On triple curves through a rational triple point of a surface

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Abstract. Let \( k \) be an algebraically closed field of characteristic 0. Let \( C \) be an irreducible nonsingular curve in \( \mathbb{P}^n \) such that \( 3C = S \cap F \), where \( S \) is a hypersurface and \( F \) is a surface in \( \mathbb{P}^n \) and \( F \) has rational triple points. We classify the rational triple points through which such a curve \( C \) can pass (Theorem 1.8), and give an example (1.12). We only consider reduced and irreducible surfaces.

On curves passing through rational triple points of surfaces

Definition 1.1. Let \( F \) be a reduced surface and \( P \) a point of \( F \). Let \((F, P)\) be a surface singularity (that is, the spectrum of an equicharacteristic local noetherian complete ring of Krull dimension 2, without zero divisors, whose closed point \( P \) is singular). Let \( \pi : \tilde{F} \to F \) be the minimal desingularization of \( F \) at \( P \). The genus of a normal singularity \( P \) is defined to be \( \dim_k (R^1 \pi_* \mathcal{O}_{\tilde{F}})_P \). If the genus is 0, the singularity is said to be rational. A rational singularity, \( P \), such that the multiplicity of the maximal ideal of the local ring \( \mathcal{O}_{F, P} \) is 3, is called a rational triple point. We are going to use configurations of dots and \( x \) \((\bullet^2 = -2, \ x^2 = -3)\) as vertices of the dual graph of the minimal desingularization of the singularity; each vertex corresponds to a curve and each arc to an intersection [1, p. 135]. We list the following singularity types of \( P \):

1. \( X_{ijk}, i, j, k \geq 1; i \) denotes the number of dots \( \bullet \) to the left of \( x \), \( j \) the number of \( \bullet \)'s above \( x \), and \( k \) the number of \( \bullet \)'s to the right of \( x \).

\[
\bullet \ \ \ \ \bullet \ \\
\bullet \ \\
\bullet \ \\ x \ \\
\bullet \ \\
\bullet \ \\
\bullet \ \\
\bullet
\]

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(2) \(Y_i, i \geq 1; i\) denotes the number of \(\bullet\)'s to the left of \(x\).

\[ \bullet \cdots \bullet \bullet \bullet \bullet \bullet \]

(3) \(R_{ij}, i,j \geq 1; i\) denotes the number of \(\bullet\)'s to the left of \(x\) and \(j\) the number of \(\bullet\) to the right of the vertical edge of the graph.

\[ \bullet \cdots \bullet x \bullet \bullet \bullet \cdots \bullet \]

(4) \(T_i, i \geq 1; i\) denotes the number of \(\bullet\)'s to the left of \(x\).

\[ \bullet \cdots \bullet x \bullet \bullet \bullet \bullet \bullet \]

(5) \(U_{ij}, i \geq 1, j \geq 2; i\) denotes the number of \(\bullet\)'s to the left of \(x\), and \(j\) the number of \(\bullet\)'s between \(x\) and the vertical edge of the graph.

\[ \bullet \cdots \bullet x \bullet \bullet \bullet \cdots \bullet \bullet \bullet \]

(6) \(V_i, i \geq 1; i\) denotes the number of \(\bullet\)'s to the left of the vertical edge of the graph.

\[ \bullet \cdots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

(7) \(W_2\).

\[ x \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

(8) \(W_3\).

\[ x \bullet \bullet \bullet \bullet \bullet \bullet \]

(9) \(W_4\).

\[ x \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

1.2. Notation. Let \(F\) be a reduced surface and \(P\) a point of \(F\). Let \((F, P)\) be a surface singularity (that is, the spectrum of an equicharacteristic complete local ring of Krull dimension 2 whose closed point \(P\) is singular). Let \(\text{Reg}(F)\) denote the regular locus of \((F, P)\). Let \(\mathcal{L}\) be the set of smooth curves \(\Gamma\) on \((F, P)\) whose generic point lies on \(\text{Reg}(F)\). Let \(\pi : \widetilde{F} \to F\) be the minimal desingularization of \(F\) at \(P\). Let \(\Phi : \mathcal{L} \to \pi^{-1}(P)\) be the map of sets which sends \(\Gamma \in \mathcal{L}\) to the exceptional point of its strict transform \(\Gamma_{\widetilde{F}}\) on \(\widetilde{F}\) (see [2]).
**Definition 1.3.**

(1) The **maximal cycle** is the cycle \( Z_{\tilde{F}} = \sum m_i E_i \), defined by the divisorial part of \( \mathcal{MO}_{\tilde{F}} \), where \( \mathcal{M} \) is the maximal ideal \( \text{Max} \mathcal{O}_{F,P} \) of \( \mathcal{O}_{F,P} \); the \( E_i \) are the irreducible components of dimension 1 of the exceptional fiber \( \pi^{-1}(P) \) and the \( m_i \) are nonnegative integers. A component \( E_j \) such that \( m_j = 1 \) is called a **reduced component** of the cycle.

