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Quasilinearization methods for nonlinear differential-functional parabolic equations: unbounded case

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Abstract. We consider the Cauchy problem for nonlinear parabolic equations with functional dependence represented by the Hale functional acting on the unknown function and its gradient. We prove convergence theorems for a general quasilinearization method in natural subclasses of unbounded solutions.

Introduction. The quasilinearization method belongs to the most effective analytical approximation techniques for a given nonlinear problem. It produces sequences of solutions to problems which are linear with respect to the unknown function. A classical version of quasilinearization methods, known as the Chaplygin method, defines two sequences of upper and lower solutions of the nonlinear problem (see [1]–[3], [7]). One of these sequences coincides with the quasilinearization method. The theory of monotone iterative techniques has been extensively described in the monograph [12].

In order to illustrate the convergence rate of the quasilinearization method, let us consider a simple nonlinear parabolic Cauchy problem without functional dependence.

Example 0.1. Let $n = 1$. We consider the Cauchy problem

$$\partial_t u(t,x) - \partial_{xx} u(t,x) = \sin u(t,x),$$
$$u(0,x) = \varphi(x),$$

where φ is a bounded continuous function. Assume that $u^{(0)}$ is any function such that

$$u^{(0)}(t,x) = \frac{1}{(2\sqrt{\pi}\,t)^n} \int_{\mathbb{R}^n} \exp\left(-\frac{(x-y)^2}{4t}\right) \varphi(y)\,dy.$$

Moreover, if $u^{(\nu)}$ is already defined then $u^{(\nu+1)}$ is a solution of the following

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linear Cauchy problem:

$$\partial_t u(t,x) - \partial_{xx} u(t,x) = \sin u^{(\nu)}(t,x) + \cos u^{(\nu)}(t,x)\cdot(u-u^{(\nu)})(t,x),$$
$$u(0,x) = \varphi(x).$$

It is obvious that there exists a unique bounded continuous solution $u = u^{(\nu+1)}$. Since $u^{(0)}$ satisfies the homogeneous equation

$$\mathcal{P}u^{(0)} = 0, \quad \text{where} \quad \mathcal{P} = \partial_t - \partial_{xx},$$

we get the differential inequality

$$\begin{aligned}|\mathcal{P}(u^{(1)}-u^{(0)})(t,x)| &\le |\sin u^{(0)}(t,x)| + |\cos u^{(0)}(t,x)|\,|(u^{(1)}-u^{(0)})(t,x)|\\ &\le 1 + |(u^{(1)}-u^{(0)})(t,x)|.\end{aligned}$$

Hence $|(u^{(1)}-u^{(0)})(t,x)| \le e^t - 1 \le te^t$ on $[0,a]\times\mathbb{R}$. Similarly, we derive the differential inequalities

$$\begin{aligned}&|\mathcal{P}(u^{(\nu+2)}-u^{(\nu+1)})(t,x)|\\ &\le |\sin u^{(\nu+1)}(t,x) - \sin u^{(\nu)}(t,x) + \cos u^{(\nu)}(t,x)(u^{(\nu+1)}-u^{(\nu)})(t,x)|\\ &\quad + |\cos u^{(\nu+1)}(t,x)|\,|(u^{(\nu+2)}-u^{(\nu+1)})(t,x)|\\ &\le |(u^{(\nu+1)}-u^{(\nu)})(t,x)|^2 + |(u^{(\nu+2)}-u^{(\nu+1)})(t,x)|.\end{aligned}$$

This leads to the integral inequalities

$$\begin{aligned}&\|(u^{(\nu+2)}-u^{(\nu+1)})(t,\cdot)\|\\ &\qquad\le \int_0^t \{\|(u^{(\nu+2)}-u^{(\nu+1)})(s,\cdot)\| + \|(u^{(\nu+1)}-u^{(\nu)})(s,\cdot)\|^2\}\,ds,\end{aligned}$$

where $\|\cdot\|$ stands for the supremum norm. Applying the Gronwall lemma we get

$$\begin{aligned}\|(u^{(\nu+2)}-u^{(\nu+1)})(t,\cdot)\| &\le \int_0^t \|(u^{(\nu+1)}-u^{(\nu)})(s,\cdot)\|^2\,e^{t-s}\,ds\\ &\le te^t \max_{s\in[0,t]} \|(u^{(\nu+1)}-u^{(\nu)})(s,\cdot)\|^2.\end{aligned}$$

From this recurrent inequality, one can prove by induction on ν that

$$\|(u^{(\nu+1)}-u^{(\nu)})(t,\cdot)\| \le (te^t)^{2^{\nu+1}-1} =: \varepsilon_\nu \quad \text{for } \nu = 0,1,\ldots.$$

Let us recall that *fast convergence* of the approximating sequence $\{u^{(\nu)}\}$ to the solution u^* means that

$$\frac{\|u^{(\nu+1)}-u^*\|}{\|u^{(\nu)}-u^*\|} \to 0 \quad \text{as } \nu\to\infty,$$

where $\|\cdot\|$ is the supremum norm.

It is easily seen that the last estimate in Example 0.1 shows fast convergence of successive approximations for $t \in [0, a]$, provided that the interval $[0, a]$ is sufficiently small.

In [5] we have generalized the result sketched in our simple Example 0.1 to differential-functional equations with bounded data and bounded solutions. In the present paper we give sufficient conditions for convergence in the unbounded case with typical a priori estimates of solutions $|u(t,x)| \le C\exp(K|x|^2)$.

The paper is organized as follows. Section 1 contains the formulation of the Cauchy problem and theorems on the existence and uniqueness of solutions in the class of continuous functions satisfying the growth condition $|u(t,x)| \le C\exp(K|x|^2)$. We consider three cases:

(i) the unknown function appears in the functional argument $u_{(t,x)}$ in the equation:
$$\mathcal{P}u(t,x) = f(t,x,u_{(t,x)}),$$
and the fast convergence rate $\varepsilon_{\nu+1}/\varepsilon_\nu \to 0$ refers to the weighted norms
$$\varepsilon_\nu = \sup \frac{|u^{(\nu+1)}(t,x) - u^{(\nu)}(t,x)|}{\exp(\psi(t)|x|^2)},$$

(ii) the functional dependence concerns both the unknown function and its derivative:
$$\mathcal{P}u(t,x) = \widetilde{f}(t,x,u_{(t,x)},\partial_x u_{(t,x)}),$$
and the fast convergence rate $\varepsilon_{\nu+1}/\varepsilon_\nu \to 0$ refers to the weighted norms
$$\varepsilon_\nu = \sup \frac{|u^{(\nu+1)}(t,x) - u^{(\nu)}(t,x)|}{\exp(\psi(t)|x|^2)} + \sup \frac{\|\partial_x u^{(\nu+1)}(t,x) - \partial_x u^{(\nu)}(t,x)\|}{\exp(\psi(t)|x|^2)},$$

(iii) the functional dependence involves the unknown function, but its derivative has the classical form
$$\mathcal{P}u(t,x) = \bar{f}(t,x,u_{(t,x)},\partial_x u(t,x)),$$
and the fast convergence rate $\varepsilon_{\nu+1}/\varepsilon_\nu \to 0$ refers to a weighted norm analogous to (ii), but taking into account the singularities of $\partial_x u$ at $t = 0^+$.

