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A global existence result for the compressible Navier–Stokes–Poisson equations in three and higher dimensions

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Abstract. The paper is dedicated to the global well-posedness of the barotropic compressible Navier–Stokes–Poisson system in the whole space $\mathbb{R}^N$ with $N \geq 3$. The global existence and uniqueness of the strong solution is shown in the framework of hybrid Besov spaces. The initial velocity has the same critical regularity index as for the incompressible homogeneous Navier–Stokes equations. The proof relies on a uniform estimate for a mixed hyperbolic/parabolic linear system with a convection term.

1. Introduction. The compressible Navier–Stokes–Poisson system takes the form of the compressible Navier–Stokes equations forced by the electric field which is governed by the self-consistent Poisson equation. In this paper, we are concerned with the Cauchy problem for the isentropic compressible Navier–Stokes–Poisson equation with external force:

$$(1.1)\quad \begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \nabla P(\rho) = \rho\nabla\Phi + \rho f, \\ \Delta\Phi = \rho - \overline{\rho}, \\ \lim\limits_{|x|\to\infty} \Phi(x,t) = 0, \\ (\rho, u)_{t=0} = (\rho_0, u_0), \end{cases}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^N, N \geq 3$. The variables are the density $\rho > 0$, the velocity u, and the electrostatic potential Φ. Furthermore, $P = P(\rho)$ is the pressure function. The viscosity coefficients satisfy $\mu > 0$ and $2\mu + N\lambda \leq 0$. Finally, $\bar{\rho} > 0$ is the background doping profile [MRS], which in this paper is taken to be a positive constant for simplicity.

We now review some previous work on related topics. Global regular small solutions to compressible Navier–Stokes were first obtained by Matsumura–

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Nishida [MN1, MN2], and qualitative properties of the solutions were studied by Valli–Zajączkowski [VZ]. The local and global existence of multidimensional renormalized solution to NSP was obtained in [Do, ZT]. The quasi-neutral limits and related combined asymptotical limits were proved in [DJL, DM, WJ]. The long time behavior of a global solution was investigated for the compressible NSP system in [LMZ, ZLZ], where the optimal decay rate of the global classical solution was obtained. The compressible Navier–Stokes–Poisson system is related to the dynamics of self-gravitating polytropic gaseous stars. There is also important recent progress on the existence of local and global weak solutions or renormalized solutions: see [De, DZ, Du, DFPS, ST] and references therein. Some results about free boundary problems were obtained in [Zaj1, Zaj2]. Furthermore, free boundary problems for the general N-S equations were considered in [Zad]. In particular, the global existence and uniqueness of strong solution in Besov type spaces was shown in [HL]:

THEOREM 1.1. *Let* $N \geq 3$, $P(\rho) = \rho^2/2$, $\mu > 0$ *and* $2\mu + N\lambda \geq 0$. *Assume* $\rho_0 - \bar{\rho} \in \widetilde{B}_{2,1}^{N/2-5/2,N/2}$ *and* $u_0 \in \widetilde{B}_{2,1}^{N/2-3/2,N/2-1}$. *Then there exist two positive constants* α *small enough and* M *such that if*

$$\|\rho_0 - \bar{\rho}\|_{\widetilde{B}_{2,1}^{N/2-5/2,N/2}} + \|u_0\|_{\widetilde{B}_{2,1}^{N/2-3/2,N/2-1}} \leq \alpha,$$

then (1.1) *has a unique global solution* (ρ, u, Φ) *such that* $(\rho - \bar{\rho}, u, \Phi)$ *belongs to*

$$\begin{aligned} E := C(\mathbb{R}^+; \widetilde{B}_{2,1}^{N/2-5/2,N/2} \times (\widetilde{B}_{2,1}^{N/2-3/2,N/2-1})^N \times \widetilde{B}_{2,1}^{N/2-1/2,N/2+2}) \\ \cap L^1(\mathbb{R}^+; \widetilde{B}_{2,1}^{N/2-1/2,N/2} \times (\widetilde{B}_{2,1}^{N/2+1/2,N/2+1})^N \times \widetilde{B}_{2,1}^{N/2+3/2,N/2+2}) \end{aligned}$$

and satisfies

$$\|(\rho - \bar{\rho}, u, \Phi)\|_E \leq M(\|\rho_0 - \bar{\rho}\|_{\widetilde{B}_{2,1}^{N/2-5/2,N/2}} + \|u_0\|_{\widetilde{B}_{2,1}^{N/2-3/2,N/2-1}}),$$

where M *is independent of the initial data, and the hybrid space* $\widetilde{B}_{2,1}^{s_1,s_2}$ *is* $\dot{B}_{2,1}^{s_1} \cap \dot{B}_{2,1}^{s_2}$ *for* $s_1 \leq s_2$.

In this paper, the initial data are supposed to be close to a stable equilibrium with constant density. Using uniform estimates for the linearized system with a convection term in the hybrid space, we prove the global existence of solution by compactness arguments as in [C, Da1, HL, P]. Define the following function space:

$$\begin{aligned} X^s := C(\mathbb{R}^+; \widetilde{B}_{2,1}^{s-2,s} \times (\dot{B}_{2,1}^{s-1})^N \times \widetilde{B}_{2,1}^{s,s+2}) \\ \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{s} \times (\dot{B}_{2,1}^{s+1})^N \times \dot{B}_{2,1}^{s+2}). \end{aligned}$$

Let us now state our main result.

MAIN THEOREM 1.2. *For* $N \geq 3$, *let* $\bar{\rho} > 0$ *be such that* $P'(\bar{\rho}) > 0$. *Then there are positive constants* α *and* M *such that for all* (ρ_0, u_0, f) *with* $\rho_0 - \bar{\rho} \in \widetilde{B}_{2,1}^{N/2-2,N/2}$, $u_0 \in \dot{B}_{2,1}^{N/2-1}$, $f \in L^1(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-1})$ *and*

$$\|\rho_0 - \bar{\rho}\|_{\widetilde{B}_{2,1}^{N/2-2,N/2}} + \|u_0\|_{\dot{B}_{2,1}^{N/2-1}} + \|f\|_{L^1(\mathbb{R}^+;\dot{B}_{2,1}^{N/2-1})} \leq \alpha,$$

(1.1) *has a unique global solution* (ρ, u, Φ) *such that* $(\rho - \bar{\rho}, u, \Phi) \in X^{N/2}$ *and satisfies*

$$\|(\rho-\bar{\rho}, u, \Phi)\|_{X^{N/2}} \leq M(\|\rho_0-\bar{\rho}\|_{\widetilde{B}_{2,1}^{N/2-2,N/2}} + \|u_0\|_{\dot{B}_{2,1}^{N/2-1}} + \|f\|_{L^1(\mathbb{R}^+;\dot{B}_{2,1}^{N/2-1})}).$$

If moreover $\rho_0 - \bar{\rho} \in \dot{B}_{2,1}^{s}$, $u_0 \in \dot{B}_{2,1}^{s-1}$ *and* $f \in L^1(\dot{B}_{2,1}^{s-1})$ *for* $s \in (N/2, N/2+1]$, *then* (1.1) *has a global solution* $(\rho - \bar{\rho}, u, \Phi) \in X^{N/2} \cap X^s$ *with*

$$\|(\rho - \bar{\rho}, u, \Phi)\|_{X^\sigma} \leq M(\|\rho_0 - \bar{\rho}\|_{\widetilde{B}_{2,1}^{\sigma-2,\sigma}} + \|u_0\|_{\dot{B}_{2,1}^{\sigma-1}} + \|f\|_{L^1(\dot{B}_{2,1}^{\sigma-1})})$$

for $\sigma = s, N/2$.

The paper is structured as follows. In Section 2, we recall some Littlewood–Paley theory for homogeneous Besov spaces and define some related function spaces. Sections 3–4 are dedicated to reformulation of the system and proving a priori estimates for a linearized system with convection terms. In Section 5, we prove the global existence and uniqueness of solution.

2. Littlewood–Paley theory and function spaces. Let us introduce the Littlewood–Paley decomposition. Choose a radial function $\varphi \in S(\mathbb{R}^N)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{if } \xi \neq 0.$$

The frequency localization operator Δ_q and S_q are defined by

$$\Delta_q f = \varphi(2^{-q}D)f, \quad S_j f = \sum_{a\leq j-1} \Delta_q f \quad \text{for } q \in \mathbb{Z}.$$

We denote the dual space of $\mathcal{Z}(\mathbb{R}^N) = \{f \in S(\mathbb{R}^N) : D^\alpha \hat{f}(0) = 0, \forall \alpha \in \mathbb{N}^d\}$ by $\mathcal{Z}'(\mathbb{R}^N)$; it can also be identified with the quotient space $S'(\mathbb{R}^N)/\mathcal{P}$ where $\mathcal{P}$ is the space of polynomials. The formal equality

$$f = \sum_{q\in\mathbb{Z}} \Delta_q f$$

holds true for $f \in \mathcal{Z}'(\mathbb{R}^N)$ and the right hand side is called the *homogeneous Littlewood–Paley decomposition.*

DEFINITION 2.1. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The *homogeneous Besov space* $\dot{B}_{p,r}^s$ is defined by

$$\dot{B}_{p,r}^s = \{f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\dot{B}_{p,r}^s} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} := \big\|2^{qs}\|\Delta_q f\|_{L^p}\big\|_{l^r}.$$

Let us now state some estimates for the product in $\dot{B}^s_{2,1}$.