(2) Consider positive cycles \( Z = \sum r_i E_i, r_i \geq 0 \), such that \( (Z, E_i) \leq 0 \) for all \( i \).

The unique componentwise smallest cycle \( Z \) satisfying this condition is called the **fundamental cycle** of \( \tilde{F} \).

**Proposition 1.4** (see [2, 1.2]). Let \( (F, P) \) be a complete surface singularity. For any irreducible component \( E \) of \( \pi^{-1}(P) \), let \( \text{ord}_E \) denote the divisorial valuation of the function field of \( (F, P) \) given by the filtration of \( \mathcal{O}_{F,E} \) by the powers of its maximal ideal. The components \( E \) such that \( L_E := \{ \Gamma \in \mathcal{L} \mid \Phi_{\tilde{F}}(\Gamma) \in E \} \neq \emptyset \) are those for which \( \text{ord}_E(\mathcal{MO}_{\tilde{F}}) = 1 \). The set \( \mathcal{L} \) is the disjoint union of the \( L_E \).

**Lemma 1.5** (see [2, 1.14]). The families of smooth curves on a normal surface singularity are in one-to-one correspondence with the reduced components of the maximal cycle of its minimal desingularization \( \pi \).

**Note 1.5.1.** For a rational surface singularity, the maximal cycle of \( \pi \) and the fundamental cycle of its weighted dual graph coincide [1].

**Corollary 1.6.** If an irreducible nonsingular curve \( C \) passes through a rational singularity \( P \) of a surface \( F \), then its strict transform must intersect transversally only one reduced component of the fundamental cycle.

**Proof.** By Lemma 1.5 and Note 1.5.1 the families of nonsingular curves on a rational surface singularity are in one-to-one correspondence with the reduced components of the fundamental cycle of its minimal desingularization. By Proposition 1.4, \( C \in L_E \) where \( E \) is an irreducible component of \( \pi^{-1}(P) \) such that \( \text{ord}_E(\mathcal{MO}_{\tilde{F}}) = 1 \); thus its strict transform must intersect \( E \) transversally, and can intersect no other irreducible exceptional curves because the set of nonsingular curves \( C \) on \( (F, P) \) whose generic point lies on \( \text{Reg}(F) \), \( \mathcal{L} \), is a disjoint union of \( L_E \) by 1.4.

**1.6.1. Fundamental cycles for rational triple singularities.** We exhibit the fundamental cycle for each singularity type.
(1) Case $X_{ijk}, i, j, k \geq 1$. The fundamental cycle is
\[
\begin{array}{c}
1 \\
.
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \cdots & \bullet & 1 1 1 1 1
\end{array}
\]

(2) Case $Y_i, i \geq 1$. The fundamental cycle is
\[
\begin{array}{c}
2 \\
.
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \bullet & \bullet & \bullet & 1 1 1 2 3 2 1
\end{array}
\]

(3) Case $R_{ij}, i, j \geq 1$. The fundamental cycle is
\[
\begin{array}{c}
1 \\
.
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & 1 1 1 2 2 2 1
\end{array}
\]

(4) Case $T_i, i \geq 1$. The fundamental cycle is
\[
\begin{array}{c}
2 \\
.
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \bullet & \bullet & \bullet & \bullet & 1 1 1 2 3 4 3 2
\end{array}
\]

(5) Case $U_{ij}, i \geq 1, j \geq 2$. The fundamental cycle is
\[
\begin{array}{c}
1 \\
.
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & 1 1 1 2 2 2 1
\end{array}
\]

(6) Case $V_1$. The fundamental cycle is
\[
\begin{array}{c}
1 \\
x
\end{array}
\]
\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet & 1 2 2 1
\end{array}
\]

(7) Case $V_i, i \geq 2$. The fundamental cycle is
\[
\begin{array}{c}
1 \\
x
\end{array}
\]
\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & 1 2 3 3 3 2 1
\end{array}
\]
(8) Case W₂. The fundamental cycle is

```
x  1  2  3  4  4  3  2
```

(9) Case W₃. The fundamental cycle is

```
x  1  3  4  3  2  1
```

(10) Case W₄. The fundamental cycle is

```
x  1  2  3  3  3  2  1
```

1.7.0. Let \( C \) be an irreducible nonsingular curve with \( 3C = S \cap F \), where \( S \) is a hypersurface and \( F \) is a surface with rational triple points. Suppose that \( C \) passes through a rational triple point \( P \) of \( F \). Let \( \tilde{F} \) be the minimal desingularization of \( F \) at \( P \), \( \pi : \tilde{F} \to F \). Let \( E_k \), \( 1 \leq k \leq n \), be the irreducible components of the exceptional divisor. The total transform \( \pi^*(3C) \) equals \( \sum_{j=1}^{n} \beta_j E_j + 3E \), where \( E \) is the strict transform of \( C \); \( \beta_j \in \mathbb{N} \).

