Invariance in the class of weighted quasi-arithmetic means

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Abstract. Under the assumption of twice continuous differentiability of some of the functions involved we determine all the weighted quasi-arithmetic means $M, N, K$ such that $K$ is $(M, N)$-invariant, that is, $K \circ (M, N) = K$. Some applications to iteration theory and functional equations are presented.

1. Introduction. In the whole paper $I \subset \mathbb{R}$ denotes an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a mean on $I$ if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

If $M : I^2 \to \mathbb{R}$ is a mean, then $M$ is reflexive, that is, $M(x, x) = x$ for all $x \in I$ and, consequently, for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$.

If $\alpha : I \to \mathbb{R}$ is a continuous strictly monotonic function and $p \in (0, 1)$ then $A_p^{[\alpha]} : I^2 \to I$, given by

$$A_p^{[\alpha]}(x, y) := \alpha^{-1}(p\alpha(x) + (1 - p)\alpha(y)), \quad x, y \in I,$$

is a mean; it is called a weighted quasi-arithmetic mean with generator $\alpha$ and weight $(p, 1 - p)$. In the case $\alpha = \text{id}|_I$ the mean $A_p^{[\alpha]}$ becomes the weighted arithmetic mean and is denoted by $A_p$; thus $A_p(x, y) = px + (1 - p)y$. We write simply $A$ instead of $A_{1/2}$.

Let $M, N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is said to be invariant with respect to $(M, N)$, briefly $(M, N)$-invariant, or $K$ is called the Gauss composition of $M$ and $N$ (cf. [3], [5]), if $K \circ (M, N) = K$.

Fix $p, q, r \in (0, 1)$. Assuming twice continuous differentiability of some of the functions involved we determine all triples $(\alpha, \beta, \gamma)$ and $(p, q, r)$ for

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which the weighted quasi-arithmetic mean \( A_p^{[\alpha]} \) is \((A_q^{[\beta]}, A_r^{[\gamma]})\)-invariant, i.e.

\[
A_p^{[\alpha]} \circ (A_q^{[\beta]}, A_r^{[\gamma]}) = A_p^{[\alpha]}
\]

(see Theorem 2 in Section 4).

The case when \( p = q = r = 1/2 \) (called simply the case of quasi-arithmetic means) has a long history. The analytic solutions were found by O. Sutô in 1914 [7]. The twice continuously differentiable solutions are given in [5]. Moreover, continuously differentiable solutions were found by Z. Daróczy and Zs. Páles [2], and finally, without any regularity assumption, the problem was solved by Z. Daróczy and Zs. Páles [3] (cf. also [1] as well as [3] for further references).

The fundamental role for Theorem 2 will be played here by Theorem 1 in Section 4, concerning the case of \( \alpha \) being the identity function. In the proof of Theorem 1 we need a characterization of conditionally homogeneous weighted quasi-arithmetic means given by Proposition 1 proved in Section 2 and by the lemmas of Section 3.

In Section 5 we also apply Theorem 1 to establish the limit of the iteration sequence of some mean type mappings (Remark) and to solve a functional equation (Theorem 3).

2. Conditional homogeneity of the quasi-arithmetic mean. Denote by \( CM(I) \) the class of all continuous strictly monotonic functions defined on \( I \). In the proof of Theorem 1 we need the following

**PROPOSITION 1.** Assume that \( I \subset (0, \infty) \). Let \( q \in (0, 1) \) and \( \sigma \in CM(I) \). The mean \( A_q^{[\sigma]} \) is conditionally homogeneous, i.e.

\[
A_q^{[\sigma]}(sx, sy) = sA_q^{[\sigma]}(x, y)
\]

for all \( x, y \in I \) and \( s > 0 \) with \( sx, sy \in I \) if, and only if, there are \( a \in \mathbb{R} \setminus \{0\} \) and \( b \in \mathbb{R} \) such that either

\[
\sigma(x) = ax^n + b, \quad x \in I,
\]

for some \( \eta \in \mathbb{R} \setminus \{0\} \), or

\[
\sigma(x) = a \ln x + b, \quad x \in I.
\]

Before proving this proposition we will find the form of so-called conditional local groups of continuous affine maps.

