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Exceptional values of meromorphic functions and of their derivatives on annuli

by Yuxian Chen (Xinyu) and Zhaojun Wu (Xianning)

Abstract. This paper is devoted to exceptional values of meromorphic functions and of their derivatives on annuli. Some facts on exceptional values for meromorphic functions in the complex plane which were established by Singh, Gopalakrishna and Bhoosnurmath [Math. Ann. 191 (1971), 121–142, and Ann. Polon. Math. 35 (1977/78), 99–105] will be considered on annuli.

1. Introduction. In [KK1] and [KK2], we can find analogues of Jensen's formula and the First Fundamental Theorem, the lemma on logarithmic derivative and the Second Fundamental Theorem of Nevanlinna theory for meromorphic functions on annuli. After [KK1] and [KK2], Fernández [F] and Cao, Yi and Xu [CY]–[CYX] studied the value distribution and uniqueness of meromorphic functions on a doubly connected domain. In this paper, we shall extend the facts which were established by Singh, Gopalakrishna and Bhoosnurmath in [SG], [GB2] to meromorphic functions on annuli.

2. Nevanlinna theory on annuli. In this section, we recall the definitions, notation and results of [KK1] and [KK2] which will be used in this paper.

Let $f(z)$ be a meromorphic function on the annulus

$$A(R_0) := \{z : 1/R_0 < |z| < R_0\},$$

where $1 < R_0 \le +\infty$. Denote

$$m\left(R, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\theta}) - a|}\, d\theta,$$

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$$m(R,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|\,d\theta,$$

where $a\in\mathbb{C}$ and $1/R_0<R<R_0$. Let

$$m_0\Big(R,\frac{1}{f-a}\Big)=m\Big(R,\frac{1}{f-a}\Big)+m\Big(\frac{1}{R},\frac{1}{f-a}\Big),\quad 1<R<R_0,$$
$$m_0(R,f)=m(R,f)+m(1/R,f),\quad 1<R<R_0.$$

Put

$$N_1\Big(R,\frac{1}{f-a}\Big)=\int_{1/R}^{1}\frac{n_1\big(t,\frac{1}{f-a}\big)}{t}\,dt,\quad N_2\Big(R,\frac{1}{f-a}\Big)=\int_{1}^{R}\frac{n_2\big(t,\frac{1}{f-a}\big)}{t}\,dt,$$

where $1<R<R_0$, $n_1\big(t,\frac{1}{f-a}\big)$ is the counting function of poles of the function $\frac{1}{f-a}$ in $\{z:t<|z|\le 1\}$ and $n_2\big(t,\frac{1}{f-a}\big)$ is the counting function of poles of the function $\frac{1}{f-a}$ in $\{z:1<|z|\le t\}$. Denote also

$$N_1(R,f)=\int_{1/R}^{1}\frac{n_1(t,f)}{t}\,dt,\quad N_2(R,f)=\int_{1}^{R}\frac{n_2(t,f)}{t}\,dt,$$

where $1<R<R_0$, $n_1(t,f)$ is the counting function of poles of f in $\{z:t<|z|\le 1\}$, and $n_2(t,f)$ is the counting function of poles of f in $\{z:1<|z|\le t\}$. Let

$$N_0(R,a,f)=N_0\Big(R,\frac{1}{f-a}\Big)=N_1\Big(R,\frac{1}{f-a}\Big)+N_2\Big(R,\frac{1}{f-a}\Big),$$
$$N_0(R,\infty,f)=N_0(R,f)=N_1(R,f)+N_2(R,f).$$

Finally, we define the *Nevanlinna characteristic* of f on $A(R_0), 1<R_0\le+\infty$, by

$$T_0(R,f)=m_0(R,f)-2m(1,f)+N_0(R,f),\quad 1<R<R_0,$$

where $R_0\le+\infty$. Suppose that f,g are two meromorphic functions on $A(R_0), 1<R_0\le+\infty$. Then

$$\max\{T_0(R,f+g),T_0(R,fg),T_0(R,f/g)\}\le T_0(R,f)+T_0(R,g)+O(1).$$

DEFINITION 2.1. Let f be a nonconstant meromorphic function on $A(\infty)$. Then the *order* of $f(z)$ is defined by

$$\lambda(f)=\limsup_{R\to+\infty}\frac{\log T_0(R,f)}{\log R}.$$

THEOREM A (The First Fundamental Theorem, see [KK1, Theorem 2]). *Let f be a nonconstant meromorphic function on $A(R_0), 1<R_0\le+\infty$, and*

let $T_0(R,f)$ be its Nevanlinna characteristic. Then

$$T_0\left(R, \frac{1}{f-a}\right) = T_0(R,f) + O(1), \quad 1 < R < R_0,$$

for every fixed $a \in \mathbb{C}$.