**Lemma 1.7.** Let \( E \) and \( E_j \), \( 1 \leq j \leq n \), be as in 1.7.0. The square of the exceptional cycle of \( C \) is

\[
\left( \sum_{j=1}^{n} \beta_j E_j \right)^2 = -3E \left( \sum_{j=1}^{n} \beta_j E_j \right) = -3 \beta_l
\]

where \( l \) is the unique natural number such that \( E_l \cap E \neq \emptyset \).

**Proof.** Since \( \sum_{j=1}^{n} \beta_j E_j + 3E \) is a Cartier divisor, it has intersection 0 with \( E_j \) for all \( j \). Thus, \( (\sum_{j=1}^{n} \beta_j E_j + 3E)(\sum_{j=1}^{n} \beta_j E_j) = 0 \).

**Theorem 1.8.** Let \( C \) be as in 1.7.0. The square of the exceptional cycle of \( C \) is \(-6\) if \( C \) passes through one singularity of type \( X_{iii} \), \( i \geq 1 \), or of type \( R_{11} \), or of type \( U_{1j} \), \( j \geq 2 \), or of type \( W_2 \), or of type \( W_3 \). If \( C \) passes through one singularity of type \( V_i \), \( i \in \mathbb{N} \), the square of the exceptional cycle is \(-9\); moreover if \( i = 3b + 2 \) for \( b \in \mathbb{Z}^+ \), the square of the exceptional cycle is \(-3b - 9 \), and if \( i = 3a - 1 \), \( a \in \mathbb{N} \), it is \(-3a - 3 \). The curve \( C \) cannot pass through any other rational triple point of the surface.

**Proof.** By Corollary 1.6, if an irreducible nonsingular curve \( C \) passes through a rational singularity of \( F \), then its strict transform must intersect transversally only one reduced component of the fundamental cycle. We
consider $3C$ and consider its total transform which must have intersection 0 with each exceptional divisor.

The numbers in the diagrams below under the dots are the multiplicities of the $E_i$’s, i.e. the $\beta_i$’s. The number assigned to the small circle is the multiplicity of $E$ in the cycle $\pi^*(3C)$.

(1) Case $X_{ijk}, i, j, k \geq 1$. Consider its fundamental cycle. Let $a$ be the multiplicity with which $E_1$ appears in the total transform.

If we could find, for $a \in \mathbb{N}, b, c \in \mathbb{Z}$, a cycle

\[
(1.8.1)\hspace{1cm} \begin{array}{cccc}
\bullet & \circ & \ldots & \bullet \\
(k+1)a-jc & (k+1)a-c & x & \ldots \\
(k+1)a-(i+1)b & (k+1)a-ib & (k+1)a-b & (k+1)a-ka \\
\end{array}
\]

then $C$ would pass through a rational singularity of type $X_{ijk}, i, j, k \geq 1$. Since

\[
(k + 1)a - (i + 1)b = 3, \\
(k + 1)a - b + ka + (k + 1)a - c + (k + 1)a(-3) = 0,
\]

we have $a = \frac{3+(i+1)b}{k+1}$, $a = -b - c$. On the other hand,

\[
2((k + 1)a - jc) = (k + 1)a - (j - 1)c
\]

implies that $a = \frac{c(j+1)}{k+1}$, so $c > 0$. Thus, $b = \frac{c(j+1)-3}{i+1}$. Therefore,

\[
c = \frac{3(k+1)}{(j+1)(i+1) + (i + j + 2)(k + 1)} \notin \mathbb{Z}.
\]

The cycle (1.8.1) cannot occur.

The cycle

\[
(1.8.2)\hspace{1cm} \begin{array}{cccc}
\bullet & \circ & \ldots & \bullet \\
(k+1)a-jc & (k+1)a-c & x & \ldots \\
a & i & (i+1)a & (i+1)a-b & (i+1)a-kb & (i+1)a-(k+1)b \\
\end{array}
\]

is symmetric to (1.8.1) ($k$ and $i$ are interchanged).
Let us see whether we can find, for \( a \in \mathbb{N}, b,c \in \mathbb{Z} \), a cycle of the form

\[
\begin{array}{c}
(k+1)a-(j+1)c \\
\circ 3 \\
\bullet (k+1)a-jc
\end{array}
\]

\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
(k+1)a-ib & (k+1)a-b & (k+1)a & ka & a
\end{array}
\]

From the equations

\[
\begin{align*}
(k + 1)a - b + (k + 1)a - c + ka + (k + 1)a(-3) &= 0, \\
(k + 1)a - (j + 1)c &= 3,
\end{align*}
\]

we obtain

\[
\begin{align*}
a &= -b - c, \\
c &= \frac{3 + (k + 1)b}{k + j + 2}.
\end{align*}
\]

Since

\[
2((k + 1)a - ib) = (k + 1)a - (i - 1)b,
\]

we get \( a = \frac{(i+1)b}{k+1} \); so \( b > 0 \). Thus,

\[
b = \frac{3k + 3}{k(i + j + 2) + ij + 2i + 2j + 3} \not\in \mathbb{Z}.
\]

Therefore, the cycle (1.8.3) cannot occur.

