty)$.

Proof. For each $x \in P_a$, we consider the following two cases:

(i) $\alpha x(0)/\beta \ge \gamma x(\infty)/\delta$;
(ii) $\alpha x(0)/\beta \le \gamma x(\infty)/\delta$.

In case (i), by Lemma 2.1, we have

$$\|x\|_2 = \alpha x(0)/\beta.$$

Then, by Lemma 2.2,

$$x(0) = \frac{\beta}{\alpha}\|x\|_2 \ge \frac{\beta}{\alpha M}\|x\|.$$

Because

$$-\frac{\gamma x(\infty)}{\delta} \le \lambda(t)^{1/(p-1)}x'(t) \le \frac{\alpha x(0)}{\beta}, \quad t \in (0,+\infty),$$

we have

$$x(\infty) = x(0) + \int\limits_0^\infty x'(s)\,ds = x(0) + \int\limits_0^\infty \lambda(s)^{1/(p-1)} x'(s)\varphi_q\Big(\frac{1}{\lambda(s)}\Big)\,ds$$
$$\geq x(0) + \int\limits_0^\infty \varphi_q\Big(\frac{1}{\lambda(s)}\Big)\Big(-\frac{\gamma x(\infty)}{\delta}\Big)\,ds = x(0) - \frac{\gamma x(\infty)}{\delta} M_1,$$

i.e.,

$$\Big(1 + \frac{\gamma M_1}{\delta}\Big) x(\infty) \geq x(0) \geq \frac{\beta}{\alpha M}\|x\|.$$

Thus,

$$x(\infty) \geq \frac{\delta}{\delta + \gamma M_1}\,\frac{\beta}{\alpha M}\|x\|.$$

By the definition of P_a, we have

$$x(t) \geq \min\{x(0), x(\infty)\} \geq \frac{\delta}{\delta + \gamma M_1}\,\frac{\beta}{\alpha M}\|x\| \geq h\|x\|, \qquad t \in [0,\infty).$$

We can deal with case (ii) in a similar way. The last two issues of the lemma can be easily obtained by the definitions of P_b and P_c. ∎

3. Existence theorems. In this section, we prove the complete continuity of the operators defined in Section 2, and then we state and prove our main results. Since the Arzelà–Ascoli theorem fails to hold in E, we need the following compactness criterion. For more general cases, we refer the readers to [3] and the references therein.

LEMMA 3.1 ([3]). *Let $V = \{x \in E : \|x\| < l\}$ $(l > 0)$. Then V is relatively compact in E if the following conditions hold:*

(a) *$\{x(t)/(1 + M_1) : x \in V\}$ is equicontinuous on any compact interval of $[0, +\infty)$ and equiconvergent at infinity, the latter meaning that for any given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that for any $t \geq T$ and $x \in V$,*
$$\Big|\frac{x(t)}{1 + M_1} - \frac{x(+\infty)}{1 + M_1}\Big| < \varepsilon;$$

(b) *$\{\lambda(t)^{1/(p-1)} x'(t) : x \in V\}$ is equicontinuous on any compact subinterval of $(0, +\infty)$ and is equiconvergent both at $t = 0$ and at infinity.*

Now we can prove the complete continuity of T_a, T_b, T_c by Lemma 3.1.

LEMMA 3.2. *$T_a : P_a \to P_a$ is completely continuous.*

Proof. Put

$$P_a^R = \{x \in P_a : \|x\| < R\},$$
$$S_R = \sup\{g(t,x) : t \in [0,\infty),\ 0 \leq x \leq (1 + M_1)R\}.$$

Firstly, we show that $T_a(P_a^R)$ is bounded. Let $x \in P_a^R$. By direct calculations, we obtain

$$\sup_{0<t<+\infty} |\lambda(t)^{1/(p-1)}(T_a x)'(t)| \le \varphi_q\Big(\mu \int_0^\infty \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big)$$
$$\le \varphi_q\Big(\mu S_R \int_0^\infty \lambda(\tau) a(\tau)\, d\tau\Big) < +\infty.$$

So there exists a constant N such that $\|T_a x\| \le N$ for all $x \in P_a^R$.