Note that there is a significant difference between cases (ii) and (iii): in case (ii) due to nontrivial functional dependence on $\partial_x u_{(t,x)}$, the assumptions on f and φ are stronger, and the solutions obtained are more regular; in case (iii) the assumptions on f and φ are weaker, and the derivatives $\partial_x u$ may admit singularities at $t = 0^+$.

Similar results concerning the existence and uniqueness of bounded solutions can be found in [6]; compare also, e.g., [17] or [18].

Section 2 contains convergence theorems for the quasilinearization method in cases (i)–(iii) of unbounded solutions of problem (1)–(2). The results of this paper are a natural continuation of [5] and [14].

1. Preliminaries

1.1. *Formulation of the problem.* Set $\mathbb{R}_+ = [0,\infty)$. The Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$. Let $a > 0$, $\tau_0, \tau_1, \ldots, \tau_n \in \mathbb{R}_+$, $\tau = (\tau_1, \ldots, \tau_n)$ and $[-\tau, \tau] = [-\tau_1, \tau_1] \times \cdots \times [-\tau_n, \tau_n]$. Define

$$E = (0,a] \times \mathbb{R}^n, \quad E_0 = [-\tau_0, 0] \times \mathbb{R}^n, \quad \widetilde{E} = E_0 \cup E, \quad B = [-\tau_0, 0] \times [-\tau, \tau].$$

The functional dependence in differential equations will be expressed in terms of Hale's operator. If $u : E_0 \cup E \to \mathbb{R}$ and $(t,x) \in E$, then the Hale-type functional $u_{(t,x)} : B \to \mathbb{R}$ is defined by

$$u_{(t,x)}(s,y) = u(t+s, x+y) \quad \text{for } (s,y) \in B.$$

An analogous one-dimensional model $z_t(s) = z(t+s)$ for $s \in [-\tau_0, 0]$ is well known for ordinary differential-functional equations (see [9]). Using it, one can generalize differential equations with delays, integrals and deviated arguments (see [11]).

Let $C(X)$ (resp. $[C(X)]^n$) be the set of all continuous functions from a metric space X into $\mathbb{R}$ (resp. $\mathbb{R}^n$), and $CB(X)$ the continuous and bounded functions from X into $\mathbb{R}$. The supremum norm in $CB(X)$ will be denoted by $\|\cdot\|$, and the norms and seminorms in function spaces by $\|\cdot\|$ with suitable indices. Let $u(t,x) = u(t, x_1, \ldots, x_n)$ be a sufficiently regular function. Write $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\partial_{jl} = \partial^2/\partial x_j \partial x_l \, (j,l = 1, \ldots, n)$. We also set $\partial_x = (\partial_1, \ldots, \partial_n)$.

Suppose that $f : E \times C(B) \to \mathbb{R}$ and $\varphi : E_0 \to \mathbb{R}$ are given functions. Denote by u an unknown function of the variables $(t,x) = (t, x_1, \ldots, x_n)$, and by $\mathcal{P}$ the differential operator

$$\mathcal{P}u(t,x) = \partial_t u(t,x) - \sum_{j,l=1}^{n} a_{jl}(t,x) \partial_{jl} u(t,x).$$

In this paper we consider the Cauchy problem for a nonhomogeneous differential-functional nonlinear parabolic equation

$$\mathcal{P}u(t,x) = f(t, x, u_{(t,x)}) \tag{1}$$

with the initial condition

$$u(t,x) = \varphi(t,x) \quad \text{on } E_0. \tag{2}$$

Condition (2) will be briefly written as $u \succ \varphi$, to be read as "u extends φ" or "φ is the restriction of u to E_0."

Among particular cases of equation (1) we distinguish the equation

$$\mathcal{P}u(t,x) = 0. \tag{3}$$

We give a basic assumption on the coefficients of the differential operator $\mathcal{P}$, which will be needed throughout the paper.

ASSUMPTION 1.1.

(1) The operator $\mathcal{P}$ is *uniformly parabolic*, i.e. there is $c' > 0$ such that

$$\sum_{j,l=1}^{n} a_{jl}(t,x)\xi_j\xi_l \ge c'|\xi|^2 \quad \text{for all } (t,x) \in E,\ \xi \in \mathbb{R}^n.$$

(2) The coefficients $a_{jl} \in CB(E)$ for $j,l = 1,\dots,n$ satisfy the Hölder condition

$$|a_{jl}(t,x) - a_{jl}(\widetilde{t},\widetilde{x})| \le c''(|t-\widetilde{t}|^{\alpha/2} + |x-\widetilde{x}|^{\alpha}) \quad (j,l = 1,\dots,n)$$

for some constants $c'' > 0$, $\alpha \in (0,1]$.

Under Assumption 1.1, there exists the fundamental solution $\Gamma(t,x;s,y)$ of (3) (see [8], [13]) and we have estimates of Γ and its derivatives.

LEMMA 1.1 ([8, p. 24]). *If Assumption* 1.1 *holds*, *then there are* $k_0, c_0, c_1, c_2 > 0$ *such that*

$$|\Gamma(t,x;s,y)| \le c_0(t-s)^{-n/2}\exp\left(-\frac{k_0|x-y|^2}{4(t-s)}\right),$$

$$|\partial_j\Gamma(t,x;s,y)| \le c_1(t-s)^{-(n+1)/2}\exp\left(-\frac{k_0|x-y|^2}{4(t-s)}\right),$$

$$|\partial_t\Gamma(t,x;s,y)|, |\partial_{jl}\Gamma(t,x;s,y)| \le c_2(t-s)^{-(n+2)/2}\exp\left(-\frac{k_0|x-y|^2}{4(t-s)}\right)$$

for all $0 \le s < t \le a$ *and* $x,y \in \mathbb{R}^n$, $j,l = 1,\dots,n$.