Let \( l \) be the unique natural number, \( 1 < l < i \), such that \( E_l \cap E \neq \emptyset \). Let us see whether we can find, for \( a \in \mathbb{N}, b,c \in \mathbb{Z} \), a cycle of the form

\[
\begin{array}{c}
(k+1)a-(j+1)c \\
\circ 3 \\
\bullet (k+1)a-jc
\end{array}
\]

\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
(k+1)a-ib & (k+1)a-lb & (k+1)a-b & (k+1)a & ka & a
\end{array}
\]

The equation

\[
(k + 1)a - (l + 1)b + (k + 1)a - (l - 1)b + 3 + ((k + 1)a - lb)(-2) = 0
\]

leads to a contradiction. Thus, the cycle (1.8.4) cannot occur.
Let us see whether we can find, for \( a \in \mathbb{N}, b, c \in \mathbb{Z} \), a cycle of the form

\[
(k+1)a-jc \quad \bullet
\]

(1.8.5)

\[
\bullet \quad \ldots \quad \bullet \quad (k+1)a-c \quad x \quad \bullet \quad \ldots \quad \bullet
\]

\[
(k+1)a-ib \quad (k+1)a-b \quad ka \quad a
\]

c3

From the equations

\[
(k + 1)a - b + 3 + (k + 1)a - c + ka + (k + 1)a(-3) = 0,
\]

\[
2((k + 1)a - jc) = (k + 1)a - (j - 1)c,
\]

\[
2((k + 1)a - ib) = (k + 1)a - (i - 1)b,
\]

we obtain

\[
a = -b - c + 3, \quad a = \frac{c(j + 1)}{(k + 1)}, \quad a = \frac{b(i + 1)}{k + 1};
\]

so, \( b > 0, c > 0 \). Since \( a = -b - c + 3 \), we get \( b = 1 \) and \( c = 1 \), which implies that \( a = 1 \). Thus, \( i = j = k \); so \( C \) can pass through a singularity of type \( X_{iii} \).

We have \((\sum_{j=1}^{i} \beta_j E_j)^2 = -6 \). In this case, \( \beta_l \) of Lemma 1.7 is 2.

We have shown that \( C \) can pass through a singularity of type \( X_{iii}, i \geq 1 \), but not through an \( X_{ijk} \) singularity with \( i \neq j \) or \( j \neq k \) or \( i \neq k \).

(2) Case \( Y_i, i \geq 1 \). Consider its fundamental cycle. If we could find, for \( 2a, 3a \in \mathbb{N}, b \in \mathbb{Z}, d = 6(i + 2)a - (3(i + 1) + 2)b \), a cycle

\[
3 \quad \bullet \quad \ldots \quad \bullet \quad x \quad \bullet \quad \ldots \quad \bullet
\]

\[
d \quad 6(i+1)a-(3i+2)b \quad 12a-5b \quad 6a-2b \quad 6a-b \quad 6a \quad 4a \quad 2a
\]

then \( C \) would pass through a rational singularity of type \( Y_i, i \geq 1 \). From the equations

\[
6a - b + 4a + 3a + 6a(-2) = 0, \quad 6(i + 2)a - (3(i + 1) + 2)b = 3,
\]

we obtain \( a = b = \frac{3}{3i+7} \notin \mathbb{Z} \) for \( i \geq 1 \).

We consider \( 3C \) and its total transform \( \pi^*(3C) \) which must have intersection 0 with each exceptional divisor. Let \( l \) be the unique natural number, \( 1 < l \leq i \), such that \( E_l \cap E \neq \emptyset \). Number the exceptional curves \( E_l \) as
follows:

\[
\begin{array}{cccccccc}
E_1 & \ldots & E_i & x & E_{i+1} & E_{i+2} & E_{i+3} & E_{i+4} & E_{i+5} \\
\end{array}
\]

Let \( a \) and \( b \) denote the respective multiplicities with which \( E_1 \) and \( E_{i+5} \) appear in \( \pi^*(3C) \). Then the equation \( \pi^*(3C) \cdot E_t = 0 \) for all \( t \) and the fact that \( E \) meets only \( E_l \) and no other exceptional curve implies that the cycle \( \pi^*(3C) \) has the form, for \( a \in \mathbb{N}, b \in 2\mathbb{N} \),

\[
(1.8.6)
\]

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

Now, \( \pi^*(3C) \cdot E_l = 0 \) reads \( 2la = (l - 1)a + 3 + (4 + i - l)b/2 \). Putting this together with the equation \( la = (7 + i - l)b/2 \), we obtain \( (l + 1)a = 3 + la - 3b/2 \) or \( a = 3 - 3b/2 \). Since no strictly positive integers \( a \) and \( b \) can satisfy the last equality, we arrive at a contradiction.