**PROPOSITION 2.** Let \( \{X_s\}_{s \in S} \), where \( S \subset \mathbb{R} \) is an interval containing 0, be a family of subsets of \( \mathbb{R} \) and let \( F : \bigcup_{s \in S} \{s\} \times X_s \to \mathbb{R} \). Assume that \( F(\cdot, u) \) and \( F(\cdot, v) \) are continuous for some different \( u, v \in \bigcap_{s \in S} X_s \) and \( F(s, \cdot) \) is continuous affine for every \( s \in S \). Assume also that for every \( s, t \in S \) with \( s + t \in S \) there is an at least two-element set \( U_{s, t} \subset X_s \cap X_{s+t} \)
such that
\[ F(\{s\} \times U_{s,t}) \subseteq X_t \]
and
\[ F(s + t, u) = F(t, F(s, u)), \quad u \in U_{s,t}. \]
Then either \( F \) is constant, or there is an interval \( S_0 \subseteq S \) containing 0 and such that
\[ F(s, u) = cs + u, \quad s \in S_0, \ u \in X_s, \]
with a \( c \in \mathbb{R} \), or
\[ F(s, u) = c_1(u - c_2) + c_2, \quad s \in S_0, \ u \in X_s, \]
with some \( c_1 \in (0, \infty) \) and \( c_2 \in \mathbb{R} \).

Proof. Let \( m : S \to \mathbb{R} \) and \( k : S \to \mathbb{R} \) be such that
\[ F(s, u) = m(s)u + k(s), \quad s \in S, \ u \in X_s. \]

Taking any \( u, v \in \bigcap_{s \in S} X_s, \ u \neq v \), such that \( F(\cdot, u) \) and \( F(\cdot, v) \) are continuous we see that \( m \) and \( k \) are continuous as linear combinations of \( F(\cdot, u) \) and \( F(\cdot, v) \).

Fix any \( s, t \in S \) with \( s + t \in S \) and choose a \( U_{s,t} \subseteq X_s \cap X_{s+t} \) with at least two points, satisfying (2) and (3). Taking an arbitrary \( u \in U_{s,t} \) and using (6), (3), and again (6), we get
\[
m(s + t)u + k(s + t) = F(s + t, u) = F(t, F(s, u)) = m(t)(m(s)u + k(s)) + k(t) = m(s)m(t)u + m(t)m(s) + k.
\]
Consequently,
\[ m(s + t) = m(s)m(t) \]
and
\[ k(s + t) = m(t)k(s) + k(t) \]
for all \( s, t \in S \) with \( s + t \in S \).

Assume that \( m(s_0) = 0 \) for some \( s_0 \in S \) with \(-s_0 \in S \). By (7) we get
\[ m(0) = m(s_0)m(-s_0) = 0, \]
whence, again according to (7), we have \( m(s) = m(s)m(0) = 0 \) for all \( s \in S \) and, consequently, \( k \) is constant by (8). Then, on account of (6) also \( F \) is constant. Now we can assume that \( m \) does not vanish in a subinterval \( S_0 \subseteq S \) containing 0 and, in addition, satisfying \( S_0 + S_0 \subseteq S \). Then, by (7), \( m|_{S_0} \) is positive.

Since (7) is a multiplicative version of the restricted Cauchy equation, with the use of continuity arguments and making use of [4, Theorem 2,
p. 327| we infer that there exists a \( c_1 \in (0, \infty) \) such that \( m(s) = c_1^s \) for all \( s \in S_0 \).

If \( c_1 = 1 \) it follows from \((8)\) that \( k(s + t) = k(s) + k(t) \) for all \( s, t \in S_0 \) and, again by \([4, \text{Theorem 2, p. 327}]\), we find a \( c \in \mathbb{R} \) such that \( k(s) = cs \) for all \( s \in S_0 \). Consequently, on account of \((6)\) we have \( F(s, u) = cs + u \) for all \( s \in S_0 \) and \( u \in X_s \).

Now assume that \( c_1 \neq 1 \). Then, by \((8)\), \( k(s + t) = c_1^t k(s) + k(t) \) for all \( s, t \in S_0 \). By symmetry \( k(s + t) = c_1^t k(t) + k(s) \) and

\[
\frac{k(s)}{1 - c_1^s} = \frac{k(t)}{1 - c_1^t}, \quad s, t \in S_0 \setminus \{0\}.
\]

Thus there exists a \( c_2 \in \mathbb{R} \) such that \( k(s) = c_2(1 - c_1^s) \) for all \( s \in S_0 \). Consequently, by \((6)\), we have \( F(s, u) = c_1^s (u - c_2) + c_2 \) for all \( s \in S_0 \) and \( u \in X_s \).