Theorem B (Lemma on the logarithmic derivative, see [KK2, Theorem 1]). *Let f be a nonconstant meromorphic function on $A(R_0), 1 < R_0 \leq +\infty$, and let $\lambda \geq 0$. Then*

(i) *in the case $R_0 = +\infty$,*

$$m_0(R, f'/f) = O(\log(RT_0(R,f)))$$

for all $R \in (1,+\infty)$ except for a set $\triangle$ such that $\int_\triangle R^{\lambda-1}dR < +\infty$;

(ii) *in the case $R_0 < +\infty$,*

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for all $R \in (1, R_0)$ except for a set $\triangle'$ such that $\int_{\triangle'} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

Theorem C (The Second Fundamental Theorem, see [KK2, Theorem 2]). *Let f be a nonconstant meromorphic function on $A(R_0), 1 < R_0 \leq +\infty$. Let $a_1, \ldots, a_p$ be distinct finite complex numbers and $\lambda \geq 0$. Then*

$$m_0(R,f) + \sum_{\nu=1}^{p} m_0\left(R, \frac{1}{f-a_\nu}\right) \leq 2T_0(R,f) - N_0^{(1)}(R,f) + S(R,f),$$

where

$$N_0^{(1)}(R,f) = N_0(R, 1/f') + 2N_0(R,f) - N_0(R,f'),$$

and

(i) *in the case $R_0 = +\infty$,*

$$S(R,f) = O(\log(RT_0(R,f)))$$

for all $R \in (1,+\infty)$ except for a set $\triangle$ such that $\int_\triangle R^{\lambda-1}dR < +\infty$;

(ii) *in the case $R_0 < +\infty$,*

$$S(R,f) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for all $R \in (1, R_0)$ except for a set $\triangle'$ such that $\int_{\triangle'} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

3. Exceptional values of a meromorphic function. Let f be a meromorphic function of order ρ on $A(\infty)$, and let $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. We denote by $\overline{n}_1(t,f,a)$ the number of distinct zeros of $f - a$ in $\{z : t < |z| \leq 1\}$

(ignoring multiplicity) and by $\overline{n}_2(t,f,a)$ the number of distinct zeros of $f-a$ in $\{z : 1 < |z| \le t\}$ (ignoring multiplicity), and

$$\overline{N}_0(R,f,a) = \int_{1/R}^{1} \frac{\overline{n}_1(t,f,a)}{t}\,dt + \int_{1}^{R} \frac{\overline{n}_2(t,f,a)}{t}\,dt.$$

For any positive integer k, we denote by $\overline{n}_1^k(t,f,a)$ the number of distinct zeros of order $\le k$ of $f-a$ in $\{z : t < |z| \le 1\}$ (ignoring multiplicity) and by $\overline{n}_2^k(t,f,a)$ the number of distinct zeros of order $\le k$ of $f-a$ in $\{z : 1 < |z| \le t\}$ (ignoring multiplicity). We define

$$\overline{N}_0^k(R,f,a) = \int_{1/R}^{1} \frac{\overline{n}_1^k(t,f,a)}{t}\,dt + \int_{1}^{R} \frac{\overline{n}_2^k(t,f,a)}{t}\,dt.$$

We also denote by $n_1^k(t,f,a)$ the number of zeros of $f-a$ in $\{z : t < |z| \le 1\}$ and by $n_2^k(t,f,a)$ the number of zeros of $f-a$ in $\{z : 1 < |z| \le t\}$, where a zero of order $< k$ is counted according to its multiplicity and a zero of order $\ge k$ is counted exactly k times. We set

$$N_0^k(R,f,a) = \int_{1/R}^{1} \frac{n_1^k(t,f,a)}{t}\,dt + \int_{1}^{R} \frac{n_2^k(t,f,a)}{t}\,dt.$$

We further define

$$\overline{\rho}_k(a,f) = \limsup_{R\to\infty} \frac{\overline{N}_0^k(R,f,a)}{\log R},$$

$$\overline{\rho}(a,f) = \limsup_{R\to\infty} \frac{\overline{N}_0(R,f,a)}{\log R},$$

$$\rho(a,f) = \limsup_{R\to\infty} \frac{N_0(R,f,a)}{\log R}.$$

Definition 3.1. Let f be a meromorphic function of order ρ on $A(\infty)$, and let $a \in \mathbb{C}_\infty$. We say that a is

(i) an *evB* (exceptional value in the sense of Borel) *for f for distinct zeros of order $\le k$* if $\overline{\rho}_k(a,f) < \rho$,
(ii) an *evB* (exceptional value in the sense of Borel) *for f for distinct zeros* if $\overline{\rho}(a,f) < \rho$,
(iii) an *evB* (Borel exceptional value) *for f* if $\rho(a,f) < \rho$.