Secondly, we show that $\{(T_a x)(t)/(1+M_1) : x \in P_a^R\}$ is equicontinuous on any compact subinterval of $[0,+\infty)$ and equiconvergent at infinity. Indeed, for any $T > 0$ and $0 \le t_1 < t_2 \le T$, we have

$$\Big|\frac{(T_a x)(t_1)}{1+M_1} - \frac{(T_a x)(t_2)}{1+M_1}\Big| = \frac{1}{1+M_1}\Big|\int_{t_1}^{t_2} (T_a x)'(s)\, ds\Big|$$
$$\le \frac{1}{1+M_1}\int_{t_1}^{t_2} |\lambda(s)^{1/(p-1)}(T_a x)'(s)|\varphi_q\Big(\frac{1}{\lambda(s)}\Big)\, ds$$
$$\le \frac{1}{1+M_1}\|T_a x\|_2 \int_{t_1}^{t_2} \varphi_q\Big(\frac{1}{\lambda(s)}\Big)\, ds$$
$$\le \frac{N}{1+M_1}\int_{t_1}^{t_2} \varphi_q\Big(\frac{1}{\lambda(s)}\Big)\, ds,$$

and for any $t > 0$,

$$\Big|\frac{(T_a x)(t)}{1+M_1} - \frac{(T_a x)(\infty)}{1+M_1}\Big| = \frac{1}{1+M_1}\Big|\int_{\infty}^{t} (T_a x)'(s)\, ds\Big|$$
$$\le \frac{1}{1+M_1}\|T_a x\|_2 \int_t^\infty \varphi_q\Big(\frac{1}{\lambda(s)}\Big)\, ds$$
$$\le \frac{N}{1+M_1}\int_t^\infty \varphi_q\Big(\frac{1}{\lambda(s)}\Big)\, ds.$$

Thirdly, we show that $\{\lambda(t)^{1/(p-1)}(T_a x)'(t) : x \in P_a^R\}$ is equicontinuous on any compact subinterval of $(0,+\infty)$ and equiconvergent both at $t = 0$ and at infinity. Indeed, for any $[a,b] \subset (0,+\infty)$ and $a \le t_1 < t_2 \le b$, we have

$$|\varphi_p(\lambda(t_1)^{1/(p-1)}(T_a x)'(t_1)) - \varphi_p(\lambda(t_2)^{1/(p-1)}(T_a x)'(t_2))|$$
$$= \Big|\mu\int_{t_1}^{A} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \mu\int_{A}^{t_2} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau\Big|$$
$$\le \mu S_R \int_{t_1}^{t_2} \lambda(\tau) a(\tau)\, d\tau \qquad \text{if } t_1 < A < t_2,$$

$$
\begin{aligned}
&|\varphi_p(\lambda(t_1)^{1/(p-1)}(T_a x)'(t_1)) - \varphi_p(\lambda(t_2)^{1/(p-1)}(T_a x)'(t_2))| \\
&\qquad = \Big|\mu \int_{t_1}^{A} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau - \mu \int_{t_2}^{A} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau\Big| \\
&\qquad \le \mu S_R \int_{t_1}^{t_2} \lambda(\tau) a(\tau)\,d\tau \quad \text{ if } t_1 < t_2 \le A, \\
&|\varphi_p(\lambda(t_1)^{1/(p-1)}(T_a x)'(t_1)) - \varphi_p(\lambda(t_2)^{1/(p-1)}(T_a x)'(t_2))| \\
&\qquad = \Big|-\mu \int_{A}^{t_1} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \mu \int_{A}^{t_2} \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau\Big| \\
&\qquad \le \mu S_R \int_{t_1}^{t_2} \lambda(\tau) a(\tau)\,d\tau \quad \text{ if } A \le t_1 < t_2.
\end{aligned}
$$

Moreover, for any $t > 0$, we have

$$
\begin{aligned}
&\Big|\varphi_p(\lambda(t)^{1/(p-1)}(T_a x)'(t)) - \varphi_p(\lim_{t\to 0^+} \lambda(t)^{1/(p-1)}(T_a x)'(t))\Big| \\
&\qquad\qquad\qquad\qquad \le \mu S_R \int_0^t \lambda(\tau) a(\tau)\,d\tau, \\
&\Big|\varphi_p(\lambda(t)^{1/(p-1)}(T_a x)'(t)) - \varphi_p(\lim_{t\to \infty} \lambda(t)^{1/(p-1)}(T_a x)'(t))\Big| \\
&\qquad\qquad\qquad\qquad \le \mu S_R \int_t^\infty \lambda(\tau) a(\tau)\,d\tau.
\end{aligned}
$$

Therefore, by Lemma 3.1, $T_a(P_a^R)$ is relatively compact.