From Lemma 1.1 the following inequalities can be derived:

$$\int_{\mathbb{R}^n} |\Gamma(t,x;s,y)|\,dy \le \widetilde{c}_0, \qquad \int_{\mathbb{R}^n} |\partial_j\Gamma(t,x;s,y)|\,dy \le \widetilde{c}_1(t-s)^{-1/2}$$

for $j = 1,\dots,n$, where $\widetilde{c}_0 = c_0(4\pi/k_0)^{n/2}$, $\widetilde{c}_1 = c_1(4\pi/k_0)^{n/2}$. The constants $\widetilde{c}_0$ and $\widetilde{c}_1$ will be frequently used.

The Cauchy problem (1)–(2) is transformed to the integral equation

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(0,y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y)f(s,y,u_{(s,y)})\,dy\,ds, \tag{4}$$

where $\Gamma(t,x;s,y)$ is the fundamental solution of (3).

1.2. *Existence and uniqueness.* We recall the main result of [4]. Let $L^1[0,a]$ be the set of all real integrable functions on $[0,a]$. If $x, y \in \mathbb{R}^n$, then we denote by $\langle x, y\rangle$ their standard scalar product.

Definition 1.1. Let $u \in C(\widetilde{E})$.

1° The function u is called a C^0 *solution* of problem (1)–(2) if u coincides with φ on E_0 and it satisfies the integral equation (4) on E.
2° The function u is called a $C^{0,1}$ *solution* of problem (1)–(2) if u is a C^0 solution whose derivatives $\partial_j u$ $(j = 1, \dots, n)$ are continuous on E.

The integral equation (4) is known as the *Cauchy formula* and in those terms our C^0 solutions can be seen as weaker than so called "mild solutions" of the differential-functional problem (1)–(2) (see [10], [16]).

Let C^+ be the set of all functions $\psi : [-\tau_0, a] \to (0,\infty)$ such that

(a) ψ is continuous and nondecreasing,
(b) $\psi(t) = \psi(0)$ for $t \in [-\tau_0, 0]$.

For any $\psi \in C^+$ define the linear space

$$\mathcal{X}_{2;\psi} = \{u \in C(\widetilde{E}) : \exists_{c\ge 0}\ \forall_{(t,x)\in\widetilde{E}}\ |u(t,x)| \le c\exp(\psi(t)|x|^2)\}$$

and the corresponding weighted norm

$$\|u\|_{2;\psi} = \sup_{(t,x)\in\widetilde{E}} \frac{|u(t,x)|}{\exp(\psi(t)|x|^2)}.$$

The set $\mathcal{X}_{2;\psi}$ with the norm $\|u\|_{2;\psi}$ is a Banach space.

Now, we formulate our fundamental assumption.

Assumption 1.2.

[ψ] The function $\psi \in C^+$ satisfies the inequality

$$\frac{k_0\psi(s)}{k_0 - 4\psi(s)(t-s)} \le \psi(t) \quad \text{for } 0 \le s \le t \le a, \tag{5}$$

where k_0 is the same constant as in Lemma 1.1.

[f] $f(\cdot, x, w) \in L^1[0,a]$, $f(t,\cdot,w) \in C(\mathbb{R}^n)$ and there is a function $m_f \in L^1[0,a]$ such that

$$|f(t,x,0)| \le m_f(t)\exp(\psi(t)|x|^2) \quad \text{on } E,$$

[φ] $\varphi \in C(E_0)$ and $|\varphi(t,x)| \le K_\varphi \exp(\psi(0)|x|^2)$ for some $K_\varphi > 0$.

Example 1.1. A simple example of a function $\psi \in C^+$ satisfying (5) is

$$\psi(t) = \begin{cases} \dfrac{k_0 C}{k_0 - 4Ct} & \text{for } 0 \le t < \dfrac{k_0}{4C}, \\ C & \text{for } t \le 0, \end{cases}$$

where $C > 0$ is such that $k_0/4C > a$ (see [15]).

We start with a theorem on existence and uniqueness in case (i), i.e. without dependence on $\partial_x u$.

PROPOSITION 1.1 ([4, Theorem 2.3]). *Suppose that Assumption* 1.2 *is satisfied and there are functions* $\lambda, \widetilde{\lambda} \in L^1[0,a]$ *such that*

$$|f(t,x,w) - f(t,x,\overline{w})| \le \lambda(t)\,|w(0,0) - \overline{w}(0,0)| + \widetilde{\lambda}(t)\,\|w - \overline{w}\|\,\exp(-2\psi(t)\langle |x|, \tau\rangle) \quad \text{on } E \times C(B).$$

Then there exists a unique C^0 *solution* u *of problem* (1)–(2) *in the class* $\mathcal{X}_{2;\psi}$.

Now, we consider equations (1) with functionals of the derivatives, which means that the right-hand side may contain not only $u_{(t,x)}$, but also $\partial_j u_{(t,x)}$; in particular $\partial_j u(\alpha(t,x), \beta(t,x))$ and $\int_B K(t,x,s,y,\partial_j u(s,y))\,dy\,ds$.

Define a new Banach space $\mathcal{X}'_{2;\psi} = \{u \in C(\widetilde{E}) : u, \partial_1 u, \ldots, \partial_n u \in \mathcal{X}_{2;\psi}\}$ with the norm

$$\|u\|'_{2;\psi} = \max\{\|u\|_{2;\psi}, \|\partial_1 u\|_{2;\psi}, \ldots, \|\partial_n u\|_{2;\psi}\}.$$

PROPOSITION 1.2 ([4, Theorem 3.1]). *Suppose that Assumption* 1.2 *is satisfied and*

(1) $\partial_x \varphi \in [C(E_0)]^n$ *and* $|\partial_x \varphi(t,x)| \le K_{\varphi'} \exp(\psi(0)|x|^2)$ *for some* $K_{\varphi'} > 0$,
(2) *there are functions* $\lambda, \widetilde{\lambda}, \lambda_1, \widetilde{\lambda}_1 \in L^1[0,a]$ *such that*

   (a) *on* $E \times C(B)$ *we have the estimates*

$$\begin{aligned}|f(t,x,w) - f(t,x,\overline{w})| \le \lambda(t)|w(0,0) - \overline{w}(0,0)| &+ \widetilde{\lambda}(t)\,\|w - \overline{w}\|\exp(-2\psi(t)\langle |x|, \tau\rangle)\\ &+ \lambda_1(t)\|\partial_x(w - \overline{w})(0,0)\|\\ &+ \widetilde{\lambda}_1(t)\|\partial_x(w - \overline{w})\|\exp(-2\psi(t)\langle |x|, \tau\rangle)\end{aligned}$$

   (b) *there are* $\theta \in (0,1)$ *and* $\gamma \in C^+$ *such that for all* $t \in [0,a]$,

$$\int_0^t \widetilde{c}_0 a^{1/2}(t-s)^{-1/2}\gamma(s)\Big\{\lambda(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2) + a^{-1/2}\frac{c_1}{c_0}\,(\lambda_1(s) + \widetilde{\lambda}_1(s)\exp(\psi(s)|\tau|^2))\Big\}\,ds \le \theta\gamma(t).$$

Then there exists a unique $C^{0,1}$ *solution of problem* (1)–(2) *in the class* $\mathcal{X}'_{2;\psi}$.