LEMMA 2.2 (see [Da2]). *Let $s > 0$, $u \in L^\infty \cap \dot{B}^s_{2,1}$ and $v \in L^\infty \cap \dot{B}^s_{2,1}$. Then $uv \in L^\infty \cap \dot{B}^s_{2,1}$ and*

$$\|uv\|^s_{\dot{B}_{2,1}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{2,1}} + \|v\|_{L^\infty}\|u\|_{\dot{B}^s_{2,1}}.$$

Let $s_1, s_2 \le N/2$ with $s_1 + s_2 > 0$, $u \in \dot{B}^{s_1}_{2,1}$ and $v \in \dot{B}^{s_2}_{2,1}$. Then $uv \in \dot{B}^{s_1+s_2-N/2}_{2,1}$ and

$$\|uv\|_{\dot{B}^{s_1+s_2-N/2}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}}\|v\|_{\dot{B}^{s_2}_{2,1}}.$$

We refer to [Da2] for the proof of the following estimates for the composition of functions.

LEMMA 2.3. *Let $s > 0$ and $u \in L^\infty \cap \dot{B}^s_{2,1}$.*

(i) *Let $F \in W^{[s]+2}_{\mathrm{loc}}(\mathbb{R}^N)$ with $F(0) = 0$. Then $F(u) \in \dot{B}^s_{2,1}$. Moreover, there exists a one-variable function C_0, depending only on s and F, such that*

$$\|F(u)\|_{\dot{B}^s_{2,1}} \le C_0(\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{2,1}}.$$

(ii) *If $u, v \in \dot{B}^{N/2}_{2,1}$ with $v - u \in \dot{B}^s_{2,1}$ for $s \in (-N/2, N/2]$ and $G \in W^{[N/2]+3,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ with $G'(0) = 0$, then $G(v) - G(u) \in \dot{B}^s_{2,1}$ and there exists a two-variable function C, depending only on s, N, and G, such that*

$$\|G(v)-G(u)\|_{\dot{B}^s_{2,1}} \le C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{\dot{B}^{N/2}_{2,1}}+\|v\|_{\dot{B}^{N/2}_{2,1}})\|v-u\|_{\dot{B}^s_{2,1}}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [Da1]. We recall the definition and main properties of these spaces.

DEFINITION 2.4. Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{\widetilde{B}^{s,t}_{2,1}} = \sum_{q\le 0} 2^{qs}\|\Delta_q f\|_{L^2} + \sum_{q>0} 2^{qs}\|\Delta_q f\|_{L^2}.$$

Let $m = -[N/2 + 1 - s]$ and define

$$\widetilde{B}^{s,t}_{2,1}(\mathbb{R}^N) = \begin{cases} \{f \in S'(\mathbb{R}^N) : \|f\|_{\widetilde{B}^{s,t}_{2,1}} < \infty\} & \text{if } m < 0,\\ \{f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\widetilde{B}^{s,t}_{2,1}} < \infty\} & \text{if } m \ge 0.\end{cases}$$

NOTATION. We set

$$u_{\mathrm{LF}} := \sum_{q\le 0} \Delta_q u \quad \text{and} \quad u_{\mathrm{HF}} := \sum_{q>0} \Delta_q u.$$

LEMMA 2.5.

(i) $\widetilde{B}^{s,s}_{2,1} = \dot{B}^{s}_{2,1}$.

(ii) *If* $s \le t$ *then* $\widetilde{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} \cap \dot{B}^{t}_{2,1}$. *Otherwise,* $\widetilde{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} + \dot{B}^{t}_{2,1}$.

(iii) $\widetilde{B}^{0,s}_{2,1}$ *coincides with the usual inhomogeneous Besov space* $B^{s}_{2,1}$.

(iv) *If* $s_1 \le s_2$ *and* $t_1 \ge t_2$ *then* $\widetilde{B}^{s_1,t_1}_{2,1} \hookrightarrow \widetilde{B}^{s_2,t_2}_{2,1}$.

Let us state some estimates for the product in hybrid Besov spaces.

LEMMA 2.6. *Let* $s_1, s_2 > 0$, *and* $f, g \in L^\infty \cap \widetilde{B}^{s_1,s_2}_{2,1}$. *Then* $fg \in \widetilde{B}^{s_1,s_2}_{2,1}$ *and*

$$\|fg\|_{\widetilde{B}^{s_1,s_2}_{2,1}} \lesssim \|f\|_{L^\infty}\|g\|_{\widetilde{B}^{s_1,s_2}_{2,1}} + \|f\|_{\widetilde{B}^{s_1,s_2}_{2,1}}\|g\|_{L^\infty}.$$

Let $s_1, s_2, t_1, t_2 \le N/2$ *with* $\min\{s_1 + s_2, t_2 + t_2\} > 0$, *and let* $f \in \widetilde{B}^{s_1,t_1}_{2,1}$ *and* $g \in \widetilde{B}^{s_2,t_2}_{2,1}$. *Then*

$$\|fg\|_{\widetilde{B}^{s_1+s_2-N/2,t_1+t_2-N/2}_{2,1}} \lesssim \|f\|_{\widetilde{B}^{s_1,t_1}_{2,1}}\|g\|_{\widetilde{B}^{s_2,t_2}_{2,1}}.$$

For $\alpha, \beta \in \mathbb{R}$, let us define the following function on $\mathbb{Z}$:

$$\phi^{\alpha,\beta}(r) = \begin{cases} \alpha & \text{if } r \le 0, \\ \beta & \text{if } r \ge 1. \end{cases}$$

Then we have the following lemma:

LEMMA 2.7. *Let* F *be a smooth homogeneous function of degree* m. *Suppose that* $-N/2 < s_1, t_1, s_2, t_2 \le 1 + N/2$. *Then*

$$|(F(D)\Delta_q(u\cdot\nabla a), F(D)\Delta_q a)| \lesssim c_q 2^{-q\phi^{s_1,s_2}(q-m)}\|u\|_{\dot{B}^{N/2+1}_{2,1}}\|a\|_{\widetilde{B}^{s_1,s_2}_{2,1}}\|F(D)\Delta_q a\|_{L^2},$$

$$|(F(D)\Delta_q(u\cdot\nabla a), \Delta_q b) + (\Delta_q(u\cdot\nabla b), F(D)\Delta_q a)| \lesssim c_q\|u\|_{\dot{B}^{N/2+1}_{2,1}} \times (2^{-q(\phi^{t_1,t_2}(q-m))}\|F(D)\Delta_q a\|_{L^2}\|b\|_{\widetilde{B}^{t_1,t_2}_{2,1}} + 2^{-q\phi^{s_1,s_2}(q-m)}\|a\|_{\widetilde{B}^{s_1,s_2}_{2,1}}\|\Delta_q b\|_{L^2}),$$

where $(\cdot,\cdot)$ *denotes the* L^2 *inner product, the operator* $F(D)$ *is defined by* $F(D)f := \mathscr{F}^{-1}F(\xi)\mathscr{F}f$, *and* $\sum_{q\in\mathbb{Z}} c_q \le 1$.

3. Reformulation of the original system. Without loss of generality, we set $\bar{\rho} = 1, P'(1) = 1$ and $2\mu + \lambda = 1$. Let $h = \rho - 1$. Then (1.1) can be rewritten as

$$\begin{cases} \partial_t h + u\cdot\nabla h + \operatorname{div} u = -h \operatorname{div} u, \\ \partial_t u + u\cdot\nabla u - \mathcal{A}u + \nabla h - \nabla\Phi = -\dfrac{h}{1+h}\mathcal{A}u - K(h)\nabla h + f, \\ \Delta\Phi = h, \qquad \lim\limits_{|x|\to\infty} \Phi(x) = 0, \\ (h,u)_{t=0} = (h_0, u_0), \end{cases} \tag{3.1}$$

where we denote $\mathcal{A} = \mu\Delta + (\lambda+\mu)\nabla\operatorname{div}$ and $K(h) = P'(1+h)/(1+h) - 1$.

For $s \in \mathbb{R}$, we denote $\Lambda^s z := \mathscr{F}^{-1}(|\xi|^s \hat{z})$. Let $\xi = \Lambda^{-1} \operatorname{div} u$ be the "compressible part" of the velocity and $\eta = \Lambda^{-1} \operatorname{curl} u$ (with $(\operatorname{curl} z)_i^j = \partial_j z^i - \partial_i z^j$) be the "incompressible part". System (3.1) can be rewritten as

$$
\begin{cases}
\partial_t h + \Lambda\xi + u\cdot\nabla h = h \operatorname{div} u,\\
\partial_t \xi - \Delta\xi - \Lambda h - \Lambda^{-1} h = -\Lambda^{-1}\operatorname{div}\big(u\cdot\nabla u + \frac{h}{1+h}\mathcal{A}u + K(h)\nabla h\big)\\
\qquad\qquad\qquad\qquad\qquad\qquad + \Lambda^{-1}\operatorname{div} f,\\
\partial_t \eta - \mu\Delta\eta = -\Lambda^{-1}\operatorname{curl}\big(u\cdot\nabla u + \frac{h}{1+h}\mathcal{A}u\big) + \Lambda^{-1}\operatorname{curl} f,\\
u = -\Lambda^{-1}\nabla\xi - \Lambda^{-1}\operatorname{div}\eta.
\end{cases} \tag{3.2}
$$

Let us observe that the third equation is, up to nonlinear terms, just the heat equation on η. We therefore expect to require the following lemmas to get appropriate estimates for the incompressible part of the velocity.