Finally, to show that $T_a : P_a \to P_a$ is continuous, let $\{x_j\}_{j=1}^\infty \subset P_a$ and $x_j \to x_0$ as $j \to \infty$. Then there exists $r > 0$ such that $\|x_j\| \le r$ for all $j \ge 1$. Hence, there exist convergent subsequences of $\{T_a x_j\}_{j=1}^\infty$. Let $\{T_a x_{j_n}\}_{n=1}^\infty$ converge to $v \in P_a$. We will prove that $v = T_a x_0$. Notice that there exists a sequence $\{A_{j_n}\}_{n=1}^\infty$ such that $A_{j_n} \in (0, +\infty)$ and

$$
\begin{aligned}
(3.1)\qquad & \frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\,d\tau\Big) \\
&\qquad + \int_0^{A_{j_n}} \varphi_q\Big(\frac{\mu}{\lambda(s)} \int_s^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\,d\tau\Big)\,ds
\end{aligned}
$$

$$= \frac{\delta}{\gamma}\varphi_q\Big(\mu \int_{A_{j_n}}^{\infty} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\, d\tau\Big) + \int_{A_{j_n}}^{\infty} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_{A_{j_n}}^{s} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\, d\tau\bigg)\, ds.$$

Moreover, we have

$$(T_a x_{j_n})(t) = \begin{cases} \dfrac{\beta}{\alpha}\varphi_q\Big(\mu \displaystyle\int_0^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\Big) \\ \quad + \displaystyle\int_0^t \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\bigg)\, ds, & 0 \le t < A_{j_n}, \\ \dfrac{\delta}{\gamma}\varphi_q\Big(\mu \displaystyle\int_{A_{j_n}}^{\infty} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\Big) \\ \quad + \displaystyle\int_t^{\infty} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_{A_{j_n}}^{s} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\bigg)\, ds, & A_{j_n} \le t. \end{cases}$$

In the following, we need to handle two cases separately.

Case I: $\{A_{j_n}\}_{n=1}^{\infty}$ is unbounded. In this case, we can find a subsequence of $\{A_{j_n}\}$, not relabeled, such that $\{A_{j_n}\}$ is strictly increasing and $A_{j_n} \to \infty$ as $n \to \infty$. Notice that

$$\int_{A_{j_n}}^{\infty} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\, d\tau \le \int_{A_{j_n}}^{\infty} \lambda(\tau) a(\tau) g(\tau, x_{j_n})\, d\tau \le \sup_{\substack{t\in[0,+\infty)\\ 0\le x\le(1+M_1)r}} g(t,x) \int_{A_{j_n}}^{\infty} \lambda(\tau) a(\tau)\, d\tau,$$

and

$$\begin{aligned} &\int_{A_{j_n}}^{\infty} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_{A_{j_n}}^{s} \lambda(\tau) f(\tau, x_{j_n}, x'_{j_n})\, d\tau\bigg)\, ds \\ &\quad\le \int_{A_{j_n}}^{\infty} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \sup_{\substack{t\in[0,+\infty)\\ 0\le x\le(1+M_1)r}} g(t,x) \int_0^{+\infty} \lambda(\tau) a(\tau)\, d\tau\bigg)\, ds \\ &\quad= \varphi_q\bigg(\mu \sup_{\substack{t\in[0,+\infty)\\ 0\le x\le(1+M_1)r}} g(t,x) \int_0^{+\infty} \lambda(\tau) a(\tau)\, d\tau\bigg) \int_{A_{j_n}}^{\infty} \varphi_q\bigg(\frac{1}{\lambda(s)}\bigg)\, ds. \end{aligned}$$

The assumptions (H2) and (H3) imply

$$\lim_{n\to+\infty}\int\limits_{A_{j_n}}^{\infty}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau=0,$$

$$\lim_{n\to+\infty}\int\limits_{A_{j_n}}^{\infty}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_{A_{j_n}}^{s}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau\bigg)\,ds=0.$$

From (3.1), it follows that

$$\begin{aligned}(3.2)\qquad \lim_{n\to+\infty}\frac{\beta}{\alpha}\varphi_q\Big(\mu\int\limits_{0}^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau\Big)&\\ +\int\limits_{0}^{A_{j_n}}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_{s}^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau\bigg)\,ds&=0.\end{aligned}$$

In particular, we have

$$\lim_{n\to+\infty}\int\limits_{0}^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau=0.$$

For any fixed $\eta>0$, there exists a positive integer N_η such that $A_{j_n}\ge\eta$ as $n\ge N_\eta$, which yields

$$\int\limits_{0}^{\eta}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau\le\int\limits_{0}^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau$$

when $n\ge N_\eta$. Thus we have

$$\lim_{n\to+\infty}\int\limits_{0}^{\eta}\lambda(\tau)f(\tau,x_{j_n},x'_{j_n})\,d\tau=0.$$