**Proof of Proposition 1.** Without loss of generality we may assume that \( \text{int} \, I \neq \emptyset \).

Assume that \( A_q^{[\sigma]} \) is conditionally homogeneous, that is,

\[
(9) \quad \sigma^{-1}(q \sigma(sx) + (1 - q) \sigma(sy)) = s \sigma^{-1}(q \sigma(x) + (1 - q) \sigma(y))
\]

for all \( x, y \in I \) and \( s > 0 \) with \( sx, sy \in I \). Fix an \( x_0 \in \text{int} \, I \) and put \( u_0 = \sigma(x_0) \). Then \( u_0 \in \text{int} \, \sigma(I) \) and there exist \( \delta > 0 \) and \( \delta_0 > 1 \) such that

\[
(10) \quad \sigma(sx_0) \in (u_0 - \delta, u_0 + \delta)
\]

for all \( s \in (1/\delta_0, \delta_0) \) and \( s \sigma^{-1}(u) \in I \) for all \( s \in (1/\delta_0, \delta_0) \) and \( u \in (u_0 - \delta, u_0 + \delta) \). In particular, \( (u_0 - \delta, u_0 + \delta) \subset \sigma(I) \). Put \( S = (-\ln \delta_0, \ln \delta_0) \). Defining \( F : S \times (u_0 - \delta, u_0 + \delta) \to \mathbb{R} \) by

\[
F(s, u) = \sigma(e^s \sigma^{-1}(u))
\]

we can rewrite \((9)\) in the form

\[
(11) \quad F(s, qu + (1 - q)v) = qF(s, u) + (1 - q)F(s, v),
\]

\[s \in S, \ u, v \in (u_0 - \delta, u_0 + \delta)\).

Fix an \( s \in S \). Then, applying the Daróczy–Páles identity

\[
q \left((1 - q) \frac{u + v}{2} + qu\right) + (1 - q) \left(q \frac{u + v}{2} + (1 - q)v\right) = qu + (1 - q)v,
\]

we get

\[
F\left(s, q \left((1 - q) \frac{u + v}{2} + qu\right) + (1 - q) \left(q \frac{u + v}{2} + (1 - q)v\right)\right) = qF(s, u) + (1 - q)F(s, v)
\]
for all \( u, v \in (u_0 - \delta, u_0 + \delta) \). Now, applying (11) twice to the left-hand side expression, for all \( u, v \in (u_0 - \delta, u_0 + \delta) \) we have
\[
q^2 F(s, u) + 2q(1-q)F\left(s, \frac{u+v}{2}\right) + (1-q)^2 F(s, v) = qF(s, u) + (1-q)F(s, v)
\]
and, consequently,
\[
2F\left(s, \frac{u+v}{2}\right) = F(s, u) + F(s, v), \quad u, v \in (u_0 - \delta, u_0 + \delta).
\]
By the continuity of \( F \) and the Jensen theorem we can find \( m(s), k(s) \in \mathbb{R} \) such that
\[
F(s, u) = m(s)u + k(s), \quad u \in (u_0 - \delta, u_0 + \delta).
\]
Thus we have shown that \( F(s, \cdot) \) is continuous affine for every \( s \in S \). Moreover, if \( s, t, s + t \in S \) then, by (10), we have \( F(s, u_0) \in (u_0 - \delta, u_0 + \delta) \), whence
\[
F(s, u) \in (u_0 - \delta, u_0 + \delta)
\]
for \( u \) from a neighbourhood \( U_{s,t} \subset (u_0 - \delta, u_0 + \delta) \) of \( u_0 \) and
\[
F(t, F(s, u)) = \sigma(e^t \sigma^{-1}\sigma(e^s \sigma^{-1}(u)))) = \sigma(e^{s+t} \sigma^{-1}(u)) = F(s + t, u)
\]
for all \( u \in U_{s,t} \). Since \( F \) is not constant, on account of Proposition 2 there is an interval \( S_0 \subset S \) containing 0 and such that \( F \) is of the form (4) or (5), where \( c_1 \in (0, \infty) \) and \( c_2, c \in \mathbb{R} \).

In this way we have shown that every point \( x \) of \( \text{int} I \) has a neighbourhood in which \( \sigma \) has one of the following forms:
\[
\sigma(x) = a \ln x + b, \quad \sigma(x) = ax^n + b
\]
with \( a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \) and \( \eta \in \mathbb{R} \setminus \{0\} \) depending on that point. By standard arguments we deduce that one of these forms holds on the whole interval \( I \). The converse assertion is obvious. ■

3. Auxiliary results. Denote by \( C^n M(I) \) the subclass of \( CM(I) \) consisting of functions which are \( n \)-times continuously differentiable.