Let us see whether we can find, for \( a, (3i + 5)a/2 \in \mathbb{N}, b \in \mathbb{Z}, d = (3i + 5)a - 3b \), a cycle of the form

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

From the equations

\[
(3i + 5)a - b + (2i + 3)a + \frac{(3i + 5)a}{2} + (3i + 5)a(-2) = 0,
\]

\[
(3i + 5)a - 3b = 3,
\]

we obtain \( (i + 1)a = 2b, a = \frac{6}{3i+7} \not\in \mathbb{N} \) for \( i \geq 1 \).

For \( 2a, 3a \in \mathbb{N}, b \in \mathbb{Z} \), consider a cycle

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

The equation \( 6a - b + 12a - 5b + 3 + (6a - 2b)(-3) = 0 \) leads to a contradiction. Thus, \( C \) cannot pass through a \( Y_i \) singularity for \( i \geq 1 \).

(3) Case \( R_{ij}, i, j \geq 1 \). Consider its fundamental cycle. If we could find, for \( 2a, (j + 1)a \in \mathbb{N}, b \in \mathbb{Z}, d = 2(i + 1)(j + 1)a - (2i + 1)b \), a cycle
then \( C \) would pass through a rational singularity of type \( R_{ij} \), \( i, j \geq 1 \). From the equations
\[
2(j + 1)a - b + (j + 1)a + 2ja + 2(j + 1)a(-2) = 0,
2(i + 2)(j + 1)a - (2(i + 1) + 1)b = 3,
\]
we obtain \((j - 1)a = b, a = \frac{3}{4j + j + 7}\), \(2a \notin \mathbb{N}\) for \(i, j \geq 1\).

Let \(2a \in \mathbb{N}, d = 2(2i + 3)a - (j + 1)b\). If we could find a cycle
\[
(2i+3)a
\]
then \( C \) would pass through a rational singularity of type \( R_{ij} \), \( i, j \geq 1 \). From the equations
\[
2(i + 1)a + 2(2i + 3)a - b + (2i + 3)a = 4(2i + 3)a,
2(2i + 3)a - (j + 1)b = 3,
\]
we obtain \(a = -b, a = \frac{3}{4i + j + 7}\), \(2a \notin \mathbb{N}\) for \(i, j \geq 1\).

Let us see whether we can find, for \(2a \in \mathbb{N}, b, c \in \mathbb{Z}, d = 2(i + 1)(j + 1)a - (2i + 1)b\), a cycle of the form
\[
2(i+1)a-2c
\]
(1.8.8)
\[
\begin{array}{cccccccc}
\circ & \bullet & \cdots & \bullet & x & \bullet & \cdots & \bullet & \circ \\
\circ & \bullet & \cdots & \bullet & x & \bullet & \cdots & \bullet & \circ \\
2a & 2i & 2(i+1)a & 2(2i+3)a & 2(2i+3)a-b & 2(2i+3)a-jb & d
\end{array}
\]
From the equations
\[
2(j + 1)a - 2c = 3,
2(j + 1)a - b + 2(j + 1)a - c + 2ja + 2(j + 1)a(-2) = 0,
\]
we obtain
\[
c = \frac{2(j + 1)a - 3}{2}, \quad b = \frac{2(j - 1)a + 3}{2}.
\]
Since
\[
2(2(i + 1)(j + 1)a - (2i + 1)b) = 2i(j + 1)a - (2i - 1)b,
\]
we get \(a = \frac{3}{8i + 6j + 10}\); so \(2a \notin \mathbb{N}\) for \(i, j \geq 1\).

Let \(l\) be the unique natural number, \(1 < l \leq i\), such that there is no cycle with \(E_l \cap E \neq \emptyset\) (see (1.8.6)).
Consider, for \( a \in \mathbb{N}, \ b, c \in \mathbb{Z}, \frac{(i+1)a-b}{2} \in \mathbb{N} \), a cycle
\[
\begin{array}{cccccccc}
3 & (i+1)a-b & (i+1)a-c & (i+1)a-jc & \cdots & \cdots & \cdots & \cdots \\
\circ & \cdots & x & \cdots & \cdots & \cdots & \cdots & \cdots \\
a & ia & (i+1)a & (i+1)a-b & (i+1)a-c & (i+1)a-jc
\end{array}
\]

From the equation
\[
(i + 1)a - b + 3 + ia + (i + 1)a(-3) = 0,
\]
we obtain \( b = 3 - (i + 2)a \). From
\[
(i + 1)a + \frac{(i + 1)a - b}{2} + (i + 1)a - c + ((i + 1)a - b)(-2) = 0,
\]
we obtain \( \frac{9-(2i+5)a}{2} = c \). On the other hand,
\[
2((i + 1)a - jc) = (i + 1)a - (j - 1)c,
\]
so \( c = \frac{9(i+1)}{2(i+1)(j+2)+3(j+1)} \geq 1 \); since \( a \in \mathbb{N} \) and \( a = \frac{9-2c}{2i+5} \), we get \( c = 1 \).
Consequently, \( i = 1 \), \( a = 1 \), \( b = 0 \), \( j = 1 \).

We have \((\sum_{j=1}^{5} \beta_j E_j)^2 = (E_1 + 2E_2 + 2E_3 + E_4 + E_5)^2 = -6\).