Using the Lebesgue dominated convergence theorem, we obtain

$$\int\limits_{0}^{\eta}\lambda(\tau)f(\tau,x_0,x'_0)\,d\tau=0.$$

Since η is arbitrary,

$$f(t,x_0,x'_0)=0,\qquad 0<t<+\infty,$$

which shows that for any $0<\bar A<+\infty$,

$$\begin{aligned}&\frac{\beta}{\alpha}\varphi_q\Big(\mu\int\limits_{0}^{\bar A}\lambda(\tau)f(\tau,x_0,x'_0)\,d\tau\Big)+\int\limits_{0}^{\bar A}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_{s}^{\bar A}\lambda(\tau)f(\tau,x_0,x'_0)\,d\tau\bigg)\,ds\\ &=\frac{\delta}{\gamma}\varphi_q\Big(\mu\int\limits_{\bar A}^{\infty}\lambda(\tau)f(\tau,x_0,x'_0)\,d\tau\Big)+\int\limits_{\bar A}^{\infty}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_{\bar A}^{s}\lambda(\tau)f(\tau,x_0,x'_0)\,d\tau\bigg)\,ds.\end{aligned}$$

Furthermore it is easy to see that $(T_a x_0)(t)\equiv 0$.

On the other hand, we will prove that $v \equiv 0$. Fix $t_0 \geq 0$. Then there exists a positive integer N_{t_0} such that $A_{j_n} \geq t_0$ for $n \geq N_{t_0}$, which yields for $n \geq N_{t_0}$,

$$\begin{aligned}(T_a x_{j_n})(t_0) &= \frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\Big) \\ &\quad + \int_0^{t_0} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\bigg) ds \\ &\leq \frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\Big) \\ &\quad + \int_0^{A_{j_n}} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^{A_{j_n}} \lambda(\tau) f(\tau, x_{j_n}(\tau), x'_{j_n}(\tau))\, d\tau\bigg) ds.\end{aligned}$$

In view of (3.2), letting $n \to \infty$ in the above inequality, we get $v(t_0) = 0$, and we conclude that

$$(T_a x_0)(t) = v(t), \qquad t \in [0, +\infty).$$

Case II: $\{A_{j_n}\}_{n=1}^{\infty}$ is bounded. In this case, there exists a subsequence, still denoted by $\{A_{j_n}\}$, and a constant $\bar{A} \in [0, +\infty)$, such that

$$\lim_{n\to+\infty} A_{j_n} = \bar{A}.$$

If $\bar{A} = 0$, by a similar argument, we can deduce that

$$(T_a x_0)(t) = 0 = v(t), \qquad t \in [0, +\infty).$$

If $0 < \bar{A} < \infty$, then by (3.1) and the Lebesgue dominated convergence theorem, letting $n \to +\infty$ yields

$$\begin{aligned}&\frac{\beta}{\alpha}\varphi_q\Big(\mu \int_0^{\bar{A}} \lambda(\tau) f(\tau, x_0(\tau), x'_0(\tau))\, d\tau\Big) \\ &\qquad\qquad + \int_0^{\bar{A}} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_s^{\bar{A}} \lambda(\tau) f(\tau, x_0(\tau), x'_0(\tau))\, d\tau\bigg) ds \\ &\qquad = \frac{\delta}{\gamma}\varphi_q\Big(\mu \int_{\bar{A}}^{\infty} \lambda(\tau) f(\tau, x_0(\tau), x'_0(\tau))\, d\tau\Big) \\ &\qquad\quad + \int_{\bar{A}}^{\infty} \varphi_q\bigg(\frac{\mu}{\lambda(s)} \int_{\bar{A}}^{s} \lambda(\tau) f(\tau, x_0(\tau), x'_0(\tau))\, d\tau\bigg) ds.\end{aligned}$$

Thus

$$(T_a x_0)(t) = \begin{cases} \dfrac{\beta}{\alpha}\varphi_q\Big(\mu\int\limits_0^{\bar A}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\Big) \\ \qquad +\int\limits_0^t\varphi_q\bigg(\dfrac{\mu}{\lambda(s)}\int\limits_s^{\bar A}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\bigg)\,ds, \quad 0\le t<\bar A, \\ \dfrac{\delta}{\gamma}\varphi_q\Big(\mu\int\limits_{\bar A}^{\infty}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\Big) \\ \qquad +\int\limits_t^{\infty}\varphi_q\bigg(\dfrac{\mu}{\lambda(s)}\int\limits_{\bar A}^{s}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\bigg)\,ds, \quad \bar A\le t. \end{cases}$$

Let $\bar t\ge 0$ be fixed. If $\bar t<\bar A$, then there exists a positive integer $N_{\bar t}$ such that $A_{j_n}\ge \bar t$ for $n\ge N_{\bar t}$, which implies, for $n\ge N_{\bar t}$,

$$\begin{aligned}(T_a x_{j_n})(\bar t) = {} & \frac{\beta}{\alpha}\varphi_q\Big(\mu\int\limits_0^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n}(\tau),x_{j_n}'(\tau))\,d\tau\Big) \\ & +\int\limits_0^{\bar t}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_s^{A_{j_n}}\lambda(\tau)f(\tau,x_{j_n}(\tau),x_{j_n}'(\tau))\,d\tau\bigg)\,ds.\end{aligned}$$