LEMMA 3.1. *Let $s \in \mathbb{R}$, $r \in [1,\infty]$, and suppose u solves*

$$
\begin{cases}
\partial_t u - \mu\Delta u = f,\\
u_{t=0} = u_0.
\end{cases}
$$

Then there exists $C > 0$, depending only on N, μ and r, such that, for all $0 < T \le \infty$,

$$\|u\|_{L^r_T(\dot{B}^{s+2/r}_{2,1})} \le C(\|u_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_T(\dot{B}^s_{2,1})}).$$

Moreover, $u \in C([0,T]; \dot{B}^s_{2,1})$.

Now, there is a linear coupling between the first two equations, which leads us to prove estimates for the following linear system:

$$
\begin{cases}
\partial_t h + u\cdot\nabla h + \Lambda\xi = F,\\
\partial_t \xi + u\cdot\nabla\xi - \Delta\xi - \Lambda h - \Lambda^{-1} h = G.
\end{cases} \tag{3.3}
$$

PROPOSITION 3.2. *Let (h,ξ) be a solution of (3.3) on $[0,T]$, assume that $1-N/2 < s \le 1+N/2$ and set $V(t) = \int_0^t \|u(\tau)\|_{\dot{B}^{N/2+1}_{2,1}}\,d\tau$. Then the following estimate holds on $[0,T]$:*

$$
\begin{aligned}
\|h(t)\|_{\widetilde{B}^{s-2,s}_{2,1}} + \|\xi(t)\|_{\dot{B}^{s-1}_{2,1}} + \int_0^t (\|h(\tau)\|_{\dot{B}^s_{2,1}} + \|\xi(\tau)\|_{\dot{B}^{s+1}_{2,1}})\,d\tau \\
\le Ce^{CV(t)}\Big(\|h_0\|_{\widetilde{B}^{s-2,s}_{2,1}} + \|\xi_0\|_{\dot{B}^{s-1}_{2,1}} \\
+ \int_0^t e^{-CV(\tau)}(\|F(\tau)\|_{\widetilde{B}^{s-2,s}_{2,1}} + \|G(\tau)\|_{\dot{B}^{s-1}_{2,1}})\,d\tau\Big),
\end{aligned}
$$

where C depends only on N and s.

Let us now introduce the function spaces which appear in the global existence theorem.

DEFINITION 3.3. For $T>0$ and $s\in\mathbb{R}$, we define E_T^s to be

$$(L^1(0,T;\dot{B}^s_{2,1})\cap C([0,T];\widetilde{B}^{s-2,s}_{2,1}))\times(L^1(0,T;\dot{B}^{s+1}_{2,1})\cap C([0,T];\dot{B}^{s-1}_{2,1}))^N$$

and set

$$\|(h,u)\|_{E_T^s}=\|h\|_{L^\infty_T(\widetilde{B}^{s-2,s}_{2,1})}+\|u\|_{L^\infty_T(\dot{B}^{s-1}_{2,1})}+\|h\|_{L^1_T(\dot{B}^s_{2,1})}+\|u\|_{L^1_T(\dot{B}^{s+1}_{2,1})};$$

we also use the notation E^s if $T=\infty$, changing $[0,T]$ into $[0,\infty)$ in the definition above.

4. Estimates for the linear model. This section is devoted to the proof of Proposition 3.2. Let (h,ξ) be a solution of (3.3) and $K>0$. Define

$$\widetilde{h}=e^{-KV(t)}h,\qquad \widetilde{\xi}=e^{-KV(t)}\xi,\qquad \widetilde{F}=e^{-KV(t)}F,\qquad \widetilde{G}=e^{-KV(t)}G.$$

Applying the operator Δ_q to (3.3), we easily infer that $(\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})$ satisfies

$$\begin{cases}\partial_t\Delta_q\widetilde{h}+\Delta_q(u\cdot\nabla\widetilde{h})+\Lambda\Delta_q\widetilde{\xi}=\Delta_q\widetilde{F}-KV'(t)\Delta_q\widetilde{h},\\ \partial_t\Delta_q\widetilde{\xi}+\Delta_q(u\cdot\nabla\widetilde{\xi})-\Delta\Delta_q\widetilde{\xi}-\Lambda\Delta_q\widetilde{h}-\Lambda^{-1}\Delta_q\widetilde{h}=\Delta_q\widetilde{G}-KV'(t)\Delta_q\widetilde{\xi}.\end{cases}\tag{4.1}$$

STEP 1: *Low frequencies* ($q\le 0$). Taking the L^2 scalar product of the first equation of (4.1) with $\Delta_q\widetilde{h}$ and of the second equation with $\Delta_q\widetilde{\xi}$, we obtain the identities

$$\begin{cases}\frac{1}{2}\frac{d}{dt}\|\Delta_q\widetilde{h}\|^2_{L^2}+(\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{h})+(\Lambda\Delta_q\widetilde{\xi},\Delta_q\widetilde{h})\\ \qquad\qquad=(\Delta_q\widetilde{F},\Delta_q\widetilde{h})-KV'(t)\|\Delta_q\widetilde{h}\|^2_{L^2},\\ \frac{1}{2}\frac{d}{dt}\|\Delta_q\widetilde{\xi}\|^2_{L^2}+\|\Lambda\Delta_q\widetilde{\xi}\|^2_{L^2}+(\Delta_q(u\cdot\nabla\widetilde{\xi}),\Delta_q\widetilde{\xi})-(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})\\ \qquad -(\Lambda^{-1}\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})=(\Delta_q\widetilde{G},\Delta_q\widetilde{\xi})-KV'(t)\|\Delta_q\widetilde{\xi}\|^2_{L^2}.\end{cases}\tag{4.2}$$

We can also obtain an identity involving $(\Lambda\Delta_q\tilde{\xi},\Delta_q\tilde{\xi})$. To achieve it, we take the L^2 scalar product of the first equation of (4.1) with $\Lambda^{-2}\Delta_q\widetilde{h}$ and $\Lambda\Delta_q\widetilde{\xi}$ and of the second equation with $\Lambda\Delta_q\widetilde{h}$ and then sum the last two resulting equalities, which yields, with the Plancherel theorem,

$$\begin{cases}\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\Delta_q\widetilde{h}\|^2_{L^2}+(\Lambda^{-1}\Delta_q(u\cdot\nabla\widetilde{h}),\Lambda^{-1}\Delta_q\widetilde{h})+(\Delta_q\widetilde{\xi},\Lambda^{-1}\Delta_q\widetilde{h})\\ \qquad\qquad=(\Lambda^{-1}\Delta_q\widetilde{F},\Lambda^{-1}\Delta_q\widetilde{h})-KV'(t)\|\Lambda^{-1}\Delta_q\widetilde{h}\|^2_{L^2},\\ \frac{d}{dt}(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})+\|\Lambda\Delta_q\widetilde{\xi}\|^2_{L^2}+(\Lambda^2\Delta_q\widetilde{\xi},\Lambda\Delta_q\widetilde{h})-\|\Lambda\Delta_q\widetilde{h}\|^2_{L^2}-\|\Delta_q\widetilde{h}\|^2_{L^2}\\ \qquad\qquad=(\Lambda\Delta_q\widetilde{F},\Delta_q\widetilde{\xi})+(\Lambda\Delta_q\widetilde{G},\Delta_q\widetilde{h})-2KV'(t)(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})\\ \qquad\qquad\quad-(\Lambda\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{\xi})-(\Lambda\Delta_q\widetilde{h},\Delta_q(u\cdot\nabla\widetilde{\xi})).\end{cases}\tag{4.3}$$

A linear combination of (4.2) and (4.3) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&[\|\Delta_q\widetilde{h}\|_{L^2}^2+\|\Lambda^{-1}\Delta_q\widetilde{h}\|_{L^2}^2+\|\Delta_q\widetilde{\xi}\|_{L^2}^2-2K_1(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})]+K_1\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2\\&+K_1\|\Delta_q\widetilde{h}\|_{L^2}^2+[1-K_1]\|\Lambda\Delta_q\tilde{\xi}\|_{L^2}^2-K_1(\Lambda^2\Delta_q\widetilde{h},\Lambda\Delta_q\widetilde{\xi})\\&=(\Delta_q\widetilde{F},\Delta_q\widetilde{h})-KV'(t)\|\Delta_q\widetilde{h}\|_{L^2}^2-(\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{h})+(\Delta_q\widetilde{G},\Delta_q\widetilde{\xi})\\&\quad-KV'(t)\|\Delta_q\tilde{\xi}\|_{L^2}^2-(\Delta_q(u\cdot\nabla\widetilde{\xi}),\Delta_q\widetilde{\xi})+(\Lambda^{-1}\Delta_q\widetilde{F},\Lambda^{-1}\Delta_q\widetilde{h})\\&\quad-KV'(t)\|\Lambda^{-1}\Delta_q\widetilde{h}\|_{L^2}^2-(\Lambda^{-1}\Delta_q(u\cdot\nabla\widetilde{h}),\Lambda^{-1}\Delta_q\widetilde{h})\\&\quad-K_1(\Lambda\Delta_q\widetilde{F},\Delta_q\widetilde{\xi})+(\Lambda\Delta_q\widetilde{G},\Delta_q\widetilde{h})-2KV'(t)(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})\\&\quad-(\Lambda\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{\xi})-(\Lambda\Delta_q\widetilde{h},\Delta_q(u\cdot\nabla\widetilde{\xi})).\end{aligned}$$

