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Invariance in the class of weighted quasi-arithmetic means

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Abstract. Under the assumption of twice continuous differentiability of some of the functions involved we determine all the weighted quasi-arithmetic means M, N, K such that K is (M, N)-invariant, that is, $K \circ (M, N) = K$. Some applications to iteration theory and functional equations are presented.

1. Introduction. In the whole paper $I \subset \mathbb{R}$ denotes an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean* on I if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

If $M : I^2 \to \mathbb{R}$ is a mean, then M is *reflexive*, that is, $M(x, x) = x$ for all $x \in I$ and, consequently, for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$.

If $\alpha : I \to \mathbb{R}$ is a continuous strictly monotonic function and $p \in (0, 1)$ then $A_p^{[\alpha]} : I^2 \to I$, given by

$$A_p^{[\alpha]}(x, y) := \alpha^{-1}(p\alpha(x) + (1 - p)\alpha(y)), \qquad x, y \in I,$$

is a mean; it is called a *weighted quasi-arithmetic mean* with *generator* α and *weight* $(p, 1 - p)$. In the case $\alpha = \mathrm{id}|_I$ the mean $A_p^{[\alpha]}$ becomes the *weighted arithmetic mean* and is denoted by A_p; thus $A_p(x, y) = px + (1 - p)y$. We write simply A instead of $A_{1/2}$.

Let $M, N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is said to be *invariant with respect to* (M, N), briefly (M, N)-*invariant*, or K is called the *Gauss composition* of M and N (cf. [3], [5]), if $K \circ (M, N) = K$.

Fix $p, q, r \in (0, 1)$. Assuming twice continuous differentiability of some of the functions involved we determine all triples (α, β, γ) and (p, q, r) for

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which the weighted quasi-arithmetic mean $A_p^{[\alpha]}$ is $(A_q^{[\beta]}, A_r^{[\gamma]})$-*invariant*, i.e.

$$A_p^{[\alpha]} \circ (A_q^{[\beta]}, A_r^{[\gamma]}) = A_p^{[\alpha]} \tag{1}$$

(see Theorem 2 in Section 4).

The case when $p = q = r = 1/2$ (called simply the case of quasi-arithmetic means) has a long history. The analytic solutions were found by O. Sutô in 1914 [7]. The twice continuously differentiable solutions are given in [5]. Moreover, continuously differentiable solutions were found by Z. Daróczy and Zs. Páles [2], and finally, without any regularity assumption, the problem was solved by Z. Daróczy and Zs. Páles [3] (cf. also [1] as well as [3] for further references).

The fundamental role for Theorem 2 will be played here by Theorem 1 in Section 4, concerning the case of α being the identity function. In the proof of Theorem 1 we need a characterization of conditionally homogeneous weighted quasi-arithmetic means given by Proposition 1 proved in Section 2 and by the lemmas of Section 3.

In Section 5 we also apply Theorem 1 to establish the limit of the iteration sequence of some mean type mappings (Remark) and to solve a functional equation (Theorem 3).

2. Conditional homogeneity of the quasi-arithmetic mean. Denote by $CM(I)$ the class of all continuous strictly monotonic functions defined on I. In the proof of Theorem 1 we need the following

PROPOSITION 1. *Assume that $I \subset (0,\infty)$. Let $q \in (0,1)$ and $\sigma \in CM(I)$. The mean $A_q^{[\sigma]}$ is conditionally homogeneous, i.e.*

$$A_q^{[\sigma]}(sx, sy) = sA_q^{[\sigma]}(x,y)$$

for all $x, y \in I$ and $s > 0$ with $sx, sy \in I$ if, and only if, there are $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$ such that either

$$\sigma(x) = ax^\eta + b, \qquad x \in I,$$

for some $\eta \in \mathbb{R}\setminus\{0\}$, or

$$\sigma(x) = a \ln x + b, \qquad x \in I.$$

Before proving this proposition we will find the form of so-called conditional local groups of continuous affine maps.