In this case, \( \beta_l \) of Lemma 1.7 is 2.

Hence, \( C \) can pass through a rational singularity of type \( R_{11} \), but not through a rational singularity of type \( R_{ij} \) with \( i \) or \( j > 1 \).

(4) Case \( T_i, i \geq 1 \). Consider its fundamental cycle. If we could find, for \( 2a, 3a \in \mathbb{N}, \ b \in \mathbb{Z}, \ d = (i + 1)6a - (5i + 2)b \), a cycle
\[
\begin{array}{cccccccc}
3 & \cdots & 3a & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \cdots & x & \cdots & \cdots & \cdots & \cdots & \cdots \\
d & 18a-12b & 12a-7b & 6a-3b & 6a-b & 6a & 4a & 2a
\end{array}
\]
then \( C \) would pass through a rational singularity of type \( T_i, i \geq 1 \). From the equations
\[
6a - b + 4a + 3a + 6a(-2) = 0, \quad (i + 2)6a - (5i + 7)b = 3,
\]
we see that \( a = b \) and \( i = -2 \), which is absurd.

Let \( l \) be the unique natural number, \( 1 < l \leq i \), such that there is no cycle such that \( E_l \cap E \neq \emptyset \).

Consider, for \( a \in \mathbb{N}, \ b, c \in \mathbb{Z}, \frac{(i+1)a-3b}{2} \in \mathbb{N}, \ d = (i + 1)a - 2c \), a cycle
\[
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \cdots & x & \cdots & \cdots & \cdots & \cdots & \cdots \\
a & ia & (i+1)a-b & (i+1)a-2b & (i+1)a-3b & (i+1)a-c & d
\end{array}
\]
From the equations
\[ ia + (i + 1)a - b + 3 + (i + 1)a(-3) = 0, \]
\[ (i + 1)a - 2b + \frac{(i + 1)a - 3b}{2} + (i + 1)a - c + ((i + 1)a - 3b)(-2) = 0, \]
we obtain \( b = 3 - (i + 2)a \) and \( c = \frac{15 - (4i + 9)a}{2} \). On the other hand,
\[ 2((i + 1)a - 2c) = (i + 1)a - c; \]
so \( a = \frac{45}{14i + 29} \notin \mathbb{N} \) for \( i \geq 1. \)

Hence, no \( C \) can pass through a \( T_i \) singularity for \( i \geq 1. \)

(5) Case \( U_{ij}, i \geq 1, j \geq 2 \). Consider its fundamental cycle. If we could find, for \( 2a \in \mathbb{N}, b \in \mathbb{Z}, d = 4(i + 1)a - ((i + 1)j + (2i + 1))b \), a cycle

\[
\begin{array}{cccccccc}
3 & \cdot & \cdots & \cdot & x & \cdots & \cdot & 2a \\
\cdot & 8a - (2j + 3)b & \cdot & 4a - (j + 1)b & \cdot & \cdots & \cdot & \cdot \\
d & 4a - b & 4a & 2a
\end{array}
\]

then \( C \) would pass through a rational singularity of type \( U_{ij}, i \geq 1, j \geq 2 \). From the equations
\[ 4a - b + 2a + 2a + 4a(-2) = 0, \]
\[ 4(i + 2)a - ((i + 2)j + (2i + 3))b = 3, \]
we obtain \( b = 0, a = \frac{3}{4(i + 2)}, 2a \notin \mathbb{N} \) for \( i \geq 1. \)

As in (1.8.6), we cannot find a cycle

\[
\begin{array}{cccccccc}
3 & \cdot & \cdots & \cdot & x & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a & (i + 1)a & (2i + 3)a & ((j + 2)i + (2j + 3))a & d
\end{array}
\]

Let us see whether we can find, for \( a, \frac{(j + 2)i + (2j + 3)a}{2} \in \mathbb{N}, b \in \mathbb{Z}, d = ((j + 2)i + (2j + 3))a - b \), a cycle

\[
\begin{array}{cccccccc}
\cdot & \cdots & \cdot & x & \cdots & \cdot & \cdot & 3 \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
a & (i + 1)a & (2i + 3)a & ((j + 2)i + (2j + 3))a & d
\end{array}
\]

From the equations
\[ ((j + 1)i + (2j + 1))a + \frac{((j + 2)i + (2j + 3))a}{2} + ((j + 2)i + (2j + 3))a - b + ((j + 2)i + (2j + 3))a(-2) = 0, \]
we obtain \( a = \frac{3}{2(i + 2)} \notin \mathbb{N} \) for \( i \geq 1. \).
Consider, for $2a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $d = 4(i+1)a - ((i+1)j + (2i+1))b$ a cycle

$$
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & 4a - 2c \\
\circ & 3
\end{array}
$$

From the equations

$$4a - b + 4a - c + 2a + 4a(-2) = 0, \quad 4a - 2c = 3,$$

we obtain $b + c = 2a, \ c = \frac{4a-3}{2}$; so $b = \frac{3}{2} \not\in \mathbb{Z}$.