Noticing that

$$\|\Delta_q\widetilde{h}\|_{L^2}\le\tfrac{8}{3}\cdot 2^q\|\Lambda^{-1}\Delta_q\widetilde{h}\|_{L^2}\le\tfrac{8}{3}\|\Delta_q\widetilde{h}\|_{L^2}$$

for $q\le 0$, we have

$$\begin{aligned}|(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})|&\le\tfrac{8}{9}\|\Delta_q\widetilde{h}\|_{L^2}^2+2\|\Delta_q\widetilde{\xi}\|_{L^2}^2,\\|(\Lambda^2\Delta_q\widetilde{h},\Lambda\Delta_q\widetilde{\xi})|&\le\tfrac{8}{9}\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2+2\|\Lambda\Delta_q\widetilde{\xi}\|_{L^2}^2.\end{aligned}$$

Hence if we take $K_1=1/8$, and denote, for $q\le 0$,

$$g_q^2:=\|\Delta_q\widetilde{h}\|_{L^2}^2+\|\Lambda^{-1}\Delta_q\widetilde{h}\|_{L^2}^2+\|\Delta_q\widetilde{\xi}\|_{L^2}^2-\tfrac{1}{4}(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi}),$$

then there exist constants C_1 and C_2 such that

$$C_1g_q^2\le\|\Lambda^{-1}\Delta_q\widetilde{h}\|_{L^2}^2+\|\Delta_q\widetilde{\xi}\|_{L^2}^2\le C_2g_q^2.$$

Thus, there exists a constant $\hat{C}$ such that, for $q\le 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}g_q^2&+(\hat{C}2^{2q}+KV')g_q^2\\&\le(\Delta_q\widetilde{F},\Delta_q\widetilde{h})-(\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{h})+(\Delta_q\widetilde{G},\Delta_q\widetilde{\xi})-(\Delta_q(u\cdot\nabla\widetilde{\xi}),\Delta_q\tilde{\xi})\\&\quad+(\Lambda^{-1}\Delta_q\widetilde{F},\Lambda^{-1}\Delta_q\widetilde{h})-(\Lambda^{-1}\Delta_q(u\cdot\nabla\widetilde{h}),\Lambda^{-1}\Delta_q\widetilde{h})-\tfrac{1}{8}[(\Lambda\Delta_q\widetilde{F},\Delta_q\widetilde{\xi})\\&\quad+(\Lambda\Delta_q\widetilde{G},\Delta_q\widetilde{h})-(\Lambda\Delta_q(u\cdot\nabla\widetilde{h}),\Delta_q\widetilde{\xi})-(\Lambda\Delta_q\widetilde{h},\Delta_q(u\cdot\nabla\widetilde{\xi}))].\end{aligned}\tag{4.4}$$

STEP 2: *High frequencies* ($q>0$). Taking the L^2 scalar product of the first equation of (4.1) with $\Lambda^2\Delta_q\widetilde{h}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2+(\Lambda\Delta_q(u\cdot\nabla\widetilde{h}),\Lambda\Delta_q\widetilde{h})+(\Lambda^2\Delta_q\widetilde{\xi},\Lambda\Delta_q\widetilde{h})\\=(\Lambda\Delta_q\widetilde{F},\Lambda\Delta_q\widetilde{\xi})-KV'(t)\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2.\end{aligned}\tag{4.5}$$

A linear combination of $(4.2)_2$, $(4.3)_2$ and (4.5) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&[\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2+2\|\Delta_q\widetilde{\xi}\|_{L^2}^2-2(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})]+\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2\\&+\|\Delta_q\widetilde{h}\|_{L^2}^2+\|\Lambda\Delta_q\widetilde{\xi}\|_{L^2}^2-2(\Lambda\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})-2(\Lambda^{-1}\Delta_q\widetilde{h},\Delta_q\widetilde{\xi})\end{aligned}$$

$$
\begin{aligned}
&= (\Lambda\Delta_q\widetilde{F}, \Lambda\Delta_q\widetilde{h}) - KV'(t)\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2 - (\Lambda\Delta_q(u\cdot\nabla\widetilde{h}), \Lambda\Delta_q\widetilde{h})\\
&\quad + 2(\Delta_q\widetilde{G}, \Delta_q\widetilde{\xi}) - 2KV'(t)\|\Delta_q\widetilde{\xi}\|_{L^2}^2 - 2(\Delta_q(u\cdot\nabla\widetilde{\xi}), \Delta_q\widetilde{\xi})\\
&\quad - [(\Lambda\Delta_q\widetilde{F}, \Delta_q\widetilde{\xi}) + (\Lambda\Delta_q\widetilde{G}, \Delta_q\widetilde{h}) - 2KV'(t)(\Lambda\Delta_q\widetilde{h}, \Delta_q\widetilde{\xi})\\
&\quad - (\Lambda\Delta_q(u\cdot\nabla\widetilde{h}), \Delta_q\widetilde{\xi}) - (\Lambda\Delta_q(u\cdot\nabla\widetilde{\xi}), \Delta_q\widetilde{h})].
\end{aligned}
$$

Denote, for $q > 0$,

$$
g_q^2 := \|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2 + 2\|\Delta_q\widetilde{\xi}\|_{L^2}^2 - 2(\Lambda\Delta_q\widetilde{h}, \Delta_q\widetilde{\xi}).
$$

There are constants C_3 and C_4 such that

$$
C_3 g_q^2 \le \|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2 + \|\Delta_q\widetilde{\xi}\|_{L^2}^2 \le C_4 g_q^2.
$$

Notice that

$$
\begin{aligned}
|2(\Lambda\Delta_q\widetilde{h}, \Delta_q\widetilde{\xi})| &\le M_1\|\Lambda\Delta_q\widetilde{\xi}\|_{L^2}^2 + \frac{4}{9M_1}\|\Lambda\Delta_q\widetilde{h}\|_{L^2}^2,\\
|2(\Lambda^{-1}\Delta_q\widetilde{h}, \Delta_q\widetilde{\xi})| &\le M_1\|\Lambda\Delta_q\widetilde{\xi}\|_{L^2}^2 + \frac{4}{9M_1}\|\Delta_q\widetilde{h}\|_{L^2}^2.
\end{aligned}
$$

Thus if we take $M_1 = \frac{17}{36}$, there exists a constant $\bar{C}$ such that

$$
\begin{aligned}
(4.6)\qquad &\frac{1}{2}\frac{d}{dt}g_q^2 + (\bar{C} + KV')g_q^2\\
&\le (\Lambda\Delta_q\widetilde{F}, \Lambda\Delta_q\widetilde{h}) - (\Lambda\Delta_q(u\cdot\nabla\widetilde{h}), \Lambda\Delta_q\widetilde{h}) + 2(\Delta_q\widetilde{G}, \Delta_q\widetilde{\xi})\\
&\quad - 2(\Delta_q(u\cdot\nabla\widetilde{\xi}), \Delta_q\widetilde{\xi}) - (\Lambda\Delta_q\widetilde{F}, \Delta_q\widetilde{\xi}) - (\Lambda\Delta_q\widetilde{G}, \Delta_q\widetilde{h})\\
&\quad + [(\Lambda\Delta_q(u\cdot\nabla\widetilde{h}), \Delta_q\widetilde{\xi}) + (\Lambda\Delta_q(u\cdot\nabla\widetilde{\xi}), \Delta_q\widetilde{h})].
\end{aligned}
$$

Now, we combine (4.4) and (4.6), and use Lemma 2.7 to estimate the terms involving convection in (4.4) and (4.6), and eventually get the existence of sequence $(\alpha_q)_{q\in\mathbb{Z}}$ such that $\sum_{q\in\mathbb{Z}}\alpha_q \le 1$ and

$$
\begin{aligned}
(4.7)\qquad &\frac{1}{2}\frac{d}{dt}g_q^2 + (\kappa\min(2^{2q},1) + KV')g_q^2\\
&\le C\alpha_q g_q^2 2^{-q(s-1)}[\|\widetilde{F}, \widetilde{G}\|_{\widetilde{B}_{2,1}^{s-2,s}\times\dot{B}_{2,1}^{s-1}} + V'\|\widetilde{h}, \widetilde{\xi}\|_{\widetilde{B}_{2,1}^{s-2,s}\times\dot{B}_{2,1}^{s-1}}],
\end{aligned}
$$

where $\kappa = \min(\hat{C}, \bar{C})$.

STEP 3: *The damping effect.* We are now going to show that inequality (4.7) entails a decay for h and ξ. We postpone the proof of smoothing properties for ξ to the next step. Let $\delta > 0$ be a small parameter and denote $\psi_q^2 = g_q^2 + \delta^2$. From (4.7) and dividing by ψ_q, we find that

$$
\begin{aligned}
(4.8)\qquad &\frac{d}{dt}\psi_q^2 + (\kappa\min(2^{2q},1) + KV')\psi_q\\
&\le C\alpha_q 2^{-q(s-1)}[\|(\widetilde{F}, \widetilde{G})\|_{\widetilde{B}_{2,1}^{s-2,s}\times\dot{B}_{2,1}^{s-1}} + V'\|(\widetilde{h}, \widetilde{\xi})\|_{\widetilde{B}_{2,1}^{s-2,s}\times\dot{B}_{2,1}^{s-1}}]\\
&\quad + \delta(\kappa\min(2^{2q},1) + KV').
\end{aligned}
$$

Integrating over $[0,t]$ and letting δ tend to 0, we get

$$(4.9)\qquad g_q(t)+\kappa\min(2^{2q},1)\int_0^t g_q(\tau)\,d\tau \le g_q(0)+C2^{-q(s-1)}\int_0^t \alpha_q(\tau)\|(\widetilde{F},\widetilde{G})\|_{\widetilde{B}^{s-2,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}\,d\tau + \int_0^t V'(\tau)[C2^{-q(s-1)}\alpha_q(\tau)\|(\widetilde{h},\widetilde{\xi})\|_{\widetilde{B}^{s-2,s}_{2,1}\times\widetilde{B}^{s-1,s}_{2,1}}-Kg_q(\tau)]\,d\tau.$$