PROPOSITION 2. *Let $\{X_s\}_{s\in S}$, where $S \subset \mathbb{R}$ is an interval containing 0, be a family of subsets of $\mathbb{R}$ and let $F : \bigcup_{s\in S}\{s\} \times X_s \to \mathbb{R}$. Assume that $F(\cdot,u)$ and $F(\cdot,v)$ are continuous for some different $u, v \in \bigcap_{s\in S} X_s$ and $F(s,\cdot)$ is continuous affine for every $s \in S$. Assume also that for every $s,t \in S$ with $s+t \in S$ there is an at least two-element set $U_{s,t} \subset X_s \cap X_{s+t}$*

such that

$$F(\{s\} \times U_{s,t}) \subset X_t \tag{2}$$

and

$$F(s+t, u) = F(t, F(s, u)), \qquad u \in U_{s,t}. \tag{3}$$

Then either F is constant, or there is an interval $S_0 \subset S$ containing 0 and such that

$$F(s, u) = cs + u, \qquad s \in S_0,\ u \in X_s, \tag{4}$$

with a $c \in \mathbb{R}$, or

$$F(s, u) = c_1^s(u - c_2) + c_2, \qquad s \in S_0,\ u \in X_s, \tag{5}$$

with some $c_1 \in (0, \infty)$ and $c_2 \in \mathbb{R}$.

Proof. Let $m : S \to \mathbb{R}$ and $k : S \to \mathbb{R}$ be such that

$$F(s, u) = m(s)u + k(s), \qquad s \in S,\ u \in X_s. \tag{6}$$

Taking any $u, v \in \bigcap_{s\in S} X_s$, $u \neq v$, such that $F(\cdot, u)$ and $F(\cdot, v)$ are continuous we see that m and k are continuous as linear combinations of $F(\cdot, u)$ and $F(\cdot, v)$.

Fix any $s, t \in S$ with $s + t \in S$ and choose a $U_{s,t} \subset X_s \cap X_{s+t}$ with at least two points, satisfying (2) and (3). Taking an arbitrary $u \in U_{s,t}$ and using (6), (3), and again (6), we get

$$\begin{aligned} m(s+t)u + k(s+t) &= F(s+t, u) = F(t, F(s, u)) \\ &= m(t)(m(s)u + k(s)) + k(t) \\ &= m(s)m(t)u + m(t)m(s) + k. \end{aligned}$$

Consequently,

$$m(s+t) = m(s)m(t) \tag{7}$$

and

$$k(s+t) = m(t)k(s) + k(t) \tag{8}$$

for all $s, t \in S$ with $s + t \in S$.

Assume that $m(s_0) = 0$ for some $s_0 \in S$ with $-s_0 \in S$. By (7) we get

$$m(0) = m(s_0)m(-s_0) = 0,$$

whence, again according to (7), we have $m(s) = m(s)m(0) = 0$ for all $s \in S$ and, consequently, k is constant by (8). Then, on account of (6) also F is constant. Now we can assume that m does not vanish in a subinterval $S_0 \subset S$ containing 0 and, in addition, satisfying $S_0 + S_0 \subset S$. Then, by (7), $m|_{S_0}$ is positive.

Since (7) is a multiplicative version of the restricted Cauchy equation, with the use of continuity arguments and making use of [4, Theorem 2,

p. 327] we infer that there exists a $c_1 \in (0, \infty)$ such that $m(s) = c_1^s$ for all $s \in S_0$.

If $c_1 = 1$ it follows from (8) that $k(s+t) = k(s) + k(t)$ for all $s, t \in S_0$ and, again by [4, Theorem 2, p. 327], we find a $c \in \mathbb{R}$ such that $k(s) = cs$ for all $s \in S_0$. Consequently, on account of (6) we have $F(s, u) = cs + u$ for all $s \in S_0$ and $u \in X_s$.

Now assume that $c_1 \neq 1$. Then, by (8), $k(s+t) = c_1^t k(s) + k(t)$ for all $s, t \in S_0$. By symmetry $k(s+t) = c_1^s k(t) + k(s)$ and

$$\frac{k(s)}{1 - c_1^s} = \frac{k(t)}{1 - c_1^t}, \qquad s, t \in S_0 \setminus \{0\}.$$

Thus there exists a $c_2 \in \mathbb{R}$ such that $k(s) = c_2(1 - c_1^s)$ for all $s \in S_0$. Consequently, by (6), we have $F(s, u) = c_1^s(u - c_2) + c_2$ for all $s \in S_0$ and $u \in X_s$. ∎

Proof of Proposition 1. Without loss of generality we may assume that $\operatorname{int} I \neq \emptyset$.