In [CYX], Cao, Yi and Xu proved

Theorem D. *Let f be a nonconstant meromorphic function on $A(R_0)$, $1 < R_0 \le +\infty$. Let $a^{[1]}, \dots, a^{[q]}$ be distinct complex numbers in $\mathbb{C}_\infty$ and k_j ($j = 1, \dots, q$) be positive integers or $+\infty$. Then*

$$\left(q-\sum_{j=1}^{q}\frac{1}{k_j+1}-2\right)T_0(R,f)\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_0^{k_j}(R,f,a^{[j]})+S(R,f). \tag{3.1}$$

Main Theorem 3.2. *Let f be a meromorphic function of order ρ on $A(\infty)$. Let $a^{[1]},\dots,a^{[q]}$ be distinct complex numbers in $\mathbb{C}_\infty$ and k_j $(j=1,\dots,q)$ be positive integers or $+\infty$. If $a^{[j]}$ is an evB for f for distinct zeros of order $\le k_j$ $(j=1,\dots,q)$, then*

$$\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)\le 2.$$

Proof. By hypothesis, we have

$$\overline{\rho}_{k_j}(a^{[j]},f)<\rho,\qquad j=1,\dots,q.$$

Then there is a positive number $\mu<\rho$ such that for $j=1,\dots,q$,

$$\overline{N}_0^{k_j}(r,f,a^{[j]})\le r^{\mu}. \tag{3.2}$$

Using (3.2) to (3.1), we have

$$\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T_0(R,f)\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}R^{\mu}+S(R,f). \tag{3.3}$$

Then, by Theorem C and (3.3),

$$\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T_0(R,f)\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}R^{\mu}+O(\log(RT_0(R,f))) \tag{3.4}$$

for all $R\in(1,+\infty)$ except a set $\triangle$ such that $\int_\triangle R^{\mu-1}\,dR<+\infty$. Suppose $I\subset\triangle$ is an interval. Let $R\in I$ and let R' is the right endpoint of I. Then

$$R'^{\mu}-R^{\mu}=\mu\int_R^{R'}r^{\mu-1}dr\le\mu\int_\triangle R^{\mu-1}dR=O(1).$$

From (3.4), we get

$$\begin{aligned}\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T_0(R,f)&\\ &\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}R^{\mu}+O(\log(RT_0(R,f)))\\ &\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}R'^{\mu}+O(\log(RT_0(R,f)))\\ &\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}R^{\mu}+O(\log R)\end{aligned} \tag{3.5}$$

for all R. Thus it follows from $0<\mu<\rho$ and (3.5) that

$$\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)\le 2. \blacksquare$$

Let $q=r+t+s$ and $k_j\equiv k$ $(j=1,\dots,r)$, $k_j\equiv l$ $(j=r+1,\dots,r+t)$ and $k_j\equiv m$ $(j=r+t+1,\dots,r+t+s)$ in Theorem 3.1. Then we get the following corollary.

Corollary 3.3. *Let f be a meromorphic function of order ρ on $A(\infty)$. If there exist distinct elements*

$$a^{[1]},\dots,a^{[r]};b^{[1]},\dots,b^{[t]};c^{[1]},\dots,c^{[s]}$$

in $\mathbb{C}_\infty$ such that $a^{[1]},\dots,a^{[r]}$ are evB for f for distinct zeros of order $\le k$, $b^{[1]},\dots,b^{[t]}$ are evB for f for distinct zeros of order $\le l$, $c^{[1]},\dots,c^{[s]}$ are evB for f for distinct zeros of order $\le m$, where k,l,m are positive integers, then

$$\frac{rk}{k+1}+\frac{tl}{l+1}+\frac{sm}{m+1}\le 2.$$

Remark. The result corresponding Main Theorem 3.2 and Corollary 3.3 in the whole complex plane was obtained by Yang [Y] and Gopalakrishna and Bhoosnurmath [GB1].

