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Heights of squares of Littlewood polynomials and infinite series

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Abstract. Let P be a unimodular polynomial of degree $d-1$. Then the height $H(P^2)$ of its square is at least $\sqrt{d/2}$ and the product $L(P^2)H(P^2)$, where L denotes the length of a polynomial, is at least d^2. We show that for any $\varepsilon > 0$ and any $d \geq d(\varepsilon)$ there exists a polynomial P with ± 1 coefficients of degree $d-1$ such that $H(P^2) < (2+\varepsilon)\sqrt{d \log d}$ and $L(P^2)H(P^2) < (16/3+\varepsilon)d^2 \log d$. A similar result is obtained for the series with ± 1 coefficients. Let A_m be the mth coefficient of the square $f(x)^2$ of a unimodular series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, where all $a_i \in \mathbb{C}$ satisfy $|a_i| = 1$. We show that then $\limsup_{m\to\infty} |A_m|/\sqrt{m} \geq 1$ and that there exist some infinite series with ± 1 coefficients and an integer $m(\varepsilon)$ such that $|A_m| < (2+\varepsilon)\sqrt{m \log m}$ for each $m \geq m(\varepsilon)$.

1. Introduction. Let

$$P(x) = a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} \in \mathbb{C}[x], \qquad a_{d-1} \neq 0,$$

be a polynomial of degree $d-1$. Its *height* is defined by the formula $H(P) := \max_{0\leq i\leq d-1} |a_i|$ and its *length* by $L(P) := \sum_{i=0}^{d-1} |a_i|$. A polynomial is called *unimodular* if $a_i \in \mathbb{C}$ and $|a_i| = 1$ for each $i = 0, \ldots, d-1$. We denote the set of unimodular polynomials of degree $d-1$ by $\mathcal{U}_d$. Its subset $\mathcal{L}_d$ of *Littlewood polynomials* of degree $d-1$ consists of polynomials with coefficients in the set $\{-1, 1\}$.

An old conjecture of Littlewood [19] is that there exist two positive constants c_1, c_2 and infinitely many $d \in \mathbb{N}$ such that for some $P \in \mathcal{L}_d$ we have

$$c_1\sqrt{d} < |P(z)| < c_2\sqrt{d} \tag{1}$$

for all z on the unit circle. Körner [18] proved that this is true for some infinite sequence of $P \in \mathcal{U}_d$ and Kahane [17] showed that there exists $P \in \mathcal{U}_d$ for which the above inequality holds with $c_1 = 1-\varepsilon$ and $c_2 = 1+\varepsilon$ provided d is large enough. Some further results in this direction have been obtained by Beck [1]. However, the conjecture of Erdős [11] that the constant c_2

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cannot be arbitrarily close to 1 for $P \in \mathcal{L}_d$ remains open (see [22], but also [23] for a disproof of the main result given in [22]). As for Littlewood's conjecture, the *Rudin–Shapiro polynomials* (whose coefficients satisfy $a_0 = 1$, $a_{2n} = a_n$, $a_{2n+1} = (-1)^n a_n$ for $n \geq 0$; see [7] and the sequence A020985 in the Online Encyclopedia of Integer Sequences) satisfy the upper bound in (1) with $c_2 = \sqrt{2}$ when d is a power of 2. No infinite sequence of Littlewood polynomials for which the lower bound (1) holds is known. In this direction, Clunie [9] gave an example showing that the lower bound in (1) is satisfied for some polynomial $\sum_{i=0}^{d-1} a_i x^i$ with $|a_i| \leq 1$.

One can also investigate how flat a Littlewood polynomial on the unit circle is with respect to other norms. Write

$$\|P\|_s := \Big(\int_0^1 |P(e^{2\pi i\theta})|^s \, d\theta\Big)^{1/s}$$

for the L_s-norm of P over unit circle. Clearly, the function $s \mapsto \|P\|_s$ is nondecreasing in s. In particular, $\|P\|_0$ is the Mahler measure of P, $\|P\|_\infty = \sup_{|z|=1} |P(z)|$ and

$$M(P) < \|P\|_2 < \|P\|_s < \|P\|_\infty$$

for every $P \in \mathcal{L}_d$ with $d \geq 2$, and every s satisfying $2 < s < \infty$. The problems on whether the quotients $\|P\|_2/\|P\|_0$ and $\|P\|_4/\|P\|_2$ are bounded away from 1 or not are known as *Mahler's problem* and the *merit factor problem* of Golay, respectively (see, e.g., [6] and [16] for two surveys describing the current status of the merit factor problem).