Hence, letting $n\to+\infty$ yields

$$\begin{aligned}v(\bar t) = {} & \frac{\beta}{\alpha}\varphi_q\Big(\mu\int\limits_0^{\bar A}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\Big) \\ & +\int\limits_0^{\bar t}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_s^{\bar A}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\bigg)\,ds.\end{aligned}$$

Similarly, one can easily prove that

$$\begin{aligned}v(\bar t) = {} & \frac{\delta}{\gamma}\varphi_q\Big(\mu\int\limits_{\bar A}^{\infty}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\Big) \\ & +\int\limits_{\bar t}^{\infty}\varphi_q\bigg(\frac{\mu}{\lambda(s)}\int\limits_{\bar A}^{s}\lambda(\tau)f(\tau,x_0(\tau),x_0'(\tau))\,d\tau\bigg)\,ds\end{aligned}$$

when $\bar t>\bar A$. Clearly, we have reached $v(t)=(T_a x_0)(t)$ for $t\in[0,+\infty)$, $t\ne\bar A$. Further, by continuity, $v(t)=(T_a x_0)(t)$ for $t\in[0,+\infty)$.

Summing up the above arguments, we conclude that $T_a x_{j_n}\to T_a x_0$ as $n\to+\infty$. Furthermore, we assert that $T_a x_j\to T_a x_0$ as $j\to+\infty$. In fact, if not, then there exist $\varepsilon_0>0$ and a subsequence $\{T_a x_{j_\kappa}\}_{\kappa=1}^{\infty}$ such that $\|T_a x_{j_\kappa}-T_a x_0\|\ge\varepsilon_0$, $\kappa\ge 1$. However, from the results we have obtained, there exist subsequences of $\{T_a x_{j_\kappa}\}_{\kappa=1}^{\infty}$ which converge to $T_a x_0$. This leads to a contradiction. Therefore, $T_a : P_a \to P_a$ is continuous. ∎

Now we can establish the existence results for positive solutions of the problem (1.1), (1.2_a).

THEOREM 3.3. *Let* (H1)–(H3) *be satisfied and suppose that*

$$(\mathrm{A}_1)\qquad \begin{aligned} &0 \le g^0 = \limsup_{x\to 0^+} \sup_{t\in[0,+\infty)} \frac{g(t,x)}{x^{p-1}} < L, \\ &0 < l < f_\infty = \liminf_{x\to+\infty} \inf_{\substack{t\in[1/k,k]\\ y\in\mathbb{R}}} \frac{f(t,x,y)}{x^{p-1}} \le \infty, \end{aligned}$$

where $L = (M^{p-1}(1+M_1)^{p-1}\int_0^\infty \lambda(\tau)a(\tau)\,d\tau)^{-1}$, $l = \varphi_p(2(M_1+1)/(h\Lambda))$. *Then the boundary value problem* (1.1), (1.2_a) *has at least one positive solution for any*

$$\mu \in (l/f_\infty, L/g^0). \tag{3.3}$$

Proof. Without loss of generality, we suppose that $0 < g^0$ and $f_\infty < \infty$. From (3.3), there exists $\varepsilon > 0$ such that

$$0 < \frac{l}{f_\infty - \varepsilon} \le \mu \le \frac{L}{g^0 + \varepsilon}. \tag{3.4}$$

By the first inequality of (A_1) and for the above ε, there exists $\sigma > 0$ such that

$$\frac{g(t,x)}{x^{p-1}} \le g^0 + \varepsilon, \quad 0 < x \le \sigma,\ t \in [0,+\infty),$$

i.e.,

$$g(t,x) \le (g^0+\varepsilon)x^{p-1}, \quad 0 < x \le \sigma,\ t\in[0,+\infty). \tag{3.5}$$

Let $P_a^{r_1} = \{x \in P_a : \|x\| < r_1\}$ $(0 < r_1 < \sigma/(1+M_1))$. From the definition of $\|\cdot\|$,

$$0 < x(t) \le (1+M_1)r_1 \le \sigma \quad \text{for all } x \in \partial P_a^{r_1},\ t\in[0,+\infty).$$

Thus,

$$g(t,x(t)) \le (g^0+\varepsilon)x(t)^{p-1} \quad \text{for all } x \in \partial P_a^{r_1},\ t \in [0,+\infty).$$