**Lemma 1.** Let \( p, q, r \in (0, 1) \). If \( \varphi, \psi \in C^1 M(I) \) satisfy
\[
(12) \quad pA_q^{[\varphi]}(x, y) + (1-p)A_r^{[\psi]}(x, y) = px + (1-p)y
\]
and \( \varphi'(x) \neq 0, \psi'(x) \neq 0 \) for all \( x \in I \), then
\[
p = \frac{r}{1 - q + r}.
\]

**Proof.** Differentiation of both sides of (12) with respect to \( x \) gives
\[
(13) \quad p \frac{q\varphi'(x)}{\varphi'(A_q^{[\varphi]}(x, y))} + (1-p) \frac{r\psi'(x)}{\psi'(A_r^{[\psi]}(x, y))} = p, \quad x, y \in I.
\]
Letting \( y \to x \) and taking into account the reflexivity of the means we obtain the desired equality. ■

**Lemma 2.** Let \( p, q, r \in (0, 1) \). If \( \varphi, \psi \in C^1 M(I) \) satisfy (12) and \( \varphi'(x) \neq 0 \) and \( \psi'(x) \neq 0 \) for all \( x \in I \), then \( f := \varphi' \circ \varphi^{-1} \) and \( g := \psi' \circ \varphi^{-1} \) satisfy the equation

\[
(14) \quad f(qu + (1-q)v)[(1-r)g(v) - (1-q)g(u)] = q(1-r)f(u)g(v) - r(1-q)f(v)g(u)
\]

for all \( u, v \in \varphi(I) \).

**Proof.** Differentiating both sides of (12), first with respect to \( x \) and then with respect to \( y \), we get (13) and

\[
(15) \quad p \frac{(1-q)\varphi'(y)}{\varphi'(A_q^{|\varphi|}(x, y))} + (1-p) \frac{(1-r)\psi'(y)}{\psi'(A_r^{|\psi|}(x, y))} = 1 - p, \quad x, y \in I.
\]

Multiplying (13) by \( (1-r)\psi'(y) \) and (15) by \( r\psi'(x) \) we get

\[
\frac{pq(1-r)\varphi'(x)\psi'(y)}{\varphi'(A_q^{|\varphi|}(x, y))} + (1-p)r(1-r)\psi'(x)\psi'(y) = p(1 - r)\psi'(y),
\]

\[
\frac{p(1-q)r\varphi'(y)\psi'(x)}{\psi'(A_r^{|\psi|}(x, y))} + (1-p)(1-r)r\psi'(x)\psi'(y) = (1 - p)r\psi'(x),
\]

for all \( x, y \in I \). Subtracting these equalities we obtain

\[
\frac{pq(1-r)\varphi'(x)\psi'(y) - p(1-q)r\varphi'(y)\psi'(x)}{\varphi'(A_q^{|\varphi|}(x, y))} = p(1-r)\psi'(y) - (1-p)r\psi'(x)
\]

for all \( x, y \in I \). Setting here \( x := \varphi^{-1}(u) \) and \( y := \varphi^{-1}(v) \) we have

\[
\frac{pq(1-r)f(u)g(v) - p(1-q)rf(v)g(u)}{f(qu + (1-q)v)} = p(1-r)g(v) - r(1-p)g(u)
\]

for all \( u, v \in \varphi(I) \). Now Lemma 1 yields the assertion. ■

**Lemma 3.** Let \( q, r \in (0, 1) \) and let \( J \subset \mathbb{R} \) be an interval. If \( f, g : J \to (0, \infty) \) are continuously differentiable and satisfy (14) for all \( u, v \in J \), then there exists a number \( c > 0 \) such that

\[
f(u)^q g(u)^{1-r} = c, \quad u \in J.
\]

**Proof.** Differentiating both sides of (14) with respect to \( u \) we get

\[
qf'(qu + (1-q)v)[(1-r)g(v) - (1-q)g(u)] - (1-q)f(qu + (1-q)v)g'(u) = q(1-r)f'(u)g(v) - (1-q)rf'(u)g'(u)
\]

for all \( u, v \in J \). Letting \( v \to u \) we obtain

\[
qf'(u)g(u) = -(1-r)f(u)g'(u), \quad u \in J.
\]
Consequently,
\[ f(u)^q g(u)^{1-r} = c, \quad u \in J, \]
for some \( c > 0 \).