Let us consider the squares of polynomials P from $\mathcal{L}_d$ and $\mathcal{U}_d$. For any given polynomial $P(x) = \sum_{j=0}^{d-1} a_j x^j$, we shall write $P(x)^2 = \sum_{j=0}^{2d-2} A_j x^j$. With this notation, for each $P \in \mathcal{U}_d$, by Parseval's formula, we have

$$\|P\|_2^2 = \int_0^1 |P(e^{2\pi i\theta})|^2 \, d\theta = \sum_{j=0}^{d-1} |a_j|^2 = d$$

and

$$\|P\|_4^4 = \int_0^1 |P(e^{2\pi i\theta})|^4 \, d\theta = \int_0^1 |P(e^{2\pi i\theta})^2|^2 \, d\theta = \sum_{j=0}^{2d-2} |A_j|^2 \leq L(P^2)H(P^2).$$

Hence

$$L(P^2)H(P^2) \geq (\|P\|_4/\|P\|_2)^4 d^2. \tag{2}$$

The *merit factor* $\mathrm{MF}(P)$ of a Littlewood polynomial can be defined by the equality

$$\|P\|_4^4 = \|P\|_2^4\Big(1 + \frac{1}{\mathrm{MF}(P)}\Big) = d^2\Big(1 + \frac{1}{\mathrm{MF}(P)}\Big)$$

(see [6]). So (2) yields

$$L(P^2)H(P^2) \geq \left(1 + \frac{1}{\mathrm{MF}(P)}\right)d^2.$$

It is known that $\limsup_{d\to\infty} \mathrm{MF}(P)$, where $P \in \mathcal{L}_d$, is at least 6 (see [13], [15]). The conjecture in [15] that 6 is the optimal bound is still open, although there is some computational evidence against it [4]. Of course, if proved, this conjecture would give a better bound, but even if $\limsup_{d\to\infty} \mathrm{MF}(P) = \infty$, from $\|P\|_4 > \|P\|_2$ when $d \geq 2$ and inequality (2) we obtain

$$L(P^2)H(P^2) > d^2$$

for each $d \geq 2$. From $L(P^2) \leq (2d-1)H(P^2) < 2dH(P^2)$ we further get

$$H(P^2) > \sqrt{d/2}$$

for $d \geq 1$. By the same argument, $L(P^k)H(P^k) \geq (\|P\|_{2k}/\|P\|_2)^{2k}d^k$ for each integer $k \geq 2$ and each $P \in \mathcal{U}_d$, $d \geq 2$, hence

$$L(P^k)H(P^k) > d^k \quad \text{and} \quad H(P^k) > \sqrt{d^{k-1}/k}.$$

How small can the quantities $H(P^2)$ and $L(P^2)H(P^2)$ (and more generally $H(P^k)$ and $L(P^k)H(P^k)$) be when $P \in \mathcal{L}_d$? The question concerning the size of $H(P^2)$ for $P \in \mathcal{L}_d$ has been raised in [2]. The present author observed that the Fekete type Littlewood polynomials $P(x) = \sum_{i=0}^{d-1} \left(\frac{i+1}{p}\right)x^i \in \mathcal{L}_d$, where p is a prime number satisfying $2d+1 \leq p < 4d+2$ and $\left(\frac{i}{p}\right)$ is the Legendre symbol, give the bound

$$H(P^2) < c\sqrt{d}\log d$$

(see Theorem 2 in [10]). The next result improves this bound by a factor of $\sqrt{\log d}$, but still leaves the gap of order $\sqrt{\log d}$ between the lower and upper bounds.