Assume that $A_q^{[\sigma]}$ is conditionally homogeneous, that is,

$$\sigma^{-1}(q\sigma(sx) + (1-q)\sigma(sy)) = s\sigma^{-1}(q\sigma(x) + (1-q)\sigma(y)) \tag{9}$$

for all $x, y \in I$ and $s > 0$ with $sx, sy \in I$. Fix an $x_0 \in \operatorname{int} I$ and put $u_0 = \sigma(x_0)$. Then $u_0 \in \operatorname{int} \sigma(I)$ and there exist $\delta > 0$ and $\delta_0 > 1$ such that

$$\sigma(sx_0) \in (u_0 - \delta, u_0 + \delta) \tag{10}$$

for all $s \in (1/\delta_0, \delta_0)$ and $s\sigma^{-1}(u) \in I$ for all $s \in (1/\delta_0, \delta_0)$ and $u \in (u_0 - \delta, u_0 + \delta)$. In particular, $(u_0 - \delta, u_0 + \delta) \subset \sigma(I)$. Put $S = (-\ln \delta_0, \ln \delta_0)$. Defining $F : S \times (u_0 - \delta, u_0 + \delta) \to \mathbb{R}$ by

$$F(s, u) = \sigma(e^s \sigma^{-1}(u))$$

we can rewrite (9) in the form

$$F(s, qu + (1-q)v) = qF(s, u) + (1-q)F(s, v), \qquad s \in S,\ u, v \in (u_0 - \delta, u_0 + \delta). \tag{11}$$

Fix an $s \in S$. Then, applying the Daróczy–Páles identity

$$q\biggl((1-q)\,\frac{u+v}{2} + qu\biggr) + (1-q)\biggl(q\,\frac{u+v}{2} + (1-q)v\biggr) = qu + (1-q)v,$$

we get

$$F\biggl(s, q\biggl((1-q)\,\frac{u+v}{2} + qu\biggr) + (1-q)\biggl(q\,\frac{u+v}{2} + (1-q)v\biggr)\biggr) = qF(s, u) + (1-q)F(s, v)$$

for all $u, v \in (u_0 - \delta, u_0 + \delta)$. Now, applying (11) twice to the left-hand side expression, for all $u, v \in (u_0 - \delta, u_0 + \delta)$ we have

$$q^2 F(s,u) + 2q(1-q)F\left(s, \frac{u+v}{2}\right) + (1-q)^2 F(s,v) = qF(s,u) + (1-q)F(s,v)$$

and, consequently,

$$2F\left(s, \frac{u+v}{2}\right) = F(s,u) + F(s,v), \qquad u, v \in (u_0 - \delta, u_0 + \delta).$$

By the continuity of F and the Jensen theorem we can find $m(s), k(s) \in \mathbb{R}$ such that

$$F(s,u) = m(s)u + k(s), \qquad u \in (u_0 - \delta, u_0 + \delta).$$

Thus we have shown that $F(s,\cdot)$ is continuous affine for every $s \in S$. Moreover, if $s, t, s+t \in S$ then, by (10), we have $F(s, u_0) \in (u_0 - \delta, u_0 + \delta)$, whence

$$F(s,u) \in (u_0 - \delta, u_0 + \delta)$$

for u from a neighbourhood $U_{s,t} \subset (u_0 - \delta, u_0 + \delta)$ of u_0 and

$$F(t, F(s,u)) = \sigma(e^t \sigma^{-1} \sigma(e^s \sigma^{-1}(u)))) = \sigma(e^{s+t} \sigma^{-1}(u)) = F(s+t, u)$$

for all $u \in U_{s,t}$. Since F is not constant, on account of Proposition 2 there is an interval $S_0 \subset S$ containing 0 and such that F is of the form (4) or (5), where $c_1 \in (0, \infty)$ and $c_2, c \in \mathbb{R}$.