**Lemma 4.** Let \( q, r \in (0, 1) \) and let \( J \subset \mathbb{R} \) be an interval. If \( f : J \to (0, \infty) \) is continuous and satisfies the equation
\[
(16) \quad f(qu + (1 - q)v)\left[(1 - r)f(v)^{-q/(1-r)} - (1 - q)f(u)^{-q/(1-r)}\right] = q(1 - r)f(u)f(v)^{-q/(1-r)} - r(1 - q)f(v)f(u)^{-q/(1-r)}
\]
for all \( u, v \in J \), then either \( f \) is constant, or \( q + r = 1 \) and there exist \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( b \in \mathbb{R} \) such that
\[ f(u) = \lambda(u - b), \quad u \in J. \]

**Proof.** First assume that \( f \) is not one-to-one. Fix \( u_0, v_0 \in J \), \( u_0 < v_0 \), such that \( f(u_0) = f(v_0) \). We will show that \( f \) is constant on \([u_0, v_0]\). Suppose, on the contrary, that
\[
(17) \quad f(v) \neq f(u_0), \quad v \in (u_0, v_0).
\]
By (16) we have
\[
(q - r)f(qu_0 + (1 - q)v_0) = (q - r)f(u_0),
\]
whence (17) gives \( q = r \). Thus (16) takes the form
\[
(18) \quad f(qu + (1 - q)v)[f(u)^{q/(1-q)} - f(v)^{q/(1-q)}] = q[f(u)^{1/(1-q)} - f(v)^{1/(1-q)}]
\]
for all \( u, v \in J \). In particular, with the use of (17) and interchanging \( u \) and \( v \), we arrive at
\[
(19) \quad f(qu_0 + (1 - q)v) = f((1 - q)u_0 + qv), \quad v \in (u_0, v_0).
\]
If \( q = 1/2 \) equality (18) gives
\[
f\left(\frac{u + v}{2}\right) = \frac{f(u) + f(v)}{2}, \quad u, v \in J,
\]
whence \( f \) is affine, which is impossible in the case of (17) with \( f(u_0) = f(v_0) \). So we may further assume that \( q \neq 1/2 \). Let, for instance, \( q \in (0, 1/2) \). Fix a \( u \in (u_0, qu_0 + (1 - q)v_0) \). Put \( u_1 := u \) and
\[
(20) \quad u_{n+1} := (1 - q)u_0 + q \frac{u_n - qu_0}{1 - q}, \quad n \in \mathbb{N}.
\]
Clearly,
\[
(21) \quad u_n = qu_0 + (1 - q) \frac{u_n - qu_0}{1 - q}, \quad n \in \mathbb{N}.
\]
Using induction we deduce from (20) that $u_n > u_0$ for all $n \in \mathbb{N}$. This implies that

$$
\frac{u_n - qu_0}{1 - q} > u_0, \quad n \in \mathbb{N}.
$$

Hence, from (20), (21) and the condition $q < 1/2$ we infer that the sequence $(u_n)_{n \in \mathbb{N}}$ is decreasing and, in particular,

$$
\frac{u_n - qu_0}{1 - q} \in (u_0, v_0), \quad n \in \mathbb{N}.
$$

Consequently, by (20) the sequence $(u_n)_{n \in \mathbb{N}}$ converges to $u_0$ and, according to (19)–(21),

$$
f(u_0) = \lim_{n \to \infty} f(u_n) = f(u_1) = f(u),
$$

which, by (17), is impossible. This contradiction shows that there are a maximal interval $J_0 \subset J$ with nonempty interior and a positive $c$ such that

$$
f(u) = c, \quad u \in J_0.
$$

We will show that $J_0 = J$. Suppose this is not the case. For instance let $\sup J_0 < \sup J$. Fix a $u \in J_0, u < \sup J_0$. Then it follows from (16) that we can find an $\varepsilon > 0$ with

$$
c[(1 - r)f(v)^{-q/(1-r)} - (1 - q)c^{-q/(1-r)}]
= q(1 - r)c f(v)^{-q/(1-r)} - (1 - q)f(v)c^{-q/(1-r)},
$$

that is,

$$
f(v)^{q/(1-r)} = \frac{c^{1 + q/(1-r)}(1 - r)}{c - rf(v)}
$$

for all $v \in (\sup J_0, \sup J_0 + \varepsilon)$. Putting here $d := c^{1 + q/(1-r)}(1 - r)$ and $y := f(v)$ we see that

$$
y^{q/(1-r)} = \frac{d}{c - ry}
$$

for all $y$ from an interval with nonempty interior, which is impossible. Thus $J_0 = J$. In other words, $f$ is constant.