THEOREM 1.1. *For each $\varepsilon > 0$ there is a constant $d(\varepsilon)$ such that for every integer $d \geq d(\varepsilon)$ there is a Littlewood polynomial $P \in \mathcal{L}_d$ for which*

$$H(P^2) < (2+\varepsilon)\sqrt{d\log d} \quad \textit{and} \quad L(P^2)H(P^2) < (16/3+\varepsilon)d^2\log d.$$

Similar questions can also be asked for infinite series. Let $\mathcal{U}_\infty$ be the collection of all series $\sum_{i=0}^{\infty} a_i x^i$, where a_i, $i = 0, 1, 2, \ldots$, are complex numbers satisfying $|a_i| = 1$, and let $\mathcal{L}_\infty$ be the subset of $\mathcal{U}_\infty$ consisting of the series $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in \{-1, 1\}$ for $i = 0, 1, 2, \ldots$.

THEOREM 1.2. *For $f(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{U}_\infty$, let $f(x)^2 = \sum_{i=0}^{\infty} A_i x^i$, i.e., $A_m = \sum_{j=0}^{m} a_j a_{m-j}$ for $m \geq 0$. Then*

$$\limsup_{m\to\infty} |A_m|/\sqrt{m} \geq 1$$

for every $f(x) \in \mathcal{U}_\infty$. *On the other hand, for each* $\varepsilon > 0$ *there exist a positive integer* $m(\varepsilon)$ *and a series* $f(x) \in \mathcal{L}_\infty$ *such that for every integer* $m \geq m(\varepsilon)$,

$$|A_m| < (2+\varepsilon)\sqrt{m \log m}.$$

We prove Theorem 1.1 and obtain the upper bound in Theorem 1.2 by a probabilistic method. In this context this approach goes back to Erdős. One should also mention the paper [24] of Salem and Zygmund. More recently, the behavior of various norms related to polynomials with coefficients restricted on average has been investigated in [3], [5], [8], [20], [21].

In fact, although most of the Littlewood polynomials and series satisfy the upper bounds given in Theorems 1.1 and 1.2, we cannot exhibit any explicit polynomial or series, because the proof is probabilistic. So it would be of interest to get bounds of the same order for some explicit polynomials and series.

2. Proof of Theorem 1.1. Consider a random polynomial

$$P(x) = \sum_{i=0}^{d-1} X_i x^i,$$

where $X_0, X_1, \ldots, X_{d-1}$ are d independent random variables satisfying

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

for each $i = 0, \ldots, d-1$. Then

$$P(x)^2 = \Big(\sum_{i=0}^{d-1} X_i x^i\Big)^2 = \sum_{m=0}^{2d-2} Z_m x^m,$$

where

$$Z_m := 2 \sum_{0 \leq k < m/2} X_k X_{m-k} + X_{m/2}^2 \tag{3}$$

for $0 \leq m \leq d-1$ and $X_{m/2} = 0$ for m odd. (Accordingly, $X_{m/2}^2 = 1$ for m even.) Similarly,

$$Z_m := 2 \sum_{m/2 < k \leq d-1} X_k X_{m-k} + X_{m/2}^2 \tag{4}$$

for $d-1 < m \leq 2d-2$.

Let $Y_1, \ldots, Y_s$ be s independent random variables such that $\mathbb{P}(Y_i \in [a_i, b_i]) = 1$ for $i = 1, \ldots, s$. Set $Y := Y_1 + \cdots + Y_s$ and write $\mathbb{E}(Y)$ for the expected value of Y. Then, by Hoeffding's inequality (see [14]),

$$\mathbb{P}(|Y - \mathbb{E}(Y)| \geq t) \leq 2 \exp\Big(-\frac{2t^2}{\sum_{i=1}^{s}(b_i - a_i)^2}\Big).$$