Then for any $x \in \partial P_a^{r_1}$,

$$\begin{aligned} \|T_a x\|_2 = \sup_{0<t<+\infty} |\lambda(t)^{1/(p-1)}(T_a x)'(t)| &\le \varphi_q\Big(\mu \int_0^\infty \lambda(\tau)a(\tau)g(\tau,x(\tau))\,d\tau\Big) \\ &\le \varphi_q\Big(\mu(g^0+\varepsilon)(1+M_1)^{p-1}{r_1}^{p-1}\int_0^\infty \lambda(\tau)a(\tau)\,d\tau\Big). \end{aligned}$$

Therefore, by (3.4),

$$\begin{aligned}\|T_a x\| &\le M\|T_a x\|_2\\ &\le r_1\varphi_q\Big(\mu(g^0+\varepsilon)(1+M_1)^{p-1}M^{p-1}\int\limits_0^\infty \lambda(\tau)a(\tau)\,d\tau\Big) \le \|x\|.\end{aligned}$$

On the other hand, by the second inequality of (A_1) and for the above ε, there exists $H>0$ such that

$$\frac{f(t,x,y)}{x^{p-1}} \ge f_\infty - \varepsilon > 0, \quad x\ge H,\ t\in[1/k,k],\ y\in\mathbb{R},$$

i.e.,

$$f(t,x,y) \ge (f_\infty-\varepsilon)x^{p-1}, \quad x\ge H,\ t\in[1/k,k],\ y\in\mathbb{R}.$$

Let $P_a^{r_2}=\{x\in P_a : \|x\|<r_2\}$ $(0<r_1<r_2,\ r_2\ge H/h)$. From Lemma 2.6, we know that

$$x(t)\ge hr_2\ge H \quad \text{for all } x\in\partial P_a^{r_2},\ t\in[1/k,k],$$

and so

$$f(t,x(t),x'(t)) \ge (f_\infty-\varepsilon)x(t)^{p-1} \ge (f_\infty-\varepsilon)(hr_2)^{p-1}$$

for all $x\in\partial P_a^{r_2}$, $t\in[1/k,k]$. Thus, for any $x\in\partial P_a^{r_2}$,

$$\begin{aligned}2\|T_a x\| &\ge 2\|T_a x\|_1 = 2\frac{(T_a x)(A)}{1+M_1}\\ &\ge \frac{1}{1+M_1}\Bigg[\int\limits_0^A \varphi_q\Big(\frac{\mu}{\lambda(s)}\int\limits_s^A \lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\Big)\,ds\\ &\quad + \int\limits_A^\infty \varphi_q\Big(\frac{\mu}{\lambda(s)}\int\limits_A^s \lambda(\tau)f(\tau,x(\tau),x'(\tau))\,d\tau\Big)\,ds\Bigg]\\ &\ge \frac{1}{1+M_1}\Bigg[\int\limits_{1/k}^A \varphi_q\Big(\frac{\mu}{\lambda(s)}\int\limits_s^A \lambda(\tau)(f_\infty-\varepsilon)(hr_2)^{p-1}\,d\tau\Big)\,ds\\ &\quad + \int\limits_A^k \varphi_q\Big(\frac{\mu}{\lambda(s)}\int\limits_A^s \lambda(\tau)(f_\infty-\varepsilon)(hr_2)^{p-1}d\tau\Big)\,ds\Bigg]\\ &= \frac{1}{1+M_1}\Bigg[\varphi_q(\mu(f_\infty-\varepsilon)(hr_2)^{p-1})\Bigg(\int\limits_{1/k}^A \varphi_q\Big(\frac{1}{\lambda(s)}\int\limits_s^A \lambda(\tau)\,d\tau\Big)\,ds\\ &\quad + \int\limits_A^k \varphi_q\Big(\frac{1}{\lambda(s)}\int\limits_A^s \lambda(\tau)\,d\tau\Big)\,ds\Bigg)\Bigg]\\ &\ge \frac{\Lambda}{1+M_1} r_2\varphi_q(\mu(f_\infty-\varepsilon)h^{p-1}) \ge 2\|x\| \quad \text{if } 1/k<A<k,\end{aligned}$$

$$\begin{aligned}
\|T_a x\| \geq \|T_a x\|_1 &= \frac{(T_a x)(A)}{1+M_1} \\
&\geq \frac{1}{1+M_1}\int_A^\infty \varphi_q\biggl(\frac{\mu}{\lambda(s)}\int_A^s \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau\biggr)\,ds \\
&\geq \frac{1}{1+M_1}\int_{1/k}^{k} \varphi_q\biggl(\frac{\mu}{\lambda(s)}\int_{1/k}^s \lambda(\tau)(f_\infty-\varepsilon)(hr_2)^{p-1}\,d\tau\biggr)\,ds \\
&= \frac{r_2}{1+M_1}\varphi_q(\mu(f_\infty-\varepsilon)h^{p-1})\int_{1/k}^{k} \varphi_q\biggl(\frac{1}{\lambda(s)}\int_{1/k}^s \lambda(\tau)\,d\tau\biggr)\,ds \\
&\geq \frac{r_2\Lambda}{1+M_1}\varphi_q(\mu(f_\infty-\varepsilon)h^{p-1}) \geq \|x\| \quad \text{if } A \leq 1/k,
\end{aligned}$$