Now, we give a theorem on the existence and uniqueness of solutions of problem (1)–(2) with the Volterra functional dependence on $u(\cdot)$ and pointwise dependence on $\partial_x u(t,x)$.

We define an operator $S : \{v \in C(E) : \lim_{t\to 0^+} \sqrt{t}\, v(t,x) = 0\} \to C(\widetilde{E})$ by

$$Sv(t,x) = \begin{cases} \sqrt{t}\, v(t,x) & \text{for } t > 0, \\ 0 & \text{for } t \le 0 \end{cases}$$

and a new Banach space $\mathcal{X}''_{2;\psi} = \{u \in C(\widetilde{E}) : u \in \mathcal{X}_{2;\psi},\ \partial_j Su \in \mathcal{X}_{2;\psi}\}$ with the norm

$$\|u\|''_{2;\psi} = \max\{\|u\|_{2;\psi}, \|\partial_{x_1} Su\|_{2;\psi}, \ldots, \|\partial_{x_n} Su\|_{2;\psi}\}.$$

PROPOSITION 1.3 ([4, Theorem 3.5]). *Suppose that Assumption* 1.2 *is satisfied and there are functions* $\lambda, \lambda_1, \widetilde{\lambda} \in L^1[0,a]$ *such that*

(1) *on* $E \times C(B)$ *we have the estimates*

$$|f(t,x,w) - f(t,x,\overline{w})| \le \lambda(t)|w(0,0) - \overline{w}(0,0)| + \widetilde{\lambda}(t)\|w - \overline{w}\| \exp(-2\psi(t)\langle |x|, \tau\rangle) + \lambda_1(t)\|\partial_x(w-\overline{w})(0,0)\|,$$

(2) *there are* $\theta \in (0,1)$ *and* $\gamma \in C^+$ *such that for all* $t \in [0,a]$,

$$\int_0^t \widetilde{c}_0 \sqrt{t}\,(t-s)^{-1/2}\gamma(s)\Big\{\lambda(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2) + \frac{c_1}{c_0}\,(\lambda_1(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2))\Big\}\, ds \le \theta\gamma(t).$$

Then there exists a unique $C^{0,1}$ *solution* u *of problem* (1)–(2) *in* $\mathcal{X}''_{2;\psi}$.

Now we state an auxiliary lemma.

LEMMA 1.2 ([15, Lemma 1.2]). *If* $0 \le B < A$, *then*

$$\int_{\mathbb{R}^n} \exp(-A|x-y|^2 + B|y|^2)\, dy = \Big(\frac{\pi}{A-B}\Big)^{n/2} \exp\Big(\frac{AB}{A-B}\,|x|^2\Big).$$

2. The quasilinearization method. In the quasilinearization method one constructs a sequence $\{u^{(\nu)}\}$ such that $u^{(0)} \in C(\widetilde{E})$ is given and $u^{(\nu+1)} \in C(\widetilde{E})$ is a solution of the Cauchy problem

$$\mathcal{P}u(t,x) = f(t,x,u^{(\nu)}_{(t,x)}) + \partial_w f(t,x,u^{(\nu)}_{(t,x)}) \cdot (u - u^{(\nu)})_{(t,x)} \tag{6}$$

$$u(t,x) = \varphi(t,x) \quad \text{on } E_0, \tag{7}$$

where $\partial_w f(t,x,u^{(\nu)}_{(t,x)})$ stands for the Fréchet derivative with respect to the functional variable. Observe that equation (6) is still differential-functional, but its right-hand side is linear with respect to u.

The convergence of the sequence $\{u^{(\nu)}\}$ depends on the initial function $u^{(0)}$ and regularity of the operator $\partial_w f$.

The functions $u = u^{(\nu+1)}$ defined as the solutions of (6)–(7) satisfy the integral formula

$$(8)\qquad u^{(\nu+1)}(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(0,y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y) \times\{f(s,y,u^{(\nu)}_{(s,y)}) + \partial_w f(s,y,u^{(\nu)}_{(s,y)})\cdot(u^{(\nu+1)} - u^{(\nu)})_{(s,y)}\}\,dy\,ds.$$

We obtain the above equation by replacing the function f in (4) by the right-hand side of (6).

We are now able to state the main result on the convergence of the quasilinearization method. First, we make further assumptions on $\partial_w f$. They feature a functional $\Lambda : C(B) \to \mathbb{R}$ that may have one of the forms below:

$$(f1)\qquad \Lambda(t,x)h = \lambda(t)|h(0,0)| + \widetilde{\lambda}(t)\|h\|\exp(-2\psi(t)\langle|x|,\tau\rangle)$$

for $h \in C(B)$ and for some $\lambda, \widetilde{\lambda} \in L^1[0,a]$;

$$(f2)\qquad \Lambda(t,x)h = \lambda(t)|h(0,0)| + \widetilde{\lambda}(t)\|h\|\exp(-2\psi(t)\langle|x|,\tau\rangle) + \lambda(t)\|\partial_x h(0,0)\| + \widetilde{\lambda}(t)\|\partial_x h\|\exp(-2\psi(t)\langle|x|,\tau\rangle)$$

for $h \in C(B)$, $\partial_x h(0,\cdot) \in C([-\tau,\tau])$ and for some $\lambda, \widetilde{\lambda} \in L^1[0,a]$;

$$(f3)\qquad \Lambda(t,x)h = \lambda(t)|h(0,0)| + \widetilde{\lambda}(t)\|h\|\exp(-2\psi(t)\langle|x|,\tau\rangle) + \lambda_1(t)\|\partial_x h(0,0)\|$$

for $h \in C(B)$, $\partial_x h(0,\cdot) \in C([-\tau,\tau])$ and for some $\lambda, \widetilde{\lambda}, \lambda_1 \in L^1[0,a]$.