In particular, in case $a_i = -1$, $b_i = 1$ for $i = 1, \ldots, s$ and $\mathbb{E}(Y) = 0$, we have

$$\mathbb{P}(|Y| \geq t) \leq 2\exp\left(-\frac{t^2}{2s}\right). \tag{5}$$

Applying inequality (5) to the sum $Y := \sum_{0 \leq k < m/2} X_k X_{m-k}$ for $m > 0$ even, we see that

$$\mathbb{P}(|Z_m - 1| \geq 2t) \leq 2\exp(-t^2/m),$$

because, by (3), $Y = (Z_m - 1)/2$ contains $s = m/2$ terms $X_k X_{m-k}$ which are independent random variables, and $\mathbb{E}(Y) = 0$, since $\mathbb{E}(X_k X_{m-k}) = 0$ when $0 \leq k < m/2$. Similarly, for m odd we see that the sum $Y = Z_m/2$ contains $s = (m+1)/2$ terms $X_k X_{m-k}$, thus

$$\mathbb{P}(|Z_m| \geq 2t) \leq 2\exp(-t^2/(m+1)).$$

Combining both these inequalities we find that

$$\mathbb{P}(|Z_m| \geq 2t + 1) \leq 2\exp(-t^2/(m+1)) \tag{6}$$

for $1 \leq m \leq d-1$. By the same argument, from (4) and (5) we obtain

$$\mathbb{P}(|Z_{2d-2-m}| \geq 2t + 1) \leq 2\exp(-t^2/(m+1)) \tag{7}$$

for $1 \leq m < d-1$.

Now, select

$$t := (1 + \varepsilon/2)\sqrt{m \log m} - 1/2,$$

so that $2t+1 = (2+\varepsilon)\sqrt{m \log m}$. Then $t^2 > (1+\varepsilon)(m+1)\log m$ for each m in the range $c_1(\varepsilon) \leq m \leq d-1$, where $c_1(\varepsilon)$ is a positive integer depending on ε only. Hence, by (6) and (7), the inequalities

$$\begin{aligned} \mathbb{P}(|Z_m| \geq (2+\varepsilon)\sqrt{m \log m}) &< 2m^{-1-\varepsilon}, \\ \mathbb{P}(|Z_{2d-2-m}| \geq (2+\varepsilon)\sqrt{m \log m}) &< 2m^{-1-\varepsilon} \end{aligned} \tag{8}$$

hold for every m in the interval $c_1(\varepsilon) \leq m \leq d-1$. It follows that the reverse inequalities

$$|Z_m| < (2+\varepsilon)\sqrt{m \log m} \tag{9}$$

for m from $c_1(\varepsilon)$ to $d-1$ and

$$|Z_{2d-2-m}| < (2+\varepsilon)\sqrt{m \log m} \tag{10}$$

for m from $c_1(\varepsilon)$ to $d-2$ hold with probability at least

$$1 - 4\sum_{m=c_1(\varepsilon)}^{d-1} m^{-1-\varepsilon} > 1 - 4\sum_{m=c_1(\varepsilon)}^{\infty} m^{-1-\varepsilon} > 0.9 \tag{11}$$

if $c_1(\varepsilon)$ is large enough.

Hence, by increasing $c_1(\varepsilon)$ if necessary (to be sure that (11) is true), we conclude that there is a positive constant $M = M(\varepsilon)$ and some Littlewood polynomial P of degree $d-1$ such that $P(x)^2 = \sum_{m=0}^{2d-2} Z_m x^m$ and inequalities (9), (10) hold for its coefficients Z_m when $M \le m \le 2d-2-M$. (Here, Z_m are integers.) Obviously, as the moduli of the coefficients of P are at most 1, we have $|Z_m| \le m+1$ for $0 \le m < M$ and $|Z_{2d-2-m}| \le m+1$ for $0 \le m < M$. Therefore,

$$H(P^2) \le \max(M, (2+\varepsilon)\sqrt{(d-1)\log(d-1)}) < (2+\varepsilon)\sqrt{d \log d}$$

for each sufficiently large d. This proves the first inequality of the theorem.

For the same polynomial P, in view of (9), (10), we also have

$$L(P^2) < 2(1+2+\cdots+M) + 2(2+\varepsilon) \sum_{m=M}^{d-1} \sqrt{m \log m}.$$

Hence

$$L(P^2) < (8/3 + 2\varepsilon) d^{3/2} \sqrt{\log d}$$

for each sufficiently large d. Multiplying this inequality with that for $H(P^2)$ we derive that

$$L(P^2)H(P^2) < (8/3+2\varepsilon)(2+\varepsilon) d^2 \log d < (16/3 + 9\varepsilon) d^2 \log d$$

for $0 < \varepsilon < 1$ and d large enough, which completes the proof of Theorem 1.1 (with the initial ε replaced by $\varepsilon/9$).