Consider, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $(\frac{i+1}{2})a - (\frac{j+1}{2})b \in \mathbb{N}$, a cycle

$$
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & x & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \frac{(i+1)a - (j+1)b}{2} \\
\circ & 3
\end{array}
$$

From the equations

$$ia + (i+1)a - b + 3 + (i+1)a(-3) = 0,$$

$$(i+1)a - jb + \frac{(i+1)a - (j + 1)b}{2} + (i+1)a - c + ((i+1)a - (j+1)b)(-2) = 0,$$

we obtain

$$b = 3 - (i + 2)a, \quad c = \frac{(i + 1)a + (j + 3)b}{2},$$

so $c = \frac{9 + 3j - ((i+2)(j+2) + 1)a}{2}$. Since

$$2((i + 1)a - c) = (i + 1)a - (j + 1)b,$$

we get $c - 2b = c$, so $b = 0$; thus $a = \frac{3}{i+2} \in \mathbb{N}$ implies that $i = 1$ and $a = 1$; so $c = 1$ for $j \geq 1$.

Hence, $C$ can pass through a rational singularity of type $U_{1j}$ for $j \geq 2$, but there are no $C$ passing through an $U_{ij}$ singularity for $i > 1, j \geq 2$.

(6) Case $V_i, \ i \geq 1$. Consider its fundamental cycle. For $2a \in \mathbb{N}$, $b \in \mathbb{Z}$, consider a cycle

$$
\begin{array}{cccccccc}
\circ & 3 \circ & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & 2a \\
\circ & x
\end{array}
$$

$$
\begin{array}{cccccccc}
6a - (i+1)b & 6a - ib & 6a - b & 6a & 4a & 2a
\end{array}
$$
From the equations
\[ 6a - b + 2a + 4a + 6a(-2) = 0, \]
\[ 6a - (i + 1)b = 3, \]
we obtain \( b = 0, 2a = 1; \) we have the cycle

\[ \begin{array}{ccccccc}
3 & & & & & & 1 \\
\circ & \cdots & \bullet & \bullet & \bullet & \bullet & \\
3 & & & 3 & 3 & 2 & 1 \\
\end{array} \]

For \( i \in \mathbb{N}, \) the singularity \( V_i \) can occur.

For \( a \in \mathbb{N}, \ b \in \mathbb{Z}, \) \( \frac{(i+1)a}{3} \in \mathbb{N}, \) consider a cycle

\[ \begin{array}{ccccccc}
\circ & \cdots & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \\
3 & & & (i+1)a & (i+1)a - b & (i+1)a - 2b & (i+1)a - 3b \\
a & i_a & (i+1)a & (i+1a - b) & (i+1a - 2b) & (i+1a - 3b) & \\
\end{array} \]

Since
\[ ia + \frac{(i+1)a}{3} + (i + 1)a - b + (-2)(i + 1)a = 0, \]
\[ (i + 1)a - 3b = 3, \]
we obtain \( a = 1 \) and \( b = \frac{i+2}{3}. \) In this case, the total transform of \( 3C, \) for \( b \in \mathbb{Z}^+, \) is

\[ b+1 \]
\[ \begin{array}{ccccccc}
\circ & \cdots & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \\
3 & & & 3b+2 & 3b+3 & 2b+3 & b+3 \\
1 & & & & & & \\
\end{array} \]

For \( b \in \mathbb{Z}^+, \) the singularity \( V_{3b+2} \) can occur.

For \( a \in \mathbb{N}, \ b, c \in \mathbb{Z}, \) consider a cycle

\[ \begin{array}{ccccccc}
\circ & \cdots & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \\
3 & & & 3a-c & 3a-c & 3a-c & \\
6a - 3c & & & & & & \\
3a - ib & 3a - b & 3a & 2a & a \\
\end{array} \]

Since \( 6a - 3c = 3, \) we have \( c = 2a - 1. \) From the equation
\[ 3a - b + 3a - c + 2a + 3a(-2) = 0, \]
we obtain \( 2a = b + c; \) so \( b = 1. \) From \( 2(3a - ib) = 3a - (i - 1)b, \) we get \( a = \frac{i+1}{3}. \) For \( a \geq 1, \) the singularity \( V_{3a-1} \) can occur.
In this case, the total transform of $3C$ is

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
1 & 3a-1 & 3a & 2a & a & & & \\
\end{array}
\]

So $C$ can pass through a rational singularity of type $V_{3a-1}$, $a \geq 1$.

(7) *Case W*$_2$. Consider its fundamental cycle. If we could find, for $2a, 3a \in \mathbb{N}, b \in \mathbb{Z}$, a cycle

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
12a-9b & 6a-4b & 6a-3b & 6a-2b & 6a-b & 6a & 4a & 2a \\
\end{array}
\]

then $C$ would pass through a rational singularity of type $W_2$. Let us see if it is possible to find $a$ and $b$ satisfying the equations $12a - 9b = 3$ and $6a - b + 4a + 3a + 6a(-2) = 0$. From the second equation we obtain $a = b$. Thus $a = 1$.