Therefore we obtain three different assumptions on $\partial_w f$:

Assumption 2.i. There is a functional $\Lambda : C(B) \to \mathbb{R}$ such that

(1) Λ has the form (fi),
(2) $|\partial_w f(t,x,w)\,h| \le \Lambda(t,x)h$ for $h \in C(B)$, $(t,x,w) \in E \times C(B)$,
(3) there is a function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, integrable with respect to the first variable, continuous and nondecreasing with respect to the last variable, such that $\sigma(t,0) = 0$ and

$$|[\partial_w f(t,x,w) - \partial_w f(t,x,\overline{w})]h| \le \Lambda(t,x)h \cdot \sigma(t, \exp(-\psi(t)|x|^2)\Lambda(t,x)(w - \overline{w}))$$

on $E \times C(B)$.

Theorem 2.1. *Suppose that Assumptions* 1.2 *and* 2.1 *are satisfied and there exists a continuous*, *nondecreasing function* $\psi_0 : [0,a] \to \mathbb{R}_+$ *which satisfies the inequalities*

$$(9)\qquad \psi_0(t) \ge \|(u^{(1)} - u^{(0)})|_{E_t}\|_{2;\psi},$$

$$(10)\qquad \psi_0(t) \ge \widetilde{c}_0\,[\psi(t)]^{n/2} \int_0^t \psi_0(s)[\psi(s)]^{-n/2}\,\widehat{\lambda}(s)\{1 + \sigma(s,\psi_0(s)\widehat{\lambda}(s))\}\,ds,$$

where $\widehat{\lambda}(s) := \lambda(s)+\widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2)$. *Then the sequence* $\{u^{(\nu)}\}$ *of solutions of problem* (6)–(7) *is well defined and almost uniformly fast convergent to* u^*, *where* u^* *is the unique unbounded* C^0 *solution of* (1)–(2). *The convergence rate is characterized by the condition*

$$\frac{\|(u^{(\nu+1)}-u^*)|_{E_t}\|_{2;\psi}}{\|(u^{(\nu)}-u^*)|_{E_t}\|_{2;\psi}} \to 0 \quad \text{as } \nu\to\infty \tag{11}$$

for $t\in[0,a]$.

REMARK 2.1. Recall that the almost uniform fast convergence means uniform convergence on compact subsets with the convergence rate satisfying $\varepsilon_{\nu+1}/\varepsilon_\nu \to 0$ as $\nu\to\infty$.

Proof of Theorem 2.1. The proof will be given in several stages. First, we observe that the existence and uniqueness of a solution of (6)–(7) follows from Proposition 1.1. Now, we estimate the differences $u^{(\nu+1)}-u^{(\nu)}$ for $\nu=0,1,\ldots$. Put $\omega^{(\nu)}=u^{(\nu+1)}-u^{(\nu)}$. Since $u^{(\nu+1)}$ satisfies the integral identity (8), and so also does $u^{(\nu+2)}$, we have the integral error equation

$$\begin{aligned}\omega^{(\nu+1)}(t,x) = \int_0^t\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\{&f(s,y,u^{(\nu+1)}_{(s,y)}) - f(s,y,u^{(\nu)}_{(s,y)})\\ &+\partial_w f(s,y,u^{(\nu+1)}_{(s,y)})\omega^{(\nu+1)}_{(s,y)} - \partial_w f(s,y,u^{(\nu)}_{(s,y)})\omega^{(\nu)}_{(s,y)}\}\,dy\,ds.\end{aligned}$$

By the Hadamard mean-value theorem, we get

$$f(s,y,u^{(\nu+1)}_{(s,y)}) - f(s,y,u^{(\nu)}_{(s,y)}) = \int_0^1 \partial_w f(s,y,u^{(\nu)}_{(s,y)}+\zeta\omega^{(\nu)}_{(s,y)})\,d\zeta\,\omega^{(\nu)}_{(s,y)}.$$

Hence, we rewrite the error equation as follows:

$$\begin{aligned}\omega^{(\nu+1)}(t,x) = \int_0^t\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\Big\{&\partial_w f(s,y,u^{(\nu+1)}_{(s,y)})\,\omega^{(\nu+1)}_{(s,y)}\\ &+\int_0^1 \partial_w f(s,y,u^{(\nu)}_{(s,y)}+\zeta\omega^{(\nu)}_{(s,y)})\omega^{(\nu)}_{(s,y)}\,d\zeta - \partial_w f(s,y,u^{(\nu)}_{(s,y)})\omega^{(\nu)}_{(s,y)}\Big\}\,dy\,ds.\end{aligned}$$

From this equation, based on Assumption 2.1, we derive

$$\begin{aligned}|\omega^{(\nu+1)}(t,x)| \le \int_0^t\int_{\mathbb{R}^n} |\Gamma(t,x;s,y)|\,\Big\{&|\partial_w f(t,x,u^{(\nu+1)}_{(s,y)})|\,\|\omega^{(\nu+1)}_{(s,y)}\|\\ &+\|\omega^{(\nu)}_{(s,y)}\|\int_0^1 |\partial_w f(s,y,u^{(\nu)}_{(s,y)}+\zeta\omega^{(\nu)}_{(s,y)}) - \partial_w f(t,x,u^{(\nu)}_{(s,y)})|\,d\zeta\Big\}\,dy\,ds\end{aligned}$$

$$\leq \int_0^t \int_{\mathbb{R}^n} |\Gamma(t,x;s,y)| \Big\{ \lambda(s)|\omega^{(\nu+1)}(s,y)| + \widetilde{\lambda}(s)\|\omega^{(\nu+1)}_{(s,y)}\| \exp(-2\psi(s)\langle |y|,\tau\rangle)$$

$$+ \int_0^1 [\lambda(s)\,|\omega^{(\nu)}(s,y)| + \widetilde{\lambda}(s)\|\omega^{(\nu)}_{(s,y)}\| \exp(-2\psi(s)\langle |y|,\tau\rangle)$$

$$\times\, \sigma(s, \lambda(s)|\zeta\omega^{(\nu)}(s,y)| \exp(-\psi(s)|y|^2) + \widetilde{\lambda}(s)\,\|\zeta\omega^{(\nu)}_{(s,y)}\|$$