By the definition of g_q^2, we have, for any $q\in\mathbb{Z}$,

$$(4.10)\qquad 2^{q(s-1)}g_q^2\approx 2^{q(s-1)}\max(2^{-q},2^q)\|\widetilde{h}_q\|_{L^2}^2+2^{q(s-1)}\|\widetilde{\xi}_q\|_{L^2}^2.$$

Thus, we can take K large enough such that

$$\sum_{q\in\mathbb{Z}}[C\alpha_q(\tau)\|(\widetilde{h}(\tau),\widetilde{\xi}(\tau))\|_{\widetilde{B}^{s-2,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}-K2^{q(s-1)}g_q(\tau)]\le 0.$$

Multiplying both sides of (4.9) by $2^{q(s-1)}$, according to the last inequality and (4.10), we conclude after summation on q in $\mathbb{Z}$ that

$$(4.11)\qquad \|\widetilde{h}(t)\|_{\widetilde{B}^{s-2,s}_{2,1}}+\|\widetilde{\xi}(t)\|_{\dot{B}^{s-1}_{2,1}} +\kappa\Big(\int_0^t\|\widetilde{h}(\tau)\|_{\dot{B}^{s}_{2,1}}\,d\tau+\sum_{q\in\mathbb{Z}}\int_0^t 2^{q(s-1)}\min(2^{2q},1)\|\Delta_q\widetilde{\xi}(\tau)\|\,d\tau\Big) \le C\Big(\|\widetilde{h}_0\|_{\widetilde{B}^{s-2,s}_{2,1}}+\|\widetilde{\xi}_0\|_{\dot{B}^{s-1}_{2,1}}+\int_0^t(\|\widetilde{F}(\tau)\|_{\widetilde{B}^{s-2,s}_{2,1}}+\|\widetilde{G}(\tau)\|_{\dot{B}^{s-1}_{2,1}})\,d\tau\Big).$$

STEP 4: *The smoothing effect.* Once having the damping effect on h, it is easy to get the smoothing effect on ξ. Thanks to (4.11), it suffices to prove it for high frequencies only. We therefore suppose in this subsection that $q>0$.

Define $g_q=\|\Delta_q\widetilde{\xi}\|_{L^2}$. From $(4.2)_2$ and using Lemma 2.7, we obtain

$$\frac{1}{2}\frac{d}{dt}g_q^2+\kappa 2^{2q}g_q^2\le g_q(\|\Lambda\widetilde{h}_q\|_{L^2}+\|\Lambda^{-1}\widetilde{h}_q\|_{L^2}+\|\widetilde{G}_q\|_{L^2}) + g_qV'(t)(C\alpha_q2^{-q(s-1)}\|\widetilde{\xi}\|_{\dot{B}^{s-1}_{2,1}}-Kg_q).$$

Standard computations therefore yield

$$\sum_{q\ge1}2^{q(s-1)}\|\Delta_q\widetilde{\xi}(t)\|_{L^2}+\kappa\int_0^t\sum_{q\ge1}2^{q(s+1)}\|\Delta_q\widetilde{\xi}(\tau)\|_{L^2}\,d\tau\le\|\widetilde{\xi}_0\|_{\dot{B}^{s-1}_{2,1}} +\int_0^t\|\widetilde{G}(\tau)\|_{\dot{B}^{s-1}_{2,1}}\,d\tau+2\int_0^t\sum_{q\ge1}2^{qs}\|\widetilde{h}_q(\tau)\|_{L^2}\,d\tau+CV(t)\sup_{[0,t]}\|\widetilde{\xi}(\tau)\|_{\dot{B}^{s-1}_{2,1}}.$$

Using (4.11), we eventually conclude that

$$\int_0^t \sum_{q\geq 1} 2^{q(s+1)}\|\Delta_q\widetilde{\xi}(\tau)\|_{L^2}\,d\tau \lesssim (1+V(t))\Big(\|h_0\|_{\widetilde{B}^{s-2,s}_{2,1}} + \|\xi_0\|_{\dot{B}^{s-1}_{2,1}} + \int_0^t (\|\widetilde{F}\|_{\widetilde{B}^{s-2,s}_{2,1}} + \|\widetilde{G}\|_{\widetilde{B}^{s-1}_{2,1}})\,d\tau\Big).$$

Combining the last inequality with (4.11), we complete the proof of Proposition 3.2.

5. A global existence and uniqueness result. This section is devoted to the proof of Theorem 1.2. The scheme of the proof is a very classical one. We use an iterative method to build approximate solutions (h^n,ξ^n) of (3.1) which are solutions of linear systems of type (3.3) coupled with a heat equation, to which we apply Proposition 3.2 and Lemma 3.1.

In the case of smooth data, that is, for u, F and G continuous in time with values in S, and for h_0 and ξ_0 in S, it is easy to prove that (3.3) has a unique global solution continuous with values in S. We set the first term (h^0,u^0) to $(0,0)$, and then define $((h^n,u^n))_{n\in\mathbb{N}}$ by induction. We choose (h^{n+1},u^{n+1}) as the unique smooth solution of the following linear system:

(5.1)
$$\begin{cases}\partial_t h^{n+1} + u\cdot\nabla h^{n+1} + \Lambda\xi^{n+1} = F^n,\\ \partial_t\xi^{n+1} + u^n\cdot\nabla\xi^{n+1} - \Delta\xi^{n+1} - \Lambda h^{n+1} - \Lambda^{-1}h^{n+1} = G^n + \Lambda^{-1}\operatorname{div} f_n,\\ \partial_t\eta^{n+1} - \mu\Delta\eta^{n+1} = H^n + \Lambda^{-1}\operatorname{curl} f_n,\\ u^{n+1} = -\Lambda^{-1}\nabla\xi^{n+1} - \Lambda^{-1}\operatorname{div}\eta^{n+1},\\ (h^{n+1},\xi^{n+1},\eta^{n+1})_{t=0} = (h_n,\Lambda^{-1}\operatorname{div} u_n,\Lambda^{-1}\operatorname{curl} u_n),\end{cases}$$

with

$$h_n = \sum_{|q|\leq n}\Delta_q h_0,\qquad u_n = \sum_{|q|\leq n}\Delta_q u_0,$$

$$f_n = \sum_{|q|\leq n}\Delta_q f,\qquad F^n = -h^n\operatorname{div} u^n,$$

$$G^n = u^n\cdot\nabla h^n - \Lambda^{-1}\operatorname{div}\Big(u^n\cdot\nabla u^n + K(h^n)\nabla h^n + \frac{h^n}{1+h^n}\mathcal{A}u^n\Big),$$

$$H^n = -\Lambda^{-1}\operatorname{curl}\Big(u^n\cdot\nabla u^n + \frac{h^n}{1+h^n}\mathcal{A}u^n\Big).$$

STEP 1: *Uniform estimates.* In this part, we prove uniform estimates in $E^{N/2}$ for (h^n,u^n). Defining

$$\alpha = \|h_0\|_{\widetilde{B}^{N/2-2,N/2}_{2,1}} + \|u_0\|_{\dot{B}^{N/2-1}_{2,1}} + \|f\|_{L^1(\dot{B}^{N/2-1}_{2,1})},$$

we are going to prove the existence of a positive M such that, if α is small enough, the following bound holds for all $n \in \mathbb{N}$:

$$(\mathcal{P}_n) \qquad \|(h_n, u_n)\|_{E^{N/2}} \le M\alpha.$$

Suppose that $(\mathcal{P}_n)$ is satisfied and let us prove $(\mathcal{P}_{n+1})$. According to Proposition 3.2 and Lemma 3.1, and the definition of (h_n, u_n, f_n), we have

$$\begin{aligned}\|(h^{n+1}, u^{n+1})\|_{E^{N/2}} \le C\exp(C\|u^n\|_{L^1(\dot{B}_{2,1}^{N/2+1})})(\|h_0\|_{\widetilde{B}_{2,1}^{N/2-2,N/2}} + \|u_0\|_{\dot{B}_{2,1}^{N/2-1}} \\ + \|f\|_{L^1(\dot{B}_{2,1}^{N/2-1})} + \|F^n\|_{L^1(\widetilde{B}_{2,1}^{N/2-2,N/2})} \\ + \|G^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})} + \|H^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})}).\end{aligned}$$

Therefore, it is only a matter of proving appropriate estimates for F^n, G^n and H^n using $(\mathcal{P}_n)$. The estimate of F^n is straightforward: according to Lemma 2.6,

$$\begin{aligned}\|F^n\|_{L^1(\widetilde{B}_{2,1}^{N/2-2,N/2})} &\le C\|h^n\|_{L^\infty(\widetilde{B}_{2,1}^{N/2-2,N/2})}\|\operatorname{div} u^n\|_{L^1(\dot{B}_{2,1}^{N/2})} \\ &\le CM^2\alpha^2.\end{aligned} \tag{5.2}$$

To estimate G^n and H^n, we make the following assumption on α:

$$\alpha \le 1/(4\mathcal{C}^2),$$

where $\mathcal{C}$ is the modulus of continuity of $\dot{B}_{2,1}^{N/2} \hookrightarrow L^\infty$. If $(\mathcal{P}_n)$ is fulfilled, this entails

$$\|h^n\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^n)} \le 1/2. \tag{5.3}$$

Indeed, we use

$$\|h^n\|_{L^\infty} \le \mathcal{C}\|h^n\|_{\dot{B}_{2,1}^{N/2}} \le \mathcal{C}\|h^n\|_{\widetilde{B}_{2,1}^{N/2-2,N/2}}.$$