In this way we have shown that every point x of $\mathrm{int} I$ has a neighbourhood in which σ has one of the following forms:

$$\sigma(x) = a \ln x + b, \qquad \sigma(x) = ax^\eta + b$$

with $a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}$ and $\eta \in \mathbb{R}\backslash\{0\}$ depending on that point. By standard arguments we deduce that one of these forms holds on the whole interval I.

The converse assertion is obvious. ∎

3. Auxiliary results. Denote by $C^n M(I)$ the subclass of $CM(I)$ consisting of functions which are n-times continuously differentiable.

LEMMA 1. *Let $p, q, r \in (0,1)$. If $\varphi, \psi \in C^1 M(I)$ satisfy*

$$pA_q^{[\varphi]}(x,y) + (1-p)A_r^{[\psi]}(x,y) = px + (1-p)y \tag{12}$$

and $\varphi'(x) \neq 0$, $\psi'(x) \neq 0$ for all $x \in I$, then

$$p = \frac{r}{1-q+r}.$$

Proof. Differentiation of both sides of (12) with respect to x gives

$$p\,\frac{q\varphi'(x)}{\varphi'(A_q^{[\varphi]}(x,y))} + (1-p)\,\frac{r\psi'(x)}{\psi'(A_r^{[\psi]}(x,y))} = p, \qquad x, y \in I. \tag{13}$$

Letting $y \to x$ and taking into account the reflexivity of the means we obtain the desired equality. ∎

Lemma 2. *Let $p, q, r \in (0,1)$. If $\varphi, \psi \in C^1M(I)$ satisfy* (12) *and $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ for all $x \in I$, then $f := \varphi' \circ \varphi^{-1}$ and $g := \psi' \circ \varphi^{-1}$ satisfy the equation*

$$(14) \qquad f(qu + (1-q)v)[(1-r)g(v) - (1-q)g(u)] = q(1-r)f(u)g(v) - r(1-q)f(v)g(u)$$

for all $u, v \in \varphi(I)$.

Proof. Differentiating both sides of (12), first with respect to x and then with respect to y, we get (13) and

$$(15) \qquad p\,\frac{(1-q)\varphi'(y)}{\varphi'(A_q^{[\varphi]}(x,y))} + (1-p)\,\frac{(1-r)\psi'(y)}{\psi'(A_r^{[\psi]}(x,y))} = 1-p, \qquad x, y \in I.$$

Multiplying (13) by $(1-r)\psi'(y)$ and (15) by $r\psi'(x)$ we get

$$\frac{pq(1-r)\varphi'(x)\psi'(y)}{\varphi'(A_q^{[\varphi]}(x,y))} + \frac{(1-p)r(1-r)\psi'(x)\psi'(y)}{\psi'(A_r^{[\psi]}(x,y))} = p(1-r)\psi'(y),$$
$$\frac{p(1-q)r\varphi'(y)\psi'(x)}{\varphi'(A_q^{[\varphi]}(x,y))} + \frac{(1-p)(1-r)r\psi'(x)\psi'(y)}{\psi'(A_r^{[\psi]}(x,y))} = (1-p)r\psi'(x),$$

for all $x, y \in I$. Subtracting these equalities we obtain

$$\frac{pq(1-r)\varphi'(x)\psi'(y) - p(1-q)r\varphi'(y)\psi'(x)}{\varphi'(A_q^{[\varphi]}(x,y))} = p(1-r)\psi'(y) - (1-p)r\psi'(x)$$

for all $x, y \in I$. Setting here $x := \varphi^{-1}(u)$ and $y := \varphi^{-1}(v)$ we have

$$\frac{pq(1-r)f(u)g(v) - p(1-q)rf(v)g(u)}{f(qu + (1-q)v)} = p(1-r)g(v) - r(1-p)g(u)$$

for all $u, v \in \varphi(I)$. Now Lemma 1 yields the assertion. ∎

Lemma 3. *Let $q, r \in (0,1)$ and let $J \subset \mathbb{R}$ be an interval. If $f, g : J \to (0,\infty)$ are continuously differentiable and satisfy* (14) *for all $u, v \in J$, then there exists a number $c > 0$ such that*