Now consider the case when $f$ is one-to-one. Putting $x = f(u)$ and $y = f(v)$ in (16) we get, for all $x, y \in f(J),$

$$
f(qf^{-1}(x) + (1 - q)f^{-1}(y))[(1 - r)y^{-q/(1-r)} - (1 - q)x^{-q/(1-r)}]
= q(1 - r)xy^{-q/(1-r)} - (1 - q)yx^{-q/(1-r)},
$$

which shows that the quasi-arithmetic mean $A_q^{[f^{-1}]}$ on $f(J)$ is conditionally homogeneous. On account of Proposition 1 there are $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that either

1° $f^{-1}(x) = ax^n + b$, $x \in f(J)$, for some $\eta \in \mathbb{R} \setminus \{0\}$,

or

2° $f^{-1}(x) = a \ln x + b$, $x \in f(J)$.
Suppose that case $2^\circ$ holds. Then we can find $c \in \mathbb{R} \setminus \{0\}$ and $\mu > 0$, $\mu \neq 1$, such that

$$f(u) = c\mu^u, \quad u \in J,$$

whence, by (16),

$$\mu^{qu+(1-q)v}[(1-r)\mu^{-qv/(1-r)} - (1-q)\mu^{-qu/(1-r)}]$$

$$= q(1-r)\mu^u \mu^{-qv/(1-r)} - r(1-q)\mu^v \mu^{-qu/(1-r)}$$

for all $u,v \in J$ and, consequently, for all $u,v \in \mathbb{R}$. Taking here $v = 0$ and putting $w := \mu^u$ we obtain

$$(1-r)w^q - (1-q)w^{-qr/(1-r)} = q(1-r)w - r(1-q)w^{-q/(1-r)}, \quad w \in (0,\infty).$$

Thus, since $q > 0$, $-qr/(1-r) < 0$ and $-q/(1-r) < 0$, we get $q = 1$ and $-qr/(1-r) = -q/(1-r)$, which is impossible. Therefore case $1^\circ$ is satisfied and it follows that

$$f(u) = c|u - u_0|^{\xi}, \quad u \in J,$$

with some $u_0 \in \mathbb{R} \setminus J$ and $c \in (0,\infty)$, $\xi \in \mathbb{R} \setminus \{0\}$. By (16) we get

$$|qu + (1-q)v - u_0|^{\xi}[(1-r)|v - u_0|^{-q\xi/(1-r)} - (1-q)|u - u_0|^{-q\xi/(1-r)}]$$

$$= q(1-r)|u - u_0|^{\xi}|v - u_0|^{-q\xi/(1-r)} - r(1-q)|v - u_0|^{\xi}|u - u_0|^{-q\xi/(1-r)}$$

for all $u,v \in J$, whence

$$|qu + (1-q)v|^{\xi}[(1-r)v^{-q\xi/(1-r)} - (1-q)v^{-q\xi/(1-r)}]$$

$$= q(1-r)v^{\xi}v^{-q\xi/(1-r)} - r(1-q)v^{\xi}v^{-q\xi/(1-r)}$$

either for all $u,v \in J - u_0$, or for all $u,v \in u_0 - J$, depending on whether $u_0 < u, u \in J$, or $u_0 > u, u \in J$. If $\xi < 0$, then (23) gives

$$u^{-\xi}v^{-\xi}[(1-r)v^{-q\xi/(1-r)} - (1-q)v^{-q\xi/(1-r)}]$$

$$= (qu + (1-q)v)^{-\xi}q(1-r)v^{-(1+\frac{q}{1-r})\xi} - r(1-q)v^{-(1+\frac{q}{1-r})\xi}$$

for all $u,v \in [0,\infty)$, whence, by setting $u = 0$,

$$q(1-r)(1-q)^{-\xi}v^{-(2+\frac{q}{1-r})\xi} = 0, \quad v \in [0,\infty),$$

which is impossible. Therefore $\xi > 0$ and now (23) gives

$$|qu + (1-q)v|^{\xi}[(1-r)v^{q\xi/(1-r)} - (1-q)v^{q\xi/(1-r)}]$$

$$= q(1-r)v^{(1+\frac{q}{1-r})\xi} - r(1-q)v^{(1+\frac{q}{1-r})\xi}$$

for all $u,v \in [0,\infty)$. Putting here $v = 0$ we arrive at

$$q^{\xi}u^{\xi}(1-r)v^{q\xi/(1-r)} = q(1-r)v^{(1+\frac{q}{1-r})\xi}, \quad u \in [0,\infty).$$

Hence $q^{\xi} = q$, i.e. $\xi = 1$. Thus, putting $u = 0$ in (24), we get

$$-(1-q)v^{(1-q)v^{q\xi/(1-r)}} = -r(1-q)v^{1+q/(1-r)}, \quad v \in [0,\infty),$$

for all $u,v \in [0,\infty)$.
and, consequently, \( r = 1 - q \). Moreover, according to (22) we have the desired affine form of \( f \). ■

4. **Main results.** Now we are in a position to prove the main result of the paper. It concerns the case of \( \alpha = \text{id} \), when equation (1) takes the form (12).