4. Exceptional values of meromorphic functions and of their derivatives. If f is a meromorphic function in the whole complex plane, Singh, Gopalakrishna and Bhoosnurmath [SG], [GB2] have proved

Theorem E (see [SG, Theorem 6]). *Let f be a meromorphic function on the plane of order ρ $(0<\rho<+\infty)$. Let $\overline{\rho}(\infty,f)<\rho$ and $\overline{\rho}(a,f)<\rho$ for some $a\in\mathbb{C}$. Then, for each integer $k\ge 1$, $\overline{\rho}_1(b,f^{(k)})=\rho$ for all $b\ne 0,\infty$.*

Theorem F (see [GB2, Theorem 1]). *Let f be a meromorphic function on the plane and k be a positive integer. Suppose that ∞ is an evB for f for distinct zeros of order $\le l$, where l is an integer ≥ 1. If there exist $a,b\in\mathbb{C}$ with $b\ne 0$ such that a is an evB for f for distinct zeros of order $\le p$, and b is an evB for $f^{(k)}$ for distinct zeros of order $\le q$, where p,q are positive integers, then*

$$\frac{q+1+k}{(q+1)(l+1)}+\frac{k+1}{p+1}+\frac{1}{q+1}\ge 1.$$

In this section, we extend Theorem F to meromorphic functions on annuli by applying the techniques of [GB2].

Main Theorem 4.1. *Let f be a meromorphic function of order ρ $(0<\rho<+\infty)$ on $A(\infty)$. Suppose that ∞ is an evB for f for distinct zeros of order $\le l$, where l is an integer ≥ 1. If there exist $a,b\in\mathbb{C}$ with $b\ne 0$ such* *that a is an evB for f for distinct zeros of order $\le p$ and b is an evB for $f^{(k)}$ for distinct zeros of order $\le q$, where p, q are positive integers, then*

$$\frac{q+1+k}{(q+1)(l+1)} + \frac{k+1}{p+1} + \frac{1}{q+1} \ge 1. \tag{4.1}$$

Proof. From [KL], we have

$$\begin{aligned} T_0(R, f') &= T_0(R, ff'/f) \le T_0(R,f) + T_0(R, f'/f) + O(1) \\ &= T_0(R,f) + m_0(R, f'/f) + N_0(R, f'/f) - 2m(1, f'/f) + O(1) \\ &\le T_0(R,f) + \overline{N}_0(R,f) + S(R,f) \\ &\le 2T_0(R,f) + S(R,f), \end{aligned} \tag{4.2}$$

and

$$m_0\left(R, \frac{f^{(k)}}{f-a}\right) = S(r,f) = S(R, f^{(k)}) \tag{4.3}$$

for any positive integer k and any $a \in \mathbb{C}$. Hence,

$$\begin{aligned} T_0\left(R, \frac{1}{f-a}\right) &= m_0\left(R, \frac{1}{f-a}\right) + N_0\left(R, \frac{1}{f-a}\right) \\ &\le N_0\left(R, \frac{1}{f-a}\right) + m_0\left(R, \frac{f^{(k)}}{f-a}\right) + m_0\left(R, \frac{1}{f^{(k)}}\right) \\ &\le N_0\left(R, \frac{1}{f-a}\right) + T_0\left(R, \frac{1}{f^{(k)}}\right) - N_0\left(R, \frac{1}{f^{(k)}}\right) + S(R,f). \end{aligned} \tag{4.4}$$

By Theorem A and (4.4), we have

$$T_0(R,f) \le N_0\left(R, \frac{1}{f-a}\right) + T_0\left(R, \frac{1}{f^{(k)}}\right) - N_0\left(R, \frac{1}{f^{(k)}}\right) + S(R,f). \tag{4.5}$$

Applying Theorems A and C to $f^{(k)}$ and invoking (4.3), we have

$$\begin{aligned} T_0(R, f^{(k)}) &\le N_0(R, f^{(k)}) + N_0\left(R, \frac{1}{f^{(k)}}\right) + N_0\left(R, \frac{1}{f^{(k)}-b}\right) \\ &\quad - \left(N_0\left(R, \frac{1}{f^{(k+1)}}\right) + 2N_0(R, f^{(k)}) - N_0(R, f^{(k+1)})\right) \\ &\quad + S(R, f^{(k)}) \\ &= N_0(R, f^{(k+1)}) - N_0(R, f^{(k)}) + N_0\left(R, \frac{1}{f^{(k)}}\right) \\ &\quad + N_0\left(R, \frac{1}{f^{(k)}-b}\right) - N_0\left(R, \frac{1}{f^{(k+1)}}\right) + S(R, f^{(k)}) \end{aligned} \tag{4.6}$$