3. Proof of Theorem 1.2. The proof of the second statement of Theorem 1.2 is essentially the same as that of the first statement of Theorem 1.1. Consider the random series

$$f(x) = \sum_{i=0}^{\infty} X_i x^i,$$

where $X_0, X_1, X_2, \dots$ are independent random variables satisfying

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

for each $i = 0, 1, 2, \dots$. Then

$$f(x)^2 = \Big(\sum_{i=0}^{\infty} X_i x^i\Big)^2 = \sum_{m=0}^{\infty} Z_m x^m,$$

where Z_m are given in (3). As above, we find that inequality (8) holds for each $m \ge c_1(\varepsilon)$. So the series

$$\sum_{m=c_1(\varepsilon)}^{\infty} \mathbb{P}(|Z_m| \ge (2+\varepsilon)\sqrt{m \log m}) \le 2 \sum_{m=c_1(\varepsilon)}^{\infty} m^{-1-\varepsilon}$$

are convergent. By the Borel–Cantelli lemma, there is an integer $m(\varepsilon)$ such that the probability of the event that $|Z_m| < (2+\varepsilon)\sqrt{m\log m}$ for $m = m(\varepsilon), m(\varepsilon)+1, \ldots$ is equal to 1. In particular, there exists a series $f(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{L}_\infty$ such that the coefficients of the series $f(x)^2 = \sum_{i=0}^{\infty} A_i x^i$ satisfy $|A_m| < (2+\varepsilon)\sqrt{m\log m}$ for every $m \ge m(\varepsilon)$.

To prove the first part of Theorem 1.2 we assume, for a contradiction, that there exist a series $f(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{U}_\infty$, a positive number ε, and an integer M such that $f(x)^2 = \sum_{i=0}^{\infty} A_i x^i$ and

$$|A_m| < (1-\varepsilon)\sqrt{m}$$

for each $m \ge M$. Note that $|A_m| = |\sum_{i=0}^{m} a_i a_{m-i}| \le \sum_{i=0}^{m} |a_i a_{m-i}| = m+1$ for every $m \ge 0$, because $|a_i| = 1$ for $i \ge 0$. By Parseval's formula, for every r satisfying $0 < r < 1$, so, in particular, for

$$r = \sqrt{1-\varepsilon/M}, \tag{12}$$

we obtain

$$\int_0^1 |f(e^{2\pi i\theta} r)|^2\, d\theta = \sum_{j=0}^{\infty} |a_j|^2 r^{2j} = \sum_{j=0}^{\infty} r^{2j} = \frac{1}{1-r^2}.$$

This does not exceed

$$\Big(\int_0^1 |f(e^{2\pi i\theta} r)^2|^2\, d\theta\Big)^{1/2} = \Big(\sum_{j=0}^{\infty} |A_j|^2 r^{2j}\Big)^{1/2}.$$

Estimating $|A_j|^2 \le (1-\varepsilon)^2(j+1)$ for $j \ge M$ and

$$|A_j|^2 \le j+1 < (1-\varepsilon)^2(j+1) + j + 1$$

for $j = 0, 1, \ldots, M-1$, we find that

$$\begin{aligned}\sum_{j=0}^{\infty} |A_j|^2 r^{2j} &< \frac{M(M+1)}{2} + \sum_{j=0}^{\infty} (1-\varepsilon)^2 (j+1) r^{2j} \\ &= \frac{M(M+1)}{2} + \frac{(1-\varepsilon)^2}{(1-r^2)^2}.\end{aligned}$$

Consequently,

$$\frac{1}{1-r^2} < \Big(\frac{M(M+1)}{2} + \frac{(1-\varepsilon)^2}{(1-r^2)^2}\Big)^{1/2}. \tag{13}$$

It is clear that $M \ge 1$, thus $\sqrt{M(M+1)/2} \le M$. Hence, applying the inequality $\sqrt{u^2+v^2} \le u+v$ for $u, v > 0$, we see that the right hand side of (13) does not exceed $M + (1-\varepsilon)/(1-r^2)$. This yields $\varepsilon/(1-r^2) < M$, contradicting (12).