We then have, according to Lemmas 2.3 and 2.6,

$$\begin{aligned}\left\|\frac{h^n}{1+h^n}\nabla^2 u^n\right\| &\le C\|\nabla^2 u^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})}\left\|\frac{h^n}{1+h^n}\right\|_{L^\infty(\dot{B}_{2,1}^{N/2})} \\ &\le C\|u^n\|_{L^1(\dot{B}_{2,1}^{N/2+1})}\|h^n\|_{L^\infty(\dot{B}_{2,1}^{N/2})} \le CM^2\alpha^2.\end{aligned} \tag{5.4}$$

We also have, using $K(0) = 0$, and Lemmas 2.2 and 2.3,

$$\begin{aligned}\|K(h^n)\nabla h^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})} &\le C\|K(h^n)\|_{L^\infty(\dot{B}_{2,1}^{N/2})}\|\nabla h^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})} \\ &\le CM^2\alpha^2.\end{aligned} \tag{5.5}$$

Thanks to Lemma 2.2, we easily infer

$$\begin{aligned}&\|u^n\cdot\nabla\xi^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})} + \|u^n\nabla u^n\|_{L^1(\dot{B}_{2,1}^{N/2-1})} \\ &\qquad \le C\|u^n\|_{L^\infty(\dot{B}_{2,1}^{N/2-1})}\|\nabla u^n\|_{L^1(\dot{B}_{2,1}^{N/2})} \le CM^2\alpha^2.\end{aligned} \tag{5.6}$$

From (5.4)–(5.6), we finally deduce

$$\|G^n\|_{L^1(\dot{B}^{N/2-1}_{2,1})} + \|H^n\|_{L^1(\dot{B}^{N/2-1}_{2,1})} \le CM^2\alpha^2, \tag{5.7}$$

whence

$$\|(h^{n+1}, u^{n+1})\|_{E^{N/2}} \le C \exp{(CM\alpha)}(\alpha + M^2\alpha^2).$$

So choosing $M = 4C$ and assuming

$$M^2\alpha \le 1, \quad \exp{(CM\alpha)} \le 2 \quad \text{and} \quad \alpha \le 1/(4\mathcal{C}^2),$$

$(\mathcal{P}_n)$ is fulfilled for all $n \in \mathbb{N}$.

Step 2: *Existence of a solution.* In this part, we shall show that, up to an extraction, the sequence $((h^n, u^n))_{n\in\mathbb{N}}$ converges in $\mathscr{D}(\mathbb{R}^+ \times \mathbb{R}^N)$ to a solution of (3.1) which has the desired regularity properties. The proof relies on compactness arguments. To begin, we show that the first time derivative of (h^n, u^n) is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit (h, u) for a subsequence. Now, the uniform bounds of Step 1 provide us with additional regularity and convergence properties so that we may pass to the limit in (5.1).

It is convenient to consider the solution of a linear system with initial data (h_n, u_n) and forcing term f_n. More precisely, we denote by (h^n_L, u^n_L) the solution to

$$\begin{cases} \partial_t h^n_L + \operatorname{div} u^n_L = 0, \\ \partial_t u^n_L - \mathcal{A}u^n_L + \nabla h^n_L - \nabla\Delta^{-1}h^n_L = f_n, \\ (h^n_L, u^n_L)_{t=0} = (h_n, u_n), \end{cases} \tag{5.8}$$

and $(\bar{h}^n, \bar{u}^n) = (h^n - h^n_L, u^n - u^n_L)$. Obviously, the definition of (h_n, u_n, f_n) entails $h_n \to h_0$ in $\widetilde{B}^{N/2-2,N/2}_{2,1}$, $u_n \to u_0$ in $\dot{B}^{N/2-1}_{2,1}$, $f_n \to f$ in $L^1(\mathbb{R}^+; \dot{B}^{N/2-1}_{2,1})$. Lemma 3.1 and Proposition 3.2 therefore ensure that

$$(h^n_L, u^n_L) \to (h_L, u_L) \quad \text{in } E^{N/2}, \tag{5.9}$$

where (h_L, u_L) is the solution of the linear system

$$\begin{cases} \partial_t h_L + \operatorname{div} u_L = 0, \\ \partial_t u_L - \mathcal{A}u_L + \nabla h_L - \nabla\Delta^{-1}h_L = f, \\ (h_L, u_L)_{t=0} = (h_0, u_0). \end{cases}$$

We now have to prove the convergence of $(\bar{h}^n, \bar{u}^n)$. This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 5.1. *For all $T > 0$, $((\bar{h}^n, \bar{u}^n))_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1/4}([0,T]; \dot{B}^{N/2-1/2}_{2,1} \times \widetilde{B}^{N/2-1/2,N/2-3/2}_{2,1})$.*

Proof. In all the proof, u.b. will stand for uniformly bounded.

We first prove that $\partial_t \bar{h}^n$ is u.b. in $L^{4/3}(\dot{B}_{2,1}^{N/2-1/2})$, which yields the desired result for $\bar{h}^n$. Let us observe that

$$\partial_t \bar{h}^{n+1} = -h^{n+1} \operatorname{div} u^n - u^n \cdot \nabla h^{n+1} - \operatorname{div} u^{n+1} + \operatorname{div} u_L^{n+1}.$$

According to Step 1, $(u^n)_{n\in\mathbb{N}}$ is u.b. in $L^{4/3}(\dot{B}_{2,1}^{N/2+1/2})$ and $(h^n)_{n\in\mathbb{N}}$ is u.b. in $L^\infty(\dot{B}_{2,1}^{N/2})$, thus $-h^n \operatorname{div} u^n - u^n\cdot\nabla h^{n+1} - \operatorname{div} u^{n+1}$ is u.b. in $L^{4/3}(\dot{B}_{2,1}^{N/2-1/2})$. The definition of u_L^n obviously provides us with uniform bounds for $\operatorname{div} u_L^n$ in $L^{4/3}(\dot{B}_{2,1}^{N/2-1/2})$, so we can conclude that $\partial_t \bar{h}^n$ is u.b. in $L^{4/3}(\dot{B}_{2,1}^{N/2-1/2})$.

Denote $\xi_L^n = -\Lambda^{-1} \operatorname{div} u_L^n$, $\bar{\xi}_L^n = -\Lambda^{-1} \operatorname{div} \bar{u}^n$, $\bar{\eta}_L^n = -\Lambda^{-1} \operatorname{curl} u_L^n$ and $\bar{\eta}^n = -\Lambda^{-1} \operatorname{curl} \bar{u}^n$.

Let us now prove that $\partial_t \bar{\xi}^n$ is u.b. in $L^{4/3}(\mathbb{R}^+; \widetilde{B}_{2,1}^{N/2-1/2,N/2-3/2})$ and that $\partial_t \bar{\eta}^n$ is u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-3/2})$, which gives the required result for $\bar{u}^n$. Let us recall that

$$\begin{aligned}
\partial_t \bar{\xi}^{n+1} &= u^n \cdot \nabla(\xi^n - \xi^{n+1}) - \Lambda^{-1} \operatorname{div}\left(u^n \cdot \nabla u^n + K(h^n)\nabla h^n + \frac{h^n}{1+h^n}\mathcal{A}u^n\right) \\
&\qquad - \Delta(\xi^{n+1} - \xi_L^{n+1}) + \Lambda(h^{n+1} - h_L^{n+1}) + \Lambda^{-1}(h^{n+1} - h_L^{n+1}), \\
\partial_t \bar{\eta}^{n+1} &= -\Lambda^{-1} \operatorname{curl}\left(u^n \cdot \nabla u^n + \frac{h^n}{1+h^n}\mathcal{A}u^n\right) + \mu\Delta(\eta^{n+1} - \eta_L^{n+1}).
\end{aligned}$$

Results of Step 1 and an interpolation argument yield uniform bounds for u^n in $L^\infty(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-1}) \cap L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2+1/2})$. As h^n is u.b. in $L^\infty(\mathbb{R}^+; \dot{B}_{2,1}^{N/2})$ and ξ_L^n is u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2+1/2})$, we easily infer that $u^n \cdot \nabla(\xi^n - \xi^{n+1}) - \Lambda^{-1} \operatorname{div}(u^n \cdot \nabla u^n + \frac{h^n}{1+h^n}\mathcal{A}u^n) + \Delta(\xi^{n+1} - \xi_L^{n+1})$ is u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-3/2})$. Using the bounds for h^n in $L^\infty(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-2}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{N/2})$, we get h^n u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-1/2})$. We also have $K(h^n)\nabla h^n$ u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-3/2})$. Of course, Λh_L^n and Λh^n are u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-3/2})$, while $\Lambda^{-1} h_L^n$ and $\Lambda^{-1} h^n$ are u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-1/2})$. The case of $\partial_t \bar{\eta}^{n+1}$ goes along the same lines. As the terms corresponding to $K(h^n)\nabla h^n$, $\Lambda(h^n - h_L^n)$ and $\Lambda^{-1}(h^n - h_L^n)$ do not appear, we easily get $\partial_t \bar{\eta}^n$ u.b. in $L^{4/3}(\mathbb{R}^+; \dot{B}_{2,1}^{N/2-3/2})$.