$$= \overline{N}_0(r,f) + N_0\left(R, \frac{1}{f^{(k)}}\right) + N_0\left(R, \frac{1}{f^{k)} - b}\right)$$
$$- N_0\left(R, \frac{1}{f^{(k+1)}}\right) + S(R,f),$$

since

$$N_0(R, f^{(k+1)}) - N_0(R, f^{(k)}) = \overline{N}_0(r, f^{(k)}) = \overline{N}_0(r,f).$$

In [GB2], Gopalakrishna and Bhoosnurmath indicated that a zero of $f-a$ of order $j>k$ is a zero of $f^{(k+1)}$ of order $j-(k+1)$ and a zero of $f^{(k)}-b$ of order m is a zero of $f^{(k+1)}$ of order $m-1$. Moreover, zeros of $f-a$ of order $>k$ are zeros of $f^{(k)}$ and so are not zeros of $f^{(k)}-b$ since $b \neq 0$. Hence

$$\begin{aligned}(4.7)\qquad N_0\left(R, \frac{1}{f-a}\right) + N_0\left(R, \frac{1}{f^{(k)}-b}\right) - N_0\left(R, \frac{1}{f^{(k+1)}}\right)& \\ \le N_0^{k+1}\left(r, \frac{1}{f-a}\right) + \overline{N}_0\left(R, \frac{1}{f^{(k)}-b}\right).&\end{aligned}$$

Substituting (4.6), (4.7) to (4.5), we obtain

$$(4.8)\quad T_0(R,f) \le \overline{N}_0(r,f) + N_0^{k+1}\left(R, \frac{1}{f-a}\right) + \overline{N}_0\left(R, \frac{1}{f^{(k)}-b}\right) + S(R,f).$$

Since

$$\begin{aligned}(4.9)\quad N_0^{k+1}\left(R, \frac{1}{f-a}\right) &\le (k+1)\overline{N}_0\left(R, \frac{1}{f-a}\right) \\ &\le \frac{k+1}{p+1}\left\{p\overline{N}_0^p\left(R, \frac{1}{f-a}\right) + N_0\left(R, \frac{1}{f-a}\right)\right\} \\ &\le \frac{k+1}{p+1}\left\{p\overline{N}_0^p\left(R, \frac{1}{f-a}\right) + T_0(R,f)\right\} + O(1),\end{aligned}$$

and

(4.10)
$$\overline{N}_0\left(R, \frac{1}{f^{(k)}-b}\right) \le \frac{1}{q+1}\left\{q\overline{N}_0^q\left(R, \frac{1}{f^{(k)}-b}\right) + T_0(R, f^{(k)})\right\} + O(1),$$

and since

$$(4.11)\qquad \overline{N}_0(r,f) \le \frac{1}{l+1}\{l\overline{N}_0^l(R,f) + T_0(R,f)\},$$

and

$$\begin{aligned}(4.12)\quad T_0(R, f^{(k)}) &= m_0(r, f^{(k)}) + m_0(R, f^{(k)}) - m(1, f^{(k)}) \\ &\le m_0(R,f) + m_0\left(R, \frac{f^{(k)}}{f}\right) + N_0(R,f) + k\overline{N}_0(R,f) + O(1) \\ &= T_0(R,f) + k\overline{N}_0(R,f) + S(R,f),\end{aligned}$$

it follows that

$$\begin{aligned}
T_0(R,f) &= \overline{N}_0(r,f) + \frac{p(k+1)}{p+1}\overline{N}_0^p\Big(R,\frac{1}{f-a}\Big) + \frac{q}{q+1}\overline{N}_0^q\Big(R,\frac{1}{f^{(k)}-b}\Big)\\
&\quad + \frac{k+1}{p+1}T_0(R,f) + \frac{1}{q+1}T_0(R,f^{(k)}) + S(R,f)\\
&\le \Big(1+\frac{k}{q+1}\Big)\overline{N}_0(r,f) + \frac{p(k+1)}{p+1}\overline{N}_0^p\Big(R,\frac{1}{f-a}\Big)\\
&\quad + \frac{q}{q+1}\overline{N}_0^q\Big(R,\frac{1}{f^{(k)}-b}\Big)\\
&\quad + \Big(\frac{k+1}{p+1}+\frac{1}{q+1}\Big)T_0(R,f) + S(R,f)\\
&\le \frac{q+1+k}{(q+1)(l+1)}\overline{N}_0^l(R,f) + \frac{p(k+1)}{p+1}\overline{N}_0^p\Big(R,\frac{1}{f-a}\Big)\\
&\quad + \frac{q}{q+1}\overline{N}_0^q\Big(R,\frac{1}{f^{(k)}-b}\Big)\\
&\quad + \Big(\frac{k+1}{p+1}+\frac{1}{q+1}+\frac{q+1+k}{(q+1)(l+1)}\Big)T_0(R,f) + S(R,f).
\end{aligned}$$