$$\begin{aligned}
\|T_a x\| \geq \|T_a x\|_1 &= \frac{(T_a x)(A)}{1+M_1} \\
&\geq \frac{1}{1+M_1}\int_0^A \varphi_q\biggl(\frac{\mu}{\lambda(s)}\int_s^A \lambda(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau\biggr)\,ds \\
&\geq \frac{1}{1+M_1}\int_{1/k}^{k} \varphi_q\biggl(\frac{\mu}{\lambda(s)}\int_s^k \lambda(\tau)(f_\infty-\varepsilon)(hr_2)^{p-1}\,d\tau\biggr)\,ds \\
&\geq \frac{r_2\Lambda}{1+M_1}\varphi_q(\mu(f_\infty-\varepsilon)h^{p-1}) \geq \|x\| \quad \text{if } k \leq A,
\end{aligned}$$

i.e.,

$$\|T_a x\| \geq \|x\| \quad \text{for all } x \in \partial P_a^{r_2}.$$

Therefore, by the Krasnosel'skiĭ fixed point theorem, T_a has a fixed point $x^* \in \overline{P_a^{r_2}} \setminus P_a^{r_1}$. Furthermore, since $0 < r_1 \leq \|x^*\| \leq r_2$, it follows that $x^*(t) > 0$ for $t \in (0,\infty)$. This shows that the fixed point x^* is a positive solution of the problem (1.1), (1.2_a). ■

Remark 3.4. In fact, Theorem 3.3 still holds if one of the following conditions is satisfied:

(1) $f_\infty = +\infty$, $g^0 > 0$, for each $\mu \in (0, L/g^0)$,
(2) $f_\infty = +\infty$, $g^0 = 0$, for each $\mu \in (0,\infty)$,
(3) $l < f_\infty < +\infty$, $g^0 = 0$, for each $\mu \in (l/f_\infty, \infty)$.

Remark 3.5. Since $l/f_\infty < 1$ and $L/g^0 > 1$, we have $1 \in (l/f_\infty, L/g^0)$. So when $\mu = 1$, Theorem 3.3 also holds.

In a similar way we can prove the following theorem.

THEOREM 3.6. *Let* (H1)–(H3) *be satisfied and suppose that*

$$(\mathrm{A}_2)\qquad \begin{aligned} &0 \le g^\infty = \limsup_{x\to+\infty} \sup_{t\in[0,+\infty)} \frac{g(t,x)}{x^{p-1}} < L,\\ &0 < l < f_0 = \liminf_{\substack{x+|y|\to 0^+\\ x\ge 0}} \inf_{t\in[1/k,k]} \frac{f(t,x,y)}{x^{p-1}+|y|^{p-1}} \le \infty, \end{aligned}$$

where $L = (M^{p-1}(1+M_1)^{p-1}\int_0^\infty \lambda(\tau)a(\tau)\,d\tau)^{-1}$, $l = \varphi_p(2(M_1+1)/(h\Lambda))$. *Then the boundary value problem* (1.1), (1.2_a) *has at least one positive solution for any*

$$\mu \in (l/f_0, L/g^\infty). \tag{3.6}$$

REMARK 3.7. Just as in Remark 3.1, Theorem 3.6 still holds if one of the following conditions is satisfied:

(1) $f_0 = +\infty$, $g^\infty > 0$, for each $\mu \in (0, L/g^\infty)$,
(2) $f_0 = +\infty$, $g^\infty = 0$, for each $\mu \in (0, +\infty)$,
(3) $l < f_0 < +\infty$, $g^\infty = 0$, for each $\mu \in (l/f_0, +\infty)$.

REMARK 3.8. Since $l/f_0 < 1$ and $L/g^\infty > 1$, we have $1 \in (l/f_0, L/g^\infty)$. So when $\mu = 1$, Theorem 3.6 also holds.