$$f(u)^q g(u)^{1-r} = c, \qquad u \in J.$$

Proof. Differentiating both sides of (14) with respect to u we get

$$qf'(qu + (1-q)v)[(1-r)g(v) - (1-q)g(u)] - (1-q)f(qu + (1-q)v)g'(u) = q(1-r)f'(u)g(v) - (1-q)rf(v)g'(u)$$

for all $u, v \in J$. Letting $v \to u$ we obtain

$$qf'(u)g(u) = -(1-r)f(u)g'(u), \qquad u \in J.$$

Consequently,

$$f(u)^q g(u)^{1-r} = c, \qquad u \in J,$$

for some $c > 0$.

LEMMA 4. *Let $q, r \in (0,1)$ and let $J \subset \mathbb{R}$ be an interval. If $f : J \to (0,\infty)$ is continuous and satisfies the equation*

$$\begin{aligned}(16)\qquad f(qu+(1-q)v)[(1-r)f(v)^{-q/(1-r)} - (1-q)f(u)^{-q/(1-r)}]\\ = q(1-r)f(u)f(v)^{-q/(1-r)} - r(1-q)f(v)f(u)^{-q/(1-r)}\end{aligned}$$

for all $u, v \in J$, then either f is constant, or $q + r = 1$ and there exist $\lambda \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$ such that

$$f(u) = \lambda(u-b), \qquad u \in J.$$

Proof. First assume that f is not one-to-one. Fix $u_0, v_0 \in J$, $u_0 < v_0$, such that $f(u_0) = f(v_0)$. We will show that f is constant on $[u_0, v_0]$. Suppose, on the contrary, that

$$(17)\qquad f(v) \neq f(u_0), \qquad v \in (u_0, v_0).$$

By (16) we have

$$(q-r)f(qu_0+(1-q)v_0) = (q-r)f(u_0),$$

whence (17) gives $q = r$. Thus (16) takes the form

$$\begin{aligned}(18)\qquad f(qu+(1-q)v)[f(u)^{q/(1-q)} - f(v)^{q/(1-q)}]\\ = q[f(u)^{1/(1-q)} - f(v)^{1/(1-q)}]\end{aligned}$$

for all $u, v \in J$. In particular, with the use of (17) and interchanging u and v, we arrive at

$$(19)\qquad f(qu_0+(1-q)v) = f((1-q)u_0+qv), \qquad v \in (u_0, v_0).$$

If $q = 1/2$ equality (18) gives

$$f\left(\frac{u+v}{2}\right) = \frac{f(u)+f(v)}{2}, \qquad u, v \in J,$$

whence f is affine, which is impossible in the case of (17) with $f(u_0) = f(v_0)$. So we may further assume that $q \neq 1/2$. Let, for instance, $q \in (0, 1/2)$. Fix a $u \in (u_0, qu_0 + (1-q)v_0)$. Put $u_1 := u$ and

$$(20)\qquad u_{n+1} := (1-q)u_0 + q\,\frac{u_n - qu_0}{1-q}, \qquad n \in \mathbb{N}.$$

Clearly,

$$(21)\qquad u_n = qu_0 + (1-q)\,\frac{u_n - qu_0}{1-q}, \qquad n \in \mathbb{N}.$$

Using induction we deduce from (20) that $u_n > u_0$ for all $n \in \mathbb{N}$. This implies that

$$\frac{u_n - qu_0}{1-q} > u_0, \qquad n \in \mathbb{N}.$$

Hence, from (20), (21) and the condition $q < 1/2$ we infer that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing and, in particular,

$$\frac{u_n - qu_0}{1-q} \in (u_0, v_0), \qquad n \in \mathbb{N}.$$

Consequently, by (20) the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u_0 and, according to (19)–(21),

$$f(u_0) = \lim_{n\to\infty} f(u_n) = f(u_1) = f(u),$$

which, by (17), is impossible. This contradiction shows that there are a maximal interval $J_0 \subset J$ with nonempty interior and a positive c such that

$$f(u) = c, \qquad u \in J_0.$$

We will show that $J_0 = J$. Suppose this is not the case. For instance let $\sup J_0 < \sup J$. Fix a $u \in J_0$, $u < \sup J_0$. Then it follows from (16) that we can find an $\varepsilon > 0$ with