$$\times\, \exp(-\psi(s)|\,|y|+\tau|^2))]\, d\zeta \Big\}\, dy\, ds.$$

By the monotonicity of σ and the inequalities

$$|u(s,y)| \leq \|u|_{E_s}\|_{2;\psi} \exp(\psi(s)|y|^2), \qquad |u_{(s,y)}| \leq \|u|_{E_s}\|_{2;\psi} \exp(\psi(s)|\,|y|+\tau|^2),$$

we have

$$|\omega^{(\nu+1)}(t,x)| \leq \int_0^t \int_{\mathbb{R}^n} |\Gamma(t,x;s,y)| \exp(\psi(s)|y|^2)\widehat{\lambda}(s)$$

$$\times \{\|\omega^{(\nu+1)}|_{E_s}\|_{2;\psi} + \|\omega^{(\nu)}|_{E_s}\|_{2;\psi}\, \sigma(s, \widehat{\lambda}(s)\|\omega^{(\nu)}|_{E_s}\|_{2;\psi})\}\, dy\, ds,$$

where $\widehat{\lambda}(s) = \lambda(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2)$. Using the estimate of Γ and Lemma 1.2 we get the recurrent integral inequality

$$\|\omega^{(\nu+1)}|_{E_t}\|_{2;\psi} \leq \widetilde{c}_0[\psi(t)]^{n/2} \int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s)\{\|\omega^{(\nu+1)}|_{E_s}\|_{2;\psi}$$

$$+ \|\omega^{(\nu)}|_{E_s}\|_{2;\psi}\, \sigma(s, \|\omega^{(\nu)}|_{E_s}\|_{2;\psi}\widehat{\lambda}(s))\}\, ds \leq \psi_{\nu+1}(t),$$

where $\psi_{\nu+1} : [0,a] \to (0,\infty)$ for $\nu = 0,1\ldots$ is defined by

$$\psi_{\nu+1}(t) = \widetilde{c}_0[\psi(t)]^{n/2}\int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s), \psi_\nu(s)\, \sigma(s, \psi_\nu(s)\widehat{\lambda}(s)) \tag{12}$$

$$\times\, \exp\Big(\widetilde{c}_0[\psi(t)]^{n/2} \int_s^t [\psi(\tau)]^{-n/2}\widehat{\lambda}(\tau)\, d\tau\Big)\, ds.$$

Applying the Gronwall lemma, we get

$$\|\omega^{(\nu+1)}|_{E_t}\|_{2;\psi} \leq \widetilde{c}_0[\psi(t)]^{n/2}\int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s)\|\omega^{(\nu)}|_{E_s}\|_{2;\psi}\, \sigma(s, \widehat{\lambda}(s)\|\omega^{(\nu)}|_{E_s}\|_{2;\psi})$$

$$\times\, \exp\Big(\widetilde{c}_0\, [\psi(t)]^{n/2}\int_s^t [\psi(\tau)]^{-n/2}\, \widehat{\lambda}(\tau)\, d\tau\Big)\, ds.$$

We now show that $\{\omega^{(\nu)}\}$ is uniformly convergent to 0. It is easy to verify that the sequence $\{\psi_\nu\}$ of continuous nondecreasing functions is nonincreasing as $\nu \to \infty$. This can be verified by induction on ν, applying the inequality

$$\psi_0(t)[\psi(t)]^{n/2} \geq \widetilde{c}_0 \int_0^t \psi_0(s)[\psi(s)]^{-n/2}\widehat{\lambda}(s)\{1+\sigma(s,\psi_0(s)\widehat{\lambda}(s))\}\,ds.$$

Furthermore, using once more induction on ν and (13) we have

$$\|\omega^{(\nu+1)}|_{E_t}\|_{2;\psi} \leq \psi_{\nu+1}(t) \quad \text{for all } \nu = 0,1,\ldots. \tag{13}$$

From this we deduce that $\{\psi_\nu\}$ converges to a limit function $\overline{\psi}$, where $0 \leq \overline{\psi}(t) \leq \psi_0(t)$. Letting $\nu \to \infty$ in (12), we get

$$\begin{aligned}0 \leq \overline{\psi}(t) = \widetilde{c}_0[\psi(t)]^{n/2} \int_0^t \overline{\psi}(s)[\psi(s)]^{-n/2}\widehat{\lambda}(s)\sigma(s,\psi(a)\widehat{\lambda}(s))\\ \times\ \exp\Big(\widetilde{c}_0[\psi(t)]^{n/2}\int_s^t [\psi(\tau)]^{-n/2}\widehat{\lambda}(\tau)\,d\tau\Big)\,ds.\end{aligned}$$

By Gronwall's lemma, we have $\overline{\psi} \equiv 0$. Since ψ_ν are nondecreasing functions and (12) holds, we have

$$\begin{aligned}\frac{\psi_{\nu+1}(t)}{\psi_\nu(t)} \leq \widetilde{c}_0[\psi(t)]^{n/2} \int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s)\sigma(s,\psi_\nu(s)\widehat{\lambda}(s))\\ \times \exp\Big(\widetilde{c}_0[\psi(t)]^{n/2}\int_s^t [\psi(\tau)]^{-n/2}\widehat{\lambda}(\tau)\,d\tau\Big)\,ds.\end{aligned} \tag{14}$$

Recalling that $\sigma(s,\cdot)$ is continuous and monotone, we observe that

$$\sigma(s,\psi_\nu(s)) \searrow 0 = \sigma(s,0) \quad \text{as } \nu \to \infty.$$

Since $\psi_\nu \searrow 0$ as $\nu \to \infty$, passing to the limit in (14) we get

$$\frac{\psi_{\nu+1}(t)}{\psi_\nu(t)} \to 0 \quad \text{as } \nu \to \infty.$$

Hence by d'Alembert's criterion, the series $\sum_{\nu=0}^{\infty} \psi_\nu(t)$ is uniformly convergent. Since $\|\omega^{(\nu)}|_{E_t}\|_{2;\psi} \leq \psi_\nu(t)$, $\{u^{(\nu)}\}$ is a Cauchy sequence. Indeed,

$$\begin{aligned}&\|(u^{(\nu)} - u^{(\nu+k)})|_{E_t}\|_{2;\psi}\\ &\quad\leq \|(u^{(\nu)} - u^{(\nu+1)})|_{E_t}\|_{2;\psi} + \cdots + \|(u^{(\nu+k-1)} - u^{(\nu+k)})|_{E_t}\|_{2;\psi}\\ &\quad\leq \psi_\nu(t) + \cdots + \psi_{\nu+k}(t).\end{aligned}$$

Consequently, $\{u^{(\nu)}\}$ uniformly converges to a continuous function u^*. We now prove that u^* satisfies (1). The initial condition (2), that is, $u^* \succ \varphi$, is fulfilled, because $u^{(\nu)} \succ \varphi$ and $u^{(\nu)} \to u^*$ as $\nu \to \infty$. It suffices to make the following observation. The integral equation (8) for the functions $u^{(\nu)}$ and $u = u^{(\nu+1)}$ is equivalent to problem (6)–(7). Then letting $\nu \to \infty$ in (8) we obtain the integral equality (4) with $u = u^*$. By Proposition 1.1, u^* is the unique solution of problem (1)–(2). The convergence rate is determined by estimate (13) and condition (14). This convergence is faster than geometric.