References

[1] J. Beck, *Flat polynomials on the unit circle—note on a problem of Littlewood*, Bull. London Math. Soc. 23 (1991), 269–277.

[2] K. S. Berenhaut and F. Saidak, *A note on the maximal coefficients of squares of Newman polynomials*, J. Number Theory 125 (2007), 285–288.

[3] P. Borwein and K.-K. S. Choi, *The average norm of polynomials of fixed height*, Trans. Amer. Math. Soc. 359 (2007), 923–936.

[4] P. Borwein, K.-K. S. Choi and J. Jedwab, *Binary sequences with merit factor greater than* 6.34, IEEE Trans. Inform. Theory 50 (2004), 3234–3249.

[5] P. Borwein, K.-K. S. Choi and I. Mercer, *Expected norms of zero-one polynomials*, Canad. Math. Bull. 51 (2008), 497–507.

[6] P. Borwein, R. Ferguson and J. Knauer, *The merit factor problem*, in: Number Theory and Polynomials, J. McKee and C. Smyth (eds.), London Math. Soc. Lecture Note Ser. 352, Cambridge Univ. Press, Cambridge, 2008, 52–70.

[7] J. Brillhart and P. Morton, *A case study in mathematical research: The Golay–Rudin–Shapiro sequence*, Amer. Math. Monthly 103 (1996), 854–869.

[8] K.-K. S. Choi and M. J. Mossinghoff, *Average Mahler's measure and L_p norms of unimodular polynomials*, Pacific J. Math. 252 (2011), 31–50.

[9] J. Clunie, *The minimum modulus of a polynomial on the unit circle*, Quart. J. Math. Oxford Ser. (2) 10 (1959), 95–98.

[10] A. Dubickas, *Heights of powers of Newman and Littlewood polynomials*, Acta Arith. 128 (2007), 167–176.

[11] P. Erdős, *An inequality for the maximum of trigonometric polynomials*, Ann. Polon. Math. 12 (1962), 151–154.

[12] M. L. Fredman, B. Saffari et B. Smith, *Polynômes réciproques: conjecture d'Erdős en norme L^4, taille des autocorrélations et inexistence des codes de Barker*, C. R. Acad. Sci. Paris Sér. I Math. 308 (1989), 461–464.

[13] M. J. E. Golay, *The merit factor of Legendre sequences*, IEEE Trans. Inform. Theory 29 (1983), 934–936.

[14] W. Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. 58 (1963), 13–30.

[15] T. Høholdt and H. E. Jensen, *Determination of the merit factor of Legendre sequences*, IEEE Trans. Inform. Theory 34 (1988), 161–164.

[16] J. Jedwab, *A survey of the merit factor problem for binary sequences*, in: Sequences and Their Applications—SETA 2004, Lecture Notes in Comput. Sci. 3486, Springer, Berlin, 2005, 30–55.

[17] J.-P. Kahane, *Sur les polynômes à coefficients unimodulaires*, Bull. London Math. Soc. 12 (1980), 321–342.

[18] T. W. Körner, *On a polynomial of Byrnes*, Bull. London Math. Soc. 12 (1980), 219–224.

[19] J. E. Littlewood, *On polynomials $\sum^n \pm z^m, \sum^n e^{\alpha_m i} z^m, z = e^{\theta i}$*, J. London Math. Soc. 41 (1966), 367–376.

[20] T. Mansour, *Average norms of polynomials*, Adv. Appl. Math. 32 (2004), 698–708.

[21] I. D. Mercer, *Autocorrelations of random binary sequences*, Combin. Probab. Comput. 15 (2006), 663–671.

[22] B. Saffari et B. Smith, *Inexistence de polynômes ultra-plats de Kahane à coefficients ± 1. Preuve de la conjecture d'Erdős*, C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), 695–698.

[23] B. Saffari et B. Smith, *Sur une note récente relative aux polynômes à coefficients* ± 1 *et à la conjecture d'Erdős*, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 541–544.

[24] R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. 91 (1954), 245–301.

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