$$\begin{aligned} &c[(1-r)f(v)^{-q/(1-r)} - (1-q)c^{-q/(1-r)}] \\ &\qquad = q(1-r)cf(v)^{-q/(1-r)} - r(1-q)f(v)c^{-q/(1-r)}, \end{aligned}$$

that is,

$$f(v)^{q/(1-r)} = \frac{c^{1+q/(1-r)}(1-r)}{c - rf(v)}$$

for all $v \in (\sup J_0, \sup J_0 + \varepsilon)$. Putting here $d := c^{1+q/(1-r)}(1-r)$ and $y := f(v)$ we see that

$$y^{q/(1-r)} = \frac{d}{c - ry}$$

for all y from an interval with nonempty interior, which is impossible. Thus $J_0 = J$. In other words, f is constant.

Now consider the case when f is one-to-one. Putting $x = f(u)$ and $y = f(v)$ in (16) we get, for all $x, y \in f(J)$,

$$\begin{aligned} &f(qf^{-1}(x) + (1-q)f^{-1}(y))[(1-r)y^{-q/(1-r)} - (1-q)x^{-q/(1-r)}] \\ &\qquad = q(1-r)xy^{-q/(1-r)} - r(1-q)yx^{-q/(1-r)}, \end{aligned}$$

which shows that the quasi-arithmetic mean $A_q^{[f^{-1}]}$ on $f(J)$ is conditionally homogeneous. On account of Proposition 1 there are $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that either

1° $f^{-1}(x) = ax^\eta + b$, $x \in f(J)$, for some $\eta \in \mathbb{R} \setminus \{0\}$,

or

2° $f^{-1}(x) = a \ln x + b$, $x \in f(J)$.

Suppose that case 2° holds. Then we can find $c \in \mathbb{R} \setminus \{0\}$ and $\mu > 0,\ \mu \neq 1$, such that

$$f(u) = c\mu^u, \qquad u \in J,$$

whence, by (16),

$$\mu^{qu+(1-q)v}[(1-r)\mu^{-qv/(1-r)} - (1-q)\mu^{-qu/(1-r)}] \\ = q(1-r)\mu^u\mu^{-qv/(1-r)} - r(1-q)\mu^v\mu^{-qu/(1-r)}$$

for all $u, v \in J$ and, consequently, for all $u, v \in \mathbb{R}$. Taking here $v = 0$ and putting $w := \mu^u$ we obtain

$$(1-r)w^q - (1-q)w^{-qr/(1-r)} = q(1-r)w - r(1-q)w^{-q/(1-r)}, \qquad w \in (0, \infty).$$

Thus, since $q > 0,\ -qr/(1-r) < 0$ and $-q/(1-r) < 0$, we get $q = 1$ and $-qr/(1-r) = -q/(1-r)$, which is impossible. Therefore case 1° is satisfied and it follows that

$$f(u) = c|u - u_0|^\xi, \qquad u \in J, \tag{22}$$

with some $u_0 \in \mathbb{R} \setminus J$ and $c \in (0, \infty),\ \xi \in \mathbb{R} \setminus \{0\}$. By (16) we get

$$|qu + (1-q)v - u_0|^\xi[(1-r)|v - u_0|^{-q\xi/(1-r)} - (1-q)|u - u_0|^{-q\xi/(1-r)}] \\ = q(1-r)|u - u_0|^\xi|v - u_0|^{-q\xi/(1-r)} - r(1-q)|v - u_0|^\xi|u - u_0|^{-q\xi/(1-r)}$$

for all $u, v \in J$, whence

$$(qu + (1-q)v)^\xi[(1-r)v^{-q\xi/(1-r)} - (1-q)u^{-q\xi/(1-r)}] \\ = q(1-r)u^\xi v^{-q\xi/(1-r)} - r(1-q)v^\xi u^{-q\xi/(1-r)} \tag{23}$$

either for all $u, v \in J - u_0$, or for all $u, v \in u_0 - J$, depending on whether $u_0 < u, u \in J$, or $u_0 > u, u \in J$. If $\xi < 0$, then (23) gives