**Theorem 1.** Functions \( \varphi, \psi \in C^2 M(I) \) and numbers \( p, q, r \in (0,1) \) satisfy (12) if, and only if, the following two conditions are fulfilled:

(i) \[ p = \frac{r}{1 - q + r}, \]

(ii) there exist \( a, c \in \mathbb{R} \setminus \{0\} \) and \( b, d \in \mathbb{R} \) such that

\[ \varphi(x) = ax + b, \quad \psi(x) = cx + d, \quad x \in I, \]

or \( p = 1/2, \quad q + r = 1 \) and

\[ \varphi(x) = ae^{\lambda x} + b, \quad \psi(x) = ce^{-\lambda x} + d, \quad x \in I, \]

with some \( \lambda \in \mathbb{R} \setminus \{0\} \).

In that case \( A_p \) is the unique \( (A_{\varphi}^q, A_{\psi}^r) \)-invariant mean. Moreover, the iterates of \( (A_{\varphi}^q, A_{\psi}^r) \) approach \( A_p \).

**Proof.** Assume that \( \varphi, \psi \) satisfy equation (12) and put \( f := \varphi' \circ \varphi^{-1}, \quad g := \psi' \circ \psi^{-1} \). First assume additionally that \( \varphi'(x) \neq 0 \) and \( \psi'(x) \neq 0 \) for all \( x \in I \).

Without loss of generality we may assume that \( \varphi' \) and \( \psi' \) are positive. By Lemmas 2 and 3 there is a \( c_0 \in (0, \infty) \) such that

(25) \[ f(u)^q g(u)^{1-r} = c_0, \quad u \in \varphi(I), \]

and, moreover, \( f \) satisfies (16). According to Lemma 4 either

\[ 1^\circ \quad f(u) = a, \quad u \in \varphi(I), \text{ for some } a \in (0, \infty), \]

or

\[ 2^\circ \quad f(u) = \lambda(u - b), \quad u \in \varphi(I), \text{ with some } \lambda \in \mathbb{R} \setminus \{0\}, \quad b \in \mathbb{R}, \text{ and } \]

\[ q + r = 1. \]

First consider case \( 1^\circ \). Then, by (25), there is a \( c \in (0, \infty) \) such that

\[ g(u) = c, \quad u \in \varphi(I). \]

Thus

\[ \varphi'(x) = a, \quad \psi'(x) = c, \quad x \in I, \]

which completes the proof in case \( 1^\circ \). Now assume \( 2^\circ \). Then, by Lemma 1, we get \( p = 1/2 \). Moreover, from the definition of \( f \), we obtain

\[ \varphi'(x) = \lambda(\varphi(x) - b), \quad x \in I, \]

whence

\[ \varphi(x) = ae^{\lambda x} + b, \quad x \in I, \]
with an \( a \in \mathbb{R} \setminus \{0\} \). Now, by (25),
\[
g(u) = -c\lambda \frac{a}{u-b}, \quad u \in \varphi(I),
\]
with a \( c \in \mathbb{R} \setminus \{0\} \), which gives
\[
\psi'(x) = g(\varphi(x)) = -c\lambda e^{-\lambda x}, \quad x \in I,
\]
that is,
\[
\psi(x) = ce^{-\lambda x} + d, \quad x \in I,
\]
with a \( d \in \mathbb{R} \).

For the proof in the general case denote by \( Z_\varphi \) and \( Z_\psi \) the sets of zeros of \( \varphi' \) and \( \psi' \), respectively. It is enough to show that these sets are empty. Since they are closed sets with empty interiors, \( I \setminus (Z_\varphi \cup Z_\psi) \) is a nonempty open subset of \( I \). Let \( I_0 \) be any of its components and suppose that \( I_0 \neq I \). Then at least one end of \( I_0 \), say \( x_0 \), belongs to \( I \). Clearly, \( x_0 \in Z_\varphi \cup Z_\psi \), that is, either \( \varphi'(x_0) = 0 \), or \( \psi'(x_0) = 0 \). On the other hand, applying the just proved case of the theorem to \( \varphi|_{I_0} \) and \( \psi|_{I_0} \), and the continuity of \( \varphi' \) and \( \psi' \) at \( x_0 \), we infer that \( \varphi'(x_0) \neq 0 \) and \( \psi'(x_0) \neq 0 \). This contradiction shows that \( I_0 = I \) and, consequently, \( Z_\varphi = Z_\psi = \emptyset \).