Now, we show that condition (11) is satisfied. Subtracting (4) with $u = u^*$ from (8) and performing similar estimations as in the case of $\omega^{(\nu+1)}$, we get

$$\|(u^{(\nu+1)}-u^*)|_{E_t}\|_{2;\psi} \le \widetilde{c}_0[\psi(t)]^{n/2}\int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s)\{\|(u^{(\nu+1)} - u^*)|_{E_s}\|_{2;\psi} + \|(u^{(\nu)}-u^*)|_{E_s}\|_{2;\psi}\,\sigma(s,\widehat{\lambda}(s)\|(u^{(\nu)}-u^*)_{E_s}\|_{2;\psi})\}\,ds.$$

By Gronwall's lemma, we have

$$\|(u^{(\nu+1)} - u^*)|_{E_t}\|_{2;\psi} \le \widetilde{c}_0[\psi(t)]^{n/2}\int_0^t [\psi(s)]^{-n/2}\|(u^{(\nu)} - u^*)_{E_s}\|_{2;\psi} \times \sigma(s, \|(u^{(\nu)} - u^*)|_{E_s}\|_{2;\psi}) \exp\Big(\widetilde{c}_0[\psi(t)]^{n/2}\int_s^t [\psi(\tau)]^{-n/2}\widehat{\lambda}(\tau)\,d\tau\Big)\,ds.$$

Since the seminorm scale $\|\cdot|_{E_t}\|_{2;\psi}$ is nondecreasing in t, we get

$$\frac{\|(u^{(\nu+1)} - u^*)|_{E_t}\|_{2;\psi}}{\|(u^{(\nu)} - u^*)|_{E_t}\|_{2;\psi}} \le \widetilde{c}_0\,[\psi(t)]^{n/2}\int_0^t [\psi(s)]^{-n/2}\widehat{\lambda}(s)\sigma(s, \|(u^{(\nu)} - u^*)|_{E_s}\|_{2;\psi}) \times\ \exp\Big(\widetilde{c}_0[\psi(t)]^{n/2}\int_s^t [\psi(\tau)]^{-n/2}\widehat{\lambda}(\tau)\,d\tau\Big)\,ds \to 0$$

as $\nu \to \infty$. This completes the proof of (11) and of Theorem 2.1. ∎

REMARK 2.2. Inequality (10) has a local solution. If the interval $[0,a]$ is sufficiently small, then there exists a solution of (10) which satisfies (9). In particular, if we put $\sigma(s,r) = L$ (or $L(s)$), then a solution of (10) exists on the whole interval $[0,a]$. If $\sigma(s,r) = Lr$, then condition (10) has the Riccati form.

Now, we discuss case (ii).

THEOREM 2.2. *Suppose that Assumptions* 1.2 *and* 2.2 *are satisfied and*

1) $\partial_x\varphi \in [C(E_0)]^n$ *and* $|\partial_x\varphi(t,x)| \le K_{\varphi'}\exp(\psi(0)|x|^2)$ *for some* $K_{\varphi'} > 0$,
2) *there are* $\theta \in (0,1)$ *and* $\gamma \in C^+$ *such that for all* $t \in [0,a]$,

$$\int_0^t \{\widetilde{c}_0\sqrt{a}\,(t-s)^{-1/2} + \widetilde{c}_1\}\gamma(s)\widehat{\lambda}(s)\,ds \le \theta\,\gamma(t),$$

3) *there exists a nondecreasing, continuous function* $\psi_0 : [0,a] \to \mathbb{R}_+$

which satisfies the inequalities

$$\psi_0(t) \geq \|(u^{(1)} - u^{(0)})|_{E_t}\|'_{2;\psi},$$

$$\psi_0(t) \geq [\psi(t)]^{n/2} \int_0^t \{\widetilde{c}_0 + \widetilde{c}_1 (t-s)^{-1/2}\} [\psi(s)]^{-n/2} \widehat{\lambda}(s) \times \{1 + \sigma(s, \psi_0(s)\widehat{\lambda}(s))\} \psi_0(s)\, ds,$$

where $\widehat{\lambda}(s) := \lambda(s) + \widetilde{\lambda}(s) \exp(\psi(s)|\tau|^2)$.

Then the sequence $\{u^{(\nu)}\}$ *of solutions of* (6)–(7) *is well defined and uniformly fast convergent to* u^* *with respect to the seminorms* $\|\cdot|_{E_t}\|'_{2;\psi}$, *where* u^* *is a unique unbounded* $C^{0,1}$ *solution of problem* (1)–(2) *in the class* $\mathcal{X}'_{2;\psi}$. *Furthermore*,

$$\frac{\|(u^{(\nu+1)} - u^*)|_{E_t}\|'_{2;\psi}}{\|(u^{(\nu)} - u^*)|_{E_t}\|'_{2;\psi}} \to 0 \quad \text{as } \nu \to \infty$$

for $t \in (0, a]$.

Proof. We only give the main ideas of the proof. Observe that

$$\begin{aligned} &\|\omega^{(\nu+1)}|_{E_t}\|_{2;\psi} \\ &\quad \leq \widetilde{c}_0 [\psi(t)]^{n/2} \int_0^t [\psi(s)]^{-n/2} \widehat{\lambda}(s) \{\|\omega^{(\nu+1)}|_{E_s}\|_{2;\psi} + \|\partial_x \omega^{(\nu+1)}|_{E_s}\|_{2;\psi} \\ &\qquad + [\|\omega^{(\nu)}|_{E_s}\|_{2;\psi} + \|\partial_x \omega^{(\nu)}|_{E_s}\|_{2;\psi}] \sigma(s, \|\omega^{(\nu)}|_{E_s}\|_{2;\psi} \widehat{\lambda}(s) \\ &\qquad + \|\partial_x \omega^{(\nu)}|_{E_s}\|_{2;\psi} \widehat{\lambda}(s))\}\, ds. \end{aligned}$$