$$u^{-\xi}v^{-\xi}[(1-r)v^{-q\xi/(1-r)} - (1-q)u^{-q\xi/(1-r)}] \\ = (qu + (1-q)v)^{-\xi}(q(1-r)v^{-(1+\frac{q}{1-r})\xi} - r(1-q)u^{-(1+\frac{q}{1-r})\xi})$$

for all $u, v \in [0, \infty)$, whence, by setting $u = 0$,

$$q(1-r)(1-q)^{-\xi}v^{-(2+\frac{q}{1-r})\xi} = 0, \qquad v \in [0, \infty),$$

which is impossible. Therefore $\xi > 0$ and now (23) gives

$$(qu + (1-q)v)^\xi[(1-r)u^{q\xi/(1-r)} - (1-q)v^{q\xi/(1-r)}] \\ = q(1-r)u^{(1+\frac{q}{1-r})\xi} - r(1-q)v^{(1+\frac{q}{1-r})\xi} \tag{24}$$

for all $u, v \in [0, \infty)$. Putting here $v = 0$ we arrive at

$$q^\xi u^\xi(1-r)u^{q\xi/(1-r)} = q(1-r)u^{(1+\frac{q}{1-r})\xi}, \qquad u \in [0, \infty).$$

Hence $q^\xi = q$, i.e. $\xi = 1$. Thus, putting $u = 0$ in (24), we get

$$-(1-q)v(1-q)v^{q/(1-r)} = -r(1-q)v^{1+q/(1-r)}, \qquad v \in [0, \infty),$$

and, consequently, $r = 1-q$. Moreover, according to (22) we have the desired affine form of f. ∎

4. Main results. Now we are in a position to prove the main result of the paper. It concerns the case of $\alpha = \mathrm{id}$, when equation (1) takes the form (12).

THEOREM 1. *Functions $\varphi, \psi \in C^2M(I)$ and numbers $p, q, r \in (0,1)$ satisfy* (12) *if*, *and only if*, *the following two conditions are fulfilled*:

(i)
$$p = \frac{r}{1-q+r},$$

(ii) *there exist $a, c \in \mathbb{R}\setminus\{0\}$ and $b, d \in \mathbb{R}$ such that*
$$\varphi(x) = ax + b, \qquad \psi(x) = cx + d, \qquad x \in I,$$
or $p = 1/2$, $q + r = 1$ and
$$\varphi(x) = ae^{\lambda x} + b, \qquad \psi(x) = ce^{-\lambda x} + d, \qquad x \in I,$$
with some $\lambda \in \mathbb{R}\setminus\{0\}$.

In that case A_p is the unique $(A_q^{[\varphi]}, A_r^{[\psi]})$-invariant mean. Moreover, *the iterates of $(A_q^{[\varphi]}, A_r^{[\psi]})$ approach A_p.*

Proof. Assume that φ, ψ satisfy equation (12) and put $f := \varphi' \circ \varphi^{-1}$, $g := \psi' \circ \varphi^{-1}$. First assume additionally that $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ for all $x \in I$. Without loss of generality we may assume that φ' and ψ' are positive. By Lemmas 2 and 3 there is a $c_0 \in (0, \infty)$ such that

$$f(u)^q g(u)^{1-r} = c_0, \qquad u \in \varphi(I), \tag{25}$$

and, moreover, f satisfies (16). According to Lemma 4 either

1° $f(u) = a$, $u \in \varphi(I)$, for some $a \in (0, \infty)$,

or

2° $f(u) = \lambda(u - b)$, $u \in \varphi(I)$, with some $\lambda \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}$, and $q + r = 1$.

First consider case 1°. Then, by (25), there is a $c \in (0, \infty)$ such that
$$g(u) = c, \qquad u \in \varphi(I).$$
Thus
$$\varphi'(x) = a, \qquad \psi'(x) = c, \qquad x \in I,$$
which completes the proof in case 1°. Now assume 2°. Then, by Lemma 1, we get $p = 1/2$. Moreover, from the definition of f, we obtain
$$\varphi'(x) = \lambda(\varphi(x) - b), \qquad x \in I,$$
whence
$$\varphi(x) = ae^{\lambda x} + b, \qquad x \in I,$$

with an $a \in \mathbb{R} \setminus \{0\}$. Now, by (25),

$$g(u) = -c\lambda \frac{a}{u-b}, \qquad u \in \varphi(I),$$

with a $c \in \mathbb{R} \setminus \{0\}$, which gives

$$\psi'(x) = g(\varphi(x)) = -c\lambda e^{-\lambda x}, \qquad x \in I,$$

that is,

$$\psi(x) = ce^{-\lambda x} + d, \qquad x \in I,$$

with a $d \in \mathbb{R}$.