We can now turn to proving the existence of a solution. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^N)$ cut-off functions supported in the ball $B(0, p+1)$ of $\mathbb{R}^N$ and equal to 1 in a neighborhood of $B(0,p)$. For any $p \in \mathbb{N}$ and $T > 0$, Lemma 5.1 and Step 1 ensure that $((\chi_p \bar{h}^n, \chi_p \bar{u}^n))_{n\in\mathbb{N}}$ is uniformly equicontinuous in $C([0,T]; \dot{B}_{2,1}^{N/2-1/2} \times (\widetilde{B}_{2,1}^{N/2-1/2,N/2-3/2})^N)$ and bounded in $E^{N/2}$. Moreover the mapping $u \mapsto \chi_p u$ is compact from $\widetilde{B}_{2,1}^{N/2-2,N/2}$ into $\dot{B}_{2,1}^{N/2-1/2}$ and from $\dot{B}_{2,1}^{N/2-1}$ into $\widetilde{B}_{2,1}^{N/2-1/2,N/2-3/2}$.

We apply Ascoli's theorem to the family $((\chi_p\bar h^n,\chi_p\bar u^n))_{n\in\mathbb N}$ on the time interval $[0,p]$, and then use Cantor's diagonal process. This finally provides us with a distribution $(\bar h,\bar u)$ continuous in time with values in $\dot B^{N/2-1/2}_{2,1}\times(\widetilde B^{N/2-1/2,N/2-3/2}_{2,1})^N$ and a subsequence (still denoted by $((\bar h^n,\bar u^n))$ for simplicity) such that for all $p\in\mathbb N$, we have

$$(\chi_p\bar h^n,\chi_p\bar u^n)\xrightarrow[n\to\infty]{}(\chi_p\bar h,\chi_p\bar u)\text{ in }C([0,p];\dot B^{N/2-1/2}_{2,1}\times(\widetilde B^{N/2-1/2,N/2-3/2}_{2,1})^N).\tag{5.10}$$

This obviously implies that $(\bar h^n,\bar u^n)$ tends to $(\bar h,\bar u)$ in $\mathscr D'(\mathbb R^+\times\mathbb R^N)$.

Coming back to the uniform estimates of Step 1, we moreover see that $(\bar h,\bar u)$ belongs to

$$L^1(\mathbb R^+;\dot B^{N/2}_{2,1}\times\dot B^{N/2+1}_{2,1})\cap L^\infty(\mathbb R^+;\widetilde B^{N/2-2,N/2}_{2,1}\times(\dot B^{N/2-1}_{2,1})^N)$$

and to $C^{1/4}(\mathbb R^+;\dot B^{N/2-1/2}_{2,1}\times(\widetilde B^{N/2-1/2,N/2-3/2}_{2,1})^N)$. Obviously, we have the bounds provided by $(\mathcal P_n)$ for this solution.

Let us now prove that $(h,u):=(h_L,u_L)+(\bar h,\bar u)$ solves (3.1). We first observe that, according to (5.1),

$$\begin{cases}\partial_t h^{n+1}+u^n\cdot\nabla h^{n+1}+\operatorname{div}u^{n+1}=-h^n\operatorname{div}u^n,\\ \partial_t u^{n+1}-\mathcal A u^{n+1}+\nabla h^{n+1}+\nabla\Delta^{-1}h^{n+1}+K(h^n)\nabla h^n+\dfrac{h^n}{1+h^n}\mathcal A u^n\\ \qquad\qquad\qquad\qquad=-\Lambda^{-1}\nabla(u^n\cdot\nabla(\xi^{n+1}-\xi^n))+f_n.\end{cases}$$

The only problem is to pass to the limit in $\mathscr D'(\mathbb R^+\times\mathbb R^N)$ in the nonlinear terms. This can be done by using the convergence results stemming from the uniform estimates of Step 1 and the convergences (5.9) and (5.10).

As it is just a matter of tedious verifications, as an example we handle the term $L(h^n)\mathcal A u^n$ (where $L(z):=z/(z+1)$). Let $\theta\in C_0^\infty(\mathbb R^+\times\mathbb R^N)$ and $p\in\mathbb N$ be such that $\operatorname{supp}\theta\subset[0,p]\times B(0,p)$. We use the decomposition

$$\begin{aligned}\theta L(h^n)\mathcal A u^n-\theta L(h)\mathcal A u=\theta L(h^n)\chi_p\mathcal A(u_L^n-u_L)+\theta L(h^n)\chi_p\mathcal A(\chi_p(\bar u^n-\bar u))\\ +\theta\mathcal A u(L(\chi_p h^n)-L(\chi_p h)).\end{aligned}$$

As $\theta L(h^n)$ is u.b. in $L^\infty(\mathbb R^+;\dot B^{N/2}_{2,1})$ and $u^n_L\to u_L$ in $L^1(\mathbb R^+;\dot B^{N/2+1}_{2,1})$, the first term tends to 0 in $L^1(\dot B^{N/2-1}_{2,1})$. According to the uniform estimates of Step 1 and (5.10), $\chi_p(\bar u^n-\bar u)\to 0$ in $L^1([0,p];\dot B^{N/2+1}_{2,1})$, so that the second terms tends to 0 in $L^1(\mathbb R^+;\dot B^{N/2-1}_{2,1})$. Clearly $L(\chi_p h^n)\to L(\chi_p h)$ in $L^\infty(\mathbb R^+;\dot B^{N/2}_{2,1})$, so that the third term also tends to 0 in $L^1(\mathbb R^+;\dot B^{N/2-1}_{2,1})$.

The other nonlinear terms can be treated the same way.

We still have to prove that h is continuous in $\widetilde B^{N/2-2,N/2}$ and that u belongs to $C(\mathbb R^+;\dot B^{N/2-1}_{2,1})$. The continuity of u is straightforward. Indeed, u

satisfies

$$\partial_t u = -u\cdot\nabla u + \mathcal{A}u - \nabla h + \nabla\Delta^{-1}h - \frac{h}{1+h}\mathcal{A}u - K(h)\nabla h + f$$

and the right-hand side belongs to $L^\infty(\mathbb{R}^+;\dot{B}^{N/2-1}_{2,1})$.

From the equation

$$h_t = -\operatorname{div}(hu) - \operatorname{div} u \tag{5.11}$$

and $u\in L^\infty(\mathbb{R}^+;\dot{B}^{N/2-1}_{2,1})$, $h\in L^\infty(\mathbb{R}^+;\dot{B}^{N/2-2}_{2,1})$, we get $h_t\in L^\infty(\mathbb{R}^+;\dot{B}^{N/2-2}_{2,1})$. It still remains to prove the continuity of h in $\dot{B}^{N/2}_{2,1}$.

Applying the operator Δ_q to (5.11), we get

$$\partial_t\Delta_q h = -\Delta_q(\operatorname{div} hu) - \Lambda\Delta_q\xi. \tag{5.12}$$

Obviously, for fixed q the right-hand side belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+;L^2)$ so that each $\Delta_q h$ is continuous in time with values in L^2 (thus in $\dot{B}^{N/2}_{2,1}$).

Now, we apply the energy method to (5.12). Thanks to Lemma 2.7, we get

$$\begin{aligned}\frac12\frac{d}{dt}\|\Delta_q h\|^2_{L^2} \le C\|\Delta_q h\|_{L^2}(C_q 2^{-qN/2}\|h\|_{\dot{B}^{N/2}_{2,1}}\|u\|_{\dot{B}^{N/2+1}_{2,1}}\\ + \|\Lambda\Delta_q\xi\|_{L^2} + \|\Delta_q(h\operatorname{div} u)\|_{L^2}).\end{aligned}$$

So time integration yields

$$\begin{aligned}2^{qN/2}\|\Delta_q h(t)\|_{L^2} \le 2^{qN/2}\|\Delta_q h_0\|_{L^2} + C\int_0^t (c_q(\tau)\|h(\tau)\|_{\dot{B}^{N/2}_{2,1}}\|u(\tau)\|_{\dot{B}^{N/2+1}_{2,1}}\\ + 2^{q(N/2+1)}\|\Delta_q\xi(\tau)\|_{L^2} + 2^{qN/2}\|\Delta_q(h\operatorname{div} u)(\tau)\|_{L^2})\,d\tau.\end{aligned}$$

Since $h\in L^\infty(\dot{B}^{N/2}_{2,1})$, $u\in L^1(\dot{B}^{N/2+1}_{2,1})$ and $h\operatorname{div} u\in L^1(\dot{B}^{N/2}_{2,1})$, we eventually get

$$\begin{aligned}&\sum_{q\in\mathbb{Z}}\sup_{t\ge 0} 2^{qN/2}\|\Delta_q h(t)\|_{L^2}\\ &\quad\lesssim \|h_0\|_{\dot{B}^{N/2}_{2,1}} + (1+\|h\|_{L^\infty(\dot{B}^{N/2}_{2,1})})\|u\|_{L^1(\dot{B}^{N/2}_{2,1}+1)} + \|h\operatorname{div} u\|_{L^1(\dot{B}^{N/2}_{2,1})} < \infty.\end{aligned}$$

In other words, $\sum_{|q|\le N}\Delta_q h$ converges uniformly in $L^\infty(\mathbb{R}^+;\dot{B}^{N/2}_{2,1})$ and we can conclude that h belongs to $C(\mathbb{R}^+;\dot{B}^{N/2}_{2,1})$.

STEP 3: *Further regularity properties for more regular data.* Let $s\in(N/2,N/2+1]$. Under the additional assumption $h_0\in\dot{B}^s_{2,1}$, $u_0\in\dot{B}^{s-1}_{2,1}$ and $f\in L^1(\dot{B}^{s-1}_{2,1})$, we shall prove that the sequence $((h^n,u^n))_{n\in\mathbb{N}}$ is uniformly bounded in E^s.