Introducing the seminorms

$$\|\cdot|_{E_t}\|'_{2;\psi} = \|\cdot|_{E_t}\|_{2;\psi} + \|\partial_x(\cdot|_{E_t})\|_{2;\psi},$$

we can write the above inequalities in the following way:

$$\begin{aligned} \|\omega^{(\nu+1)}|_{E_t}\|_{2;\psi} &\leq \widetilde{c}_0 [\psi(t)]^{n/2} \int_0^t [\psi(s)]^{-n/2} \widehat{\lambda}(s) \\ &\quad \times \{\|\omega^{(\nu+1)}|_{E_s}\|'_{2;\psi} + \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi} \sigma(s, \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi} \widehat{\lambda}(s))\}\, ds \end{aligned}$$

and

$$\begin{aligned} \|\partial_x \omega^{(\nu+1)}|_{E_t}\|_{2;\psi} &\leq \widetilde{c}_1\, [\psi(t)]^{n/2} \int_0^t (t-s)^{-1/2} [\psi(s)]^{-n/2} \widehat{\lambda}(s) \\ &\quad \times \{\|\omega^{(\nu+1)}|_{E_s}\|'_{2;\psi} + \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi} \sigma(s, \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi} \widehat{\lambda}(s))\}\, ds. \end{aligned}$$

Adding these inequalities and applying the definition of $\|\cdot|_{E_t}\|'_{2;\psi}$ we get

$$\|\omega^{(\nu+1)}|_{E_t}\|'_{2;\psi} \le [\psi(t)]^{n/2}\int_0^t \{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\}[\psi(s)]^{-n/2}\,\widehat{\lambda}(s) \times \{\|\omega^{(\nu+1)}|_{E_s}\|'_{2;\psi} + \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi}\,\sigma(s, \|\omega^{(\nu)}|_{E_s}\|'_{2;\psi}\widehat{\lambda}(s))\}\,ds.$$

Now, we define $\psi_{\nu+1} : [0,a] \to \mathbb{R}_+$ by

$$\psi_{\nu+1}(t) = [\psi(t)]^{n/2}\int_0^t \{\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}\}[\psi(s)]^{-n/2}\,\widehat{\lambda}(s) \times \{\psi_{\nu+1}(s) + \psi_\nu(s)\sigma(s, \psi_\nu(s)\widehat{\lambda}(s))\}\,ds,$$

and repeat the arguments in the proof of the previous theorem. ∎

REMARK 2.3. The function Λ of Theorem 2.2 can be replaced by a more general one,

$$\widetilde{\Lambda}(t,x)h = \lambda(t)|h(0,0)| + \widetilde{\lambda}(t)\|h\|\exp(-2\psi(t)\langle|x|,\tau\rangle) + \lambda_1(t)\|\partial_x h(0,0)\| + \widetilde{\lambda}_1(t)\|\partial_x h\|\exp(-2\psi(t)\langle|x|,\tau\rangle)$$

for $\lambda, \lambda_1, \widetilde{\lambda}, \widetilde{\lambda}_1 \in L^1[0,a]$. The proof of such a generalization of Theorem 2.2 is more technical and complicated. In fact, it can be reduced to Theorem 2.2 on a shorter interval $[0,a]$ by taking a new λ, equal to the sum of λ, λ_1, $\widetilde{\lambda}$, $\widetilde{\lambda}_1$.

It is easy to formulate results on the convergence of the quasilinearization method for unbounded solutions in case (iii). The proof is based on the same idea as for the previous theorems, the main difference being that the weaker assumptions on φ and f lead to singularities of $\partial_x u$ at $t = 0^+$ and the norm $\|\cdot\|''_{2,\psi}$ takes account of these singularities.

THEOREM 2.3. *Suppose that Assumptions* 1.2 *and* 2.3 *are satisfied and*

1) *there are $\theta \in (0,1)$ and $\gamma \in C^+$ such that for all $t \in [0,a]$,*

$$\int_0^t \widetilde{c}_0\sqrt{t}\,(t-s)^{-1/2}\gamma(s)\Big\{\lambda(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2) + \frac{c_1}{c_0}\,(\lambda_1(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2))\Big\}\,ds \le \theta\gamma(t),$$

2) *there exists a nondecreasing continuous function $\psi_0 : [0,a] \to \mathbb{R}_+$ such*

that

$$\psi_0(t) \geq \|(u^{(1)} - u^{(0)})|_{E_t}\|''_{2;\psi},$$

$$\psi_0(t) \geq [\psi(t)]^{n/2} \int_0^t \{\widetilde{c}_0 + \sqrt{t}\,\widetilde{c}_1(t-s)^{-1/2}\}[\psi(s)]^{-n/2}\widehat{\lambda}_1(s) \times \{1 + \sigma(s, \psi_0(s)\widehat{\lambda}_1(s))\}\psi_0(s)\,ds,$$

where $\lambda_1(t) = \lambda(t)\sqrt{t}$ *and* $\widehat{\lambda}_1(t) = \lambda(t) + \widetilde{\lambda}(t)\exp(\psi(t)|x|^2) + \lambda_1(t)$.

Then the sequence $\{u^{(\nu)}\}$ *of solutions of* (6)–(7) *is well defined and uniformly fast convergent to* u^* *with respect to the seminorms* $\|\cdot|_{E_t}\|''_{2;\psi}$, *where* u^* *is a unique unbounded* $C^{0,1}$ *solution of problem* (1)–(2) *in the class* $\mathcal{X}''_{2;\psi}$. *Moreover*,

$$\frac{\|(u^{(\nu+1)} - u^*)|_{E_t}\|''_{2;\psi}}{\|(u^{(\nu)} - u^*)|_{E_t}\|''_{2;\psi}} \to 0 \quad \text{as } \nu \to \infty$$

for $t \in (0, a]$.

The proof of the above theorem is similar to the proof of Theorem 2.1. We omit the details.

Remark 2.4. Theorems 2.1–2.3 have a similar structure with different functionals Λ which estimate the derivative $\partial_w f$. If Λ has the form $(f1)$, then $\partial_w f$ is weighted Lipschitzean with respect to $u(t,x)$ and $u_{(t,x)}$. If Λ has the form $(f2)$ or $(f3)$, then the right-hand side is additionally Lipschitz continuous with respect to $\partial_x u(t,x)$, and (only in the case $(f2)$) it is weighted Lipschitzean with respect to $\partial_x u_{(t,x)}$. The same comment is valid concerning the differences of $\partial_w f$.

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