REMARK 3.9. If we set $l = (\alpha M/\beta)^{p-1}(\int_{1/k}^k \lambda(\tau)\,d\tau)^{-1}$, then all the results above hold for the problem (1.1), (1.2_b). If we set

$$l = (\gamma\tilde{M}/\delta)^{p-1}\Big(\int_{1/k}^{k} \lambda(\tau)\,d\tau\Big)^{-1} \text{ and } L = \Big(\tilde{M}^{p-1}(1+M_1)^{p-1}\int_0^\infty \lambda(\tau)a(\tau)\,d\tau\Big)^{-1},$$

then all the results above also hold for the problem (1.1), (1.2_c).

4. Examples. In this section we present some examples to illustrate our main results. Set

$$a(t) = \begin{cases} 1/t, & 0 < t \le 1,\\ e^{(-2/p)(t-1)}, & 1 < t, \end{cases} \qquad \lambda(t) = \begin{cases} t^{1/p}, & 0 \le t \le 1,\\ e^{(t-1)/p}, & 1 < t. \end{cases}$$

For f and g, we can give two pairs of examples. One pair is

$$g_1(t,x) = \begin{cases} 2e^{(-2/p)(t-1)}x^p, & 0 \le t \le 1,\ x \ge 0,\\ 2t^{-1/p}x^p, & 1 < t,\ x \ge 0, \end{cases}$$

$$f_1(t,x,y) = t^{-1/p}e^{(-2/p)(t-1)}x^p(|\sin y|+1), \quad t > 0,\ x \ge 0,\ y \in \mathbb{R},$$

and the other pair is

$$g_2(t,x) = \begin{cases} 2He^{(-2/p)(t-1)}(x^{p-1}+1), & 0 \le t \le 1, x \ge 0,\\ 2Ht^{-1/p}(x^{p-1}+1), & 1 < t, x \ge 0, \end{cases}$$

$$f_2(t,x,y) = Ht^{-1/p}e^{(-2/p)(t-1)}(x^{p-1}+1)(|\sin y|+1), \quad t > 0,\ x \ge 0,\ y \in \mathbb{R}.$$

where H is any positive constant. We can verify that when $p > (1+\sqrt{5})/2$, all of the assumptions (H1)–(H3) are satisfied for f_i, $a(t)$, g_i and $\lambda(t)$, $i = 1, 2$.

By simple calculations, we obtain

$$\int_0^{+\infty} \varphi_q\left(\frac{1}{\lambda(t)}\right) dt = \frac{p^2(p-1)^2}{p(p-1)-1},$$

$$\int_0^{+\infty} \lambda(t)a(t)\, dt = 2p,$$

$$\sup_{t\in[0,+\infty)} \frac{g_1(t,x)}{x^{p-1}} = 2e^{2/p}x \quad \text{for } x > 0,$$

$$\inf_{\substack{t\in[1/k,k]\\ y\in\mathbb{R}}} \frac{f_1(t,x,y)}{x^{p-1}} = k^{-1/p}e^{(-2/p)(k-1)}x \quad \text{for } x > 0,$$

$$\sup_{t\in[0,+\infty)} \frac{g_2(t,x)}{x^{p-1}} = \frac{2He^{2/p}(x^{p-1}+1)}{x^{p-1}} \quad \text{for } x > 0,$$

$$\inf_{t\in[1/k,k]} \frac{f_2(t,x,y)}{x^{p-1}+|y|^{p-1}} = \frac{Hk^{-1/p}e^{(-2/p)(k-1)}(x^{p-1}+1)(|\sin y|+1)}{x^{p-1}+|y|^{p-1}} \quad \text{for } x+|y| > 0 \text{ and } x \geq 0.$$

Thus,

$$\limsup_{x\to 0^+} \sup_{t\in[0,+\infty)} \frac{g_1(t,x)}{x^{p-1}} = 0,$$

$$\liminf_{x\to+\infty} \inf_{\substack{t\in[1/k,k]\\ y\in\mathbb{R}}} \frac{f_1(t,x,y)}{x^{p-1}} = +\infty,$$

$$\limsup_{x\to+\infty} \sup_{t\in[0,+\infty)} \frac{g_2(t,x)}{x^{p-1}} = 2He^{2/p},$$

$$\liminf_{\substack{x+|y|\to 0^+\\ x\geq 0}} \inf_{t\in[1/k,k]} \frac{f_2(t,x,y)}{x^{p-1}+|y|^{p-1}} = +\infty.$$

Therefore from Remark 3.4, we see that for each $\mu \in (0,\infty)$, the boundary value problem (1.1), (1.2_a) with f replaced by f_1 (and g replaced by g_1) has at least one positive solution. Let $H < \frac{1}{2}e^{-2/p}L$. Then, by an application of Remark 3.7, we know that for each $\mu \in \left(0, \frac{1}{2H}e^{-2/p}L\right)$, the boundary value problem (1.1), (1.2_a) with f replaced by f_2 (and g replaced by g_2) has at least one positive solution.

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