We have $(\sum_{j=1}^{8} \beta_j E_j)^2 = -6$. In this case, $\beta_l$ of Lemma 1.7 is 2.

Hence, $C$ can pass through a rational singularity of type $W_2$.

(8) *Case W*$_3$. Consider its fundamental cycle. If we could find, for $2a \in \mathbb{N}, b \in \mathbb{Z}$, a cycle

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
16a-5b & 8a-2b & 8a-b & 8a & 6a & 4a & 2a \\
\end{array}
\]

then $C$ would pass through a rational singularity of type $W_3$. From the equations

\[
8a - b + 6a + 4a + 8a(-2) = 0, \quad 16a - 5b = 3,
\]

we obtain $2a = b$ and $a = \frac{1}{2}$, so $2a = 1$.

We have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$. In this case, $\beta_l$ of Lemma 1.7 is 2.

Hence, $C$ can pass through a rational singularity of type $W_3$.

Let us see whether we can find, for $2a, 5a \in \mathbb{N}, b \in \mathbb{Z}$, a cycle

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
2a & 6a & 10a & 10a-b & 10a-2b & 10a-3b & 10a-4b & \\
\end{array}
\]

From the equations

\[
6a + 5a + 10a - b + 10a(-2) = 0, \quad 10a - 4b = 3,
\]
we obtain \( a = b = \frac{1}{2} \), so \( 2a = 1 \) but \( 5a \notin \mathbb{N} \). Thus, this possibility cannot occur.

(9) Case \( W_4 \). If \( C \) passed through a rational singularity of type \( W_4 \), we would be able to find, for \( 2a, 5a \in \mathbb{N}, b \in \mathbb{Z} \), a cycle

\[
\begin{array}{ccccccc}
3 & \circ & x & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
20a-5b & 10a-2b & 10a-b & 10a & 8a & 6a & 4a & 2a
\end{array}
\]

From the equations
\[
10a - b + 8a + 5a + 10a(-2) = 0, \quad 20a - 5b = 3,
\]
we obtain \( 3a = b, a = \frac{3}{5}, b \notin \mathbb{Z} \).

Let us see whether we can find, for \( 2a, 5a \in \mathbb{N}, b \in \mathbb{Z} \), a cycle

\[
\begin{array}{ccccccc}
5a & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\
x & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 3 \\
2a & 6a & 10a & 10a-b & 10a-2b & 10a-3b & 10a-4b & 10a-5b
\end{array}
\]

From the equations
\[
6a + 5a + 10a - b + 10a(-2) = 0, \quad 10a - 5b = 3,
\]
we obtain \( a = b = \frac{3}{5}, b \notin \mathbb{Z} \). Thus, \( C \) cannot pass through a \( W_4 \) singularity.

**Definition 1.9.** Let \( R \) be a regular 2-dimensional local noetherian ring. A *sandwiched singularity* is the singularity of a blowing-up of \( \text{Spec} \ R \) along a complete ideal [4, p. 432].

**Note 1.10.** A normal surface singularity is minimal if and only if it is rational with reduced fundamental cycle [3, 4.4.10]

**Proposition 1.11 ([4, Proposition 2.4]).** Every minimal singularity is sandwiched.

**Example 1.12.** A singularity of type \( X_{111} \) is a sandwiched singularity because it is minimal (1.11). It is minimal because it is a rational singularity with reduced fundamental cycle (1.10). We consider the blowing-up of \( \text{Spec} \ k[x, y] \) along the complete ideal \((yx^4, x^6, y^2 + yx^2)\). It has a unique singularity with local coordinates \((x, y, t', u')\), where
\[
t' = \frac{yx^4}{y^2 + yx^2}, \quad u' = \frac{x^6}{y^2 + yx^2}.
\]
This blowing-up is a surface \( F \) defined in \( \mathbb{A}^4_k \) by the relations
\[
y(t' + u') = x^4, \quad u'y = t'x^2, \quad u'x^2 = t'^2 + t'u'.
\]
Let $I$ be the ideal generated in $k[x, y, t', u']$ by the relations above. Then $k[x, y, t', u']/I$ is a free module over $k[y, u']$ generated by $(1, x, x^2, x^3, t', t'x)$; it is the affine coordinate ring of the surface $F$ with a rational triple point $X_{111}$ at the origin. In $k[x, y, t', u']$ we consider the ideal $J$ generated by the irreducible polynomial $f(x, y, t', u') = 7t' - u' + y - 4x^2$. This polynomial is obtained as follows. Let us look at the dual graph (1.8.6). Then $f(x, y, t', u')$ equals $(y - x^2)^3 y (y + x^2)$ modulo $I$. The quotient $k[x, y, t', u']/J$ is the affine coordinate ring of a hypersurface $S$. After projectivization, the intersection of $S$ and $F$ is a multiplicity-three structure on a curve $C$ passing through the singularity $X_{111}$ of $F$.

References