Applying Lemma 3.1 and Proposition 3.2 yields

$$\|(h^{n+1},u^{n+1})\|_{E^s} \le C\exp\big(C\|u^n\|_{L^1(\dot B^{N/2+1}_{2,1})}\big)\big(\|h_n\|_{\widetilde B^{s-2,s}_{2,1}}+\|u_n\|_{\dot B^{s-1}_{2,1}} + \|f_n\|_{L^1(\dot B^{s-1}_{2,1})}+\|F^n\|_{L^1(\widetilde B^{s-2,s}_{2,1})}+\|G^n\|_{L^1(\dot B^{s-1}_{2,1})} + \|H^n\|_{L^1(\dot B^{s-1}_{2,1})}\big).$$

According to (5.3) and Lemmas 2.3, 2.6 and 2.7, we can write

$$\|F^n\|_{L^1(\widetilde B^{s-2,s}_{2,1})} \lesssim \|h^n\|_{L^\infty(\widetilde B^{N/2-2,N/2}_{2,1})}\|\operatorname{div} u^n\|_{L^1(\dot B^{s}_{2,1})} + \|h^n\|_{L^\infty(\widetilde B^{s-2,s}_{2,1})}\|\operatorname{div} u^n\|_{L^1(\widetilde B^{N/2}_{2,1})},$$

$$\|G^n\|_{L^1(\dot B^{s-1}_{2,1})}+\|H^n\|_{L^1(\dot B^{s-1}_{2,1})} \lesssim \|u^n\|_{L^2(\dot B^{N/2}_{2,1})}\|\nabla u^n\|_{L^2(\dot B^{s-1}_{2,1})} + \|h^n\|_{L^2(\dot B^{N/2}_{2,1})}\|h^n\|_{L^2(\dot B^{s}_{2,1})} + \|h^n\|_{L^\infty(\dot B^{N/2}_{2,1})}\|\nabla^2 u^n\|_{L^1(\dot B^{s-1}_{2,1})}.$$

We thus have

$$\|(h^{n+1},u^{n+1})\|_{E^s} \le C\exp(C\|u^n\|_{L^1(\dot B^{N/2+1}_{2,1})})(\|h_0\|_{\widetilde B^{s-2,s}_{2,1}}+\|u_0\|_{\dot B^{s-1}_{2,1}} + \|f\|_{L^1(\dot B^{s-1}_{2,1})}+\|(h^n,u^n)\|_{E^{N/2}}\|(h^n,u^n)\|_{E^s}. \tag{5.13}$$

Now, we conclude that $((h_n,u_n))_{n\in\mathbb N}$ is uniformly bounded in E^s. Indeed, we can deduce by induction from (5.13) and assumption $(\mathcal P_n)$ that

$$\|(h^{n+1},u^{n+1})\|_{E^s} \le 2C\exp\big(CM\alpha\big)\big(\|h_0\|_{\widetilde B^{s-2,s}_{2,1}}+\|u_0\|_{\dot B^{s-1}_{2,1}}+\|f\|_{L^1(\dot B^{s-1}_{2,1})}\big).$$

This clearly enables us to prove that the solution (h,u) built in the previous section also belongs to E^s.

STEP 4: *Uniqueness.* Suppose that (h_1,u_1) and (h_2,u_2) solve (3.1) with the same initial data. If we define $\delta h = h_2-h_1$ and $\delta u = u_2-u_1$, then $(\delta h,\delta u)$ solves

$$\begin{cases}\partial_t\delta h + u_2\cdot\nabla\delta h + \Lambda\delta\xi = \delta F,\\ \partial_t\delta\xi + u_2\cdot\nabla\delta\xi - \Delta\delta\xi - \Lambda\delta h - \Lambda^{-1}\delta h = \delta G,\\ \partial_t\delta\eta - \mu\Delta\delta\eta = \delta H,\\ \delta u = -\Lambda^{-1}\nabla\delta\xi + \Lambda^{-1}\operatorname{div}\delta\eta,\end{cases} \tag{5.14}$$

with

$$\delta F = -\,\delta u\cdot\nabla h_1 - \delta h\operatorname{div} u_2 - h_1\operatorname{div}\delta u,$$

$$\delta G = u_2\cdot\nabla\delta\xi - \Lambda^{-1}\operatorname{div}\bigg(u_2\cdot\nabla\delta u + \delta u\cdot\nabla u_1 + \Big(\frac{h_2}{1+h_2}-\frac{h_1}{1+h_1}\Big)\mathcal A u_2 + \frac{h_1}{1+h_1}\mathcal A\delta u\bigg) + \Lambda(\mathscr K(h_2)-\mathscr K(h_1))$$

$$\delta H = -\Lambda^{-1}\operatorname{curl}\bigg(u_2\cdot\nabla\delta u + \delta u\cdot\nabla u_1 + \Big(\frac{h_2}{1+h_2} - \frac{h_1}{1+h_1}\Big)\mathcal{A}u_2 + \frac{h_1}{1+h_1}\mathcal{A}\delta u\bigg),$$

and $\mathscr{K}(z) = \int_0^z K(y)\,dy$.

Since $(h_1,u_1),(h_2,u_2),(\delta h,\delta u)\in E_T^{N/2}$, we can easily prove that $\delta F\in L_T^1(\widetilde{B}_{2,1}^{N/2-5/2,N/2-1/2})$ and $\delta G,\delta H\in L_T^1(\dot{B}_{2,1}^{N/2-3/2})$ for any finite T. Apply Lemma 3.1 and Proposition 3.2 to (5.14) with $s = N/2-1/2$ to get

$$\text{(5.15)}\quad \|(\delta h,\delta u)\|_{E_T^{N/2-1/2}} \le C\exp(c\|u_2\|_{L_T^1(\dot{B}_{2,1}^{N/2+1})}) \times (\|\delta F\|_{L_T^1(\widetilde{B}_{2,1}^{N/2-5/2,N/2-1/2})} + \|\delta G\|_{L_T^1(\dot{B}_{2,1}^{N/2-3/2})} + \|\delta H\|_{L_T^1(\dot{B}_{2,1}^{N/2-3/2})}).$$

Noticing that

$$h^1,h^2\in L_T^\infty(\widetilde{B}_{2,1}^{N/2-2,N/2})\cap L_T^1(\dot{B}_{2,1}^{N/2}),$$
$$u^1,u^2\in L_T^\infty(\dot{B}_{2,1}^{N/2-1})\cap L_T^1(\dot{B}_{2,1}^{N/2+1})$$

and

$$\|h_1\|_{L^\infty([0,T]\times\mathbb{R}^N)}\le 1/2,\qquad \|h_2\|_{L^\infty([0,T]\times\mathbb{R}^N)}\le 1/2,$$

by the construction of solutions, we have with the help of interpolation arguments

$$\begin{aligned}\|\delta F\|_{L_T^1(\widetilde{B}_{2,1}^{N/2-5/2,N/2-1/2})} \lesssim{}& \|h_{1\mathrm{LF}}\|_{L_T^1(\dot{B}_{2,1}^{N/2})}\|\delta u_{\mathrm{LF}}\|_{L^\infty(\dot{B}_{2,1}^{N/2-3/2})}\\ &+ \|h_{1\mathrm{HF}}\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})}\|\delta u_{\mathrm{HF}}\|_{L_T^1(\dot{B}_{2,1}^{N/2+1/2})}\\ &+ \|u_2\|_{L_T^1(\dot{B}_{2,1}^{N/2+1})}\|\delta h\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-5/2,N/2-1/2})},\end{aligned}$$

$$\begin{aligned}&\|\delta G\|_{L_T^1(\dot{B}_{2,1}^{N/2-3/2})} + \|\delta H\|_{L_T^1(\dot{B}_{2,1}^{N/2-3/2})}\\ &\quad\lesssim \|u_2\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-1})}\|\delta u\|_{L_T^1(\dot{B}_{2,1}^{N/2+1/2})} + \|u_1\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-1})}\|\delta u\|_{L_T^1(\dot{B}_{2,1}^{N/2+1/2})}\\ &\qquad+ (1+\|h_1\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})} + \|h_2\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})})\|u_2\|_{L_T^1(\dot{B}_{2,1}^{N/2+1})}\|\delta h\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-1/2})}\\ &\qquad+ \|h_1\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})}\|\delta u\|_{L_T^1(\dot{B}_{2,1}^{N/2+1/2})} + T(\|h_1\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})})\\ &\qquad+ \|h_2\|_{L_T^\infty(\dot{B}_{2,1}^{N/2})}\|\delta h\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-1/2})}.\end{aligned}$$

Coming back to (5.15), we eventually get

$$\|(\delta h,\delta u)\|_{E_T^{N/2-1/2}} \le Z(T)\|(\delta h,\delta u)\|_{E_T^{N/2-1/2}},$$

with $\limsup_{T\to 0^+} Z(T)\le CE(0)$.

Supposing that $C\alpha<1$, we get $\|(\delta h,\delta u)\|_{E_T^{N/2-1/2}} = 0$ for a $T>0$ small enough, hence $(h_2,u_2)\equiv(h_1,u_1)$ on $[0,T]$.

Let T_m (supposedly finite) be the largest time such that the two solutions coincide on $[0, T_m]$. If we denote

$$(\widetilde{h}_i(t), \widetilde{u}_i(t)) := (h_i(t - T_m), u_i(t - T_m)), \qquad i = 1, 2,$$

we can use the above arguments and the fact that $\|\widetilde{h}_i(0)\|_{L^\infty} \le 1/4$ to prove that $(\widetilde{h}_2, \widetilde{u}_2) = (\widetilde{h}_1, \widetilde{u}_1)$ on a suitable small interval $[0, \varepsilon]$ ($\varepsilon > 0$). This completes the proof.

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