Hence,

$$\begin{aligned}
(4.13)\quad &\Big\{1-\frac{k+1}{p+1}-\frac{1}{q+1}-\frac{q+1+k}{(q+1)(l+1)}\Big\}T_0(R,f)\\
&\qquad \le \frac{q+1+k}{(q+1)(l+1)}\overline{N}_0^l(R,f) + \frac{p(k+1)}{p+1}\overline{N}_0^p\Big(R,\frac{1}{f-a}\Big)\\
&\qquad\quad + \frac{q}{q+1}\overline{N}_0^q\Big(R,\frac{1}{f^{(k)}-b}\Big) + S(R,f).
\end{aligned}$$

Since ∞ is an evB for f for distinct zeros of order $\le l$, and a is an evB for f for distinct zeros of order $\le p$, and since b is an evB for $f^{(k)}$ for distinct zeros of order $\le q$, it follows that there is a positive number $\mu < \rho$ such that

$$(4.14)\quad \overline{N}_0^l(R,f) \le R^\mu, \quad \overline{N}_0^p\Big(R,\frac{1}{f-a}\Big) \le R^\mu, \quad \overline{N}_0^q\Big(R,\frac{1}{f^{(k)}-b}\Big) \le R^\mu.$$

Substituting (4.14) to (4.13) and invoking Theorem B, we have

(4.15)

$$\Big\{1-\frac{k+1}{p+1}-\frac{1}{q+1}-\frac{q+1+k}{(q+1)(l+1)}\Big\}T_0(R,f) \le O(R^\mu)+O(\log(RT_0(R,f)))$$

for all $R \in (1,+\infty)$ except for a set $\triangle$ such that $\int_\triangle R^{\mu-1}\,dR < +\infty$. Suppose $I \subset \triangle$ is an interval. Let $R \in I$ and let R' be the right endpoint of I.

Then

$$R'^{\mu} - R^{\mu} = \mu \int_{R}^{R'} r^{\mu-1}\,dr \le \mu \int_{\triangle} R^{\mu-1}\,dR = O(1).$$

From (4.15), we can get

$$\left\{1 - \frac{k+1}{p+1} - \frac{1}{q+1} - \frac{q+1+k}{(q+1)(l+1)}\right\}T_0(R,f) = O(R^{\mu}) + O(\log(RT_0(R,f))) = O(R'^{\mu}) + O(\log(RT_0(R,f))) = O(R^{\mu}) + O(\log R) \tag{4.16}$$

for all R. It follows from $0 < \mu < \rho$ and (4.16) that (4.1) holds. ∎

Remark. If ∞, a are evB for f for distinct zeros, i.e. letting l, p tend to infinity in (4.2), we can get $\frac{1}{q+1} \ge 1$. This means that for each integer $k, q \ge 1$, $\overline{\rho}_q(b, f^{(k)}) = \rho$ for all $b \ne 0, \ne \infty$. Hence, we get

Corollary 4.2. *Let f be a meromorphic function of order ρ $(0 < \rho < +\infty)$ on $A(\infty)$. Let $\overline{\rho}(\infty, f) < \rho$ and $\overline{\rho}(a, f) < \rho$ for some $a \in \mathbb{C}$. Then, for each integer $k \ge 1$, $\overline{\rho}_1(b, f^{(k)}) = \rho$ for all $b \ne 0, \ne \infty$. Consequently, the order of $f^{(k)}$ is ρ in this case.*

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Yuxian Chen
School of Mathematics
and Computer Science
Xinyu University
Xinyu, 338004, P.R. China
E-mail: xygzcyx@126.com

Zhaojun Wu (corresponding author)
School of Mathematics and Statistics
Hubei University of Science and Technology
Xianning, 437100, P.R. China
E-mail: wuzj52@hotmail.com

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