For the proof in the general case denote by Z_φ and Z_ψ the sets of zeros of φ' and ψ', respectively. It is enough to show that these sets are empty. Since they are closed sets with empty interiors, $I \setminus (Z_\varphi \cup Z_\psi)$ is a nonempty open subset of I. Let I_0 be any of its components and suppose that $I_0 \neq I$. Then at least one end of I_0, say x_0, belongs to I. Clearly, $x_0 \in Z_\varphi \cup Z_\psi$, that is, either $\varphi'(x_0) = 0$, or $\psi'(x_0) = 0$. On the other hand, applying the just proved case of the theorem to $\varphi|_{I_0}$ and $\psi|_{I_0}$, and the continuity of φ' and ψ' at x_0, we infer that $\varphi'(x_0) \neq 0$ and $\psi'(x_0) \neq 0$. This contradiction shows that $I_0 = I$ and, consequently, $Z_\varphi = Z_\psi = \emptyset$.

The converse implication can be easily verified. The last paragraph of Theorem 1 is an immediate consequence of Theorem 1 in [6]. ∎

Theorem 1 implies the following result concerning equation (1) for general α.

THEOREM 2. *Let $\alpha, \beta, \gamma \in CM(I)$. Assume that $\beta \circ \alpha^{-1}, \gamma \circ \alpha^{-1} \in C^2M(I)$. The functions α, β, γ and numbers $p, q, r \in (0,1)$ satisfy* (1) *if, and only if, the following two conditions are fulfilled:*

(i)
$$p = \frac{r}{1-q+r},$$

(ii) *there exist $a, c \in \mathbb{R} \setminus \{0\}$ and $b, d \in \mathbb{R}$ such that*

$$\beta(x) = a\alpha(x) + b, \qquad \gamma(x) = c\alpha(x) + d, \qquad x \in I,$$

or $p = 1/2,\ q + r = 1$ and

$$\beta(x) = ae^{\lambda\alpha(x)} + b, \qquad \gamma(x) = ce^{-\lambda\alpha(x)} + d, \qquad x \in I,$$

with some $\lambda \in \mathbb{R} \setminus \{0\}$.

In that case $A_p^{[\alpha]}$ is the unique $(A_q^{[\beta]}, A_r^{[\gamma]})$-invariant mean. Moreover, the iterates of $(A_q^{[\beta]}, A_r^{[\gamma]})$ approach $A_p^{[\alpha]}$.

Proof. It is enough to observe that α, β, γ satisfy (1) if, and only if, $\varphi := \beta \circ \alpha^{-1}$ and $\psi := \gamma \circ \alpha^{-1}$ satisfy (12) and next use Theorem 1. ∎

5. An application

Remark. It can be seen from Theorem 1 (or by a direct substitution) that for $q \in (0,1)$ and $\varphi, \psi : I \to \mathbb{R}$ given by

$$\varphi(x) = e^{\lambda x}, \qquad \psi(x) = e^{-\lambda x} \tag{26}$$

with arbitrary $\lambda \in \mathbb{R} \setminus \{0\}$ we have

$$A_q^{[\varphi]}(x,y) + A_{1-q}^{[\psi]}(x,y) = x + y, \qquad x, y \in I, \tag{27}$$

that is, the arithmetic mean is the (unique) $(A_q^{[\varphi]}, A_{1-q}^{[\psi]})$-invariant mean. Moreover, we have

$$\lim_{n\to\infty} (A_q^{[\varphi]}, A_{1-q}^{[\psi]})^n(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right), \qquad x, y \in I. \tag{28}$$

The above Remark allows us to obtain

Theorem 3. *