The converse implication can be easily verified. The last paragraph of Theorem 1 is an immediate consequence of Theorem 1 in [6].

Theorem 1 implies the following result concerning equation (1) for general \( \alpha \).

**Theorem 2.** Let \( \alpha, \beta, \gamma \in CM(I) \). Assume that \( \beta \circ \alpha^{-1}, \gamma \circ \alpha^{-1} \in C^2M(I) \). The functions \( \alpha, \beta, \gamma \) and numbers \( p, q, r \in (0,1) \) satisfy (1) if, and only if, the following two conditions are fulfilled:

(i) \[
p = \frac{r}{1-q+r},
\]

(ii) there exist \( a, c \in \mathbb{R} \setminus \{0\} \) and \( b, d \in \mathbb{R} \) such that
\[
\beta(x) = a\alpha(x) + b, \quad \gamma(x) = c\alpha(x) + d, \quad x \in I,
\]
or \( p = 1/2 \), \( q + r = 1 \) and
\[
\beta(x) = ae^{\lambda \alpha(x)} + b, \quad \gamma(x) = ce^{-\lambda \alpha(x)} + d, \quad x \in I,
\]
with some \( \lambda \in \mathbb{R} \setminus \{0\} \).

In that case \( A_p^{[\alpha]} \) is the unique \( (A_q^{[\beta]}, A_r^{[\gamma]}) \)-invariant mean. Moreover, the iterates of \( (A_q^{[\beta]}, A_r^{[\gamma]}) \) approach \( A_p^{[\alpha]} \).

**Proof.** It is enough to observe that \( \alpha, \beta, \gamma \) satisfy (1) if, and only if, \( \varphi := \beta \circ \alpha^{-1} \) and \( \psi := \gamma \circ \alpha^{-1} \) satisfy (12) and next use Theorem 1.
5. An application

Remark. It can be seen from Theorem 1 (or by a direct substitution) that for \( q \in (0, 1) \) and \( \varphi, \psi : I \to \mathbb{R} \) given by

\[
\varphi(x) = e^{\lambda x}, \quad \psi(x) = e^{-\lambda x}
\]

with arbitrary \( \lambda \in \mathbb{R} \setminus \{0\} \) we have

\[
A_q^{[\varphi]}(x, y) + A_{1-q}^{[\psi]}(x, y) = x + y, \quad x, y \in I,
\]

that is, the arithmetic mean is the (unique) \( (A_q^{[\varphi]}, A_{1-q}^{[\psi]}) \)-invariant mean. Moreover, we have

\[
\lim_{n \to \infty} (A_q^{[\varphi]}, A_{1-q}^{[\psi]})^n(x, y) = \left( \frac{x + y}{2}, \frac{x + y}{2} \right), \quad x, y \in I.
\]

The above Remark allows us to obtain

Theorem 3. Let \( q \in (0, 1) \) and let \( \varphi, \psi : I \to \mathbb{R} \) be given by (26). A function \( F : I^2 \to \mathbb{R} \) is a solution of the equation

\[
F(A_q^{[\varphi]}(x, y), A_{1-q}^{[\psi]}(x, y)) = F(x, y),
\]

continuous at every point of the diagonal \( \{(x, y) \in I^2 : x = y\} \), if, and only if, there is a continuous function \( f : I \to \mathbb{R} \) such that

\[
F(x, y) = f \left( \frac{x + y}{2} \right), \quad x, y \in I.
\]

Proof. Assume that \( F : I^2 \to \mathbb{R} \) satisfies (29) and is continuous on the diagonal. By induction we get

\[
F(x, y) = F((A_q^{[\varphi]}, A_{1-q}^{[\psi]})^n(x, y)), \quad x, y \in I, \ n \in \mathbb{N}.
\]

Letting \( n \to \infty \) and making use of (28) and of the continuity of \( F \) at the diagonal we obtain

\[
F(x, y) = F(A(x, y), A(x, y)), \quad x, y \in I.
\]

Putting

\[
f(u) := F(u, u), \quad u \in I,
\]

we get the desired form of \( F \). In view of (27) the converse is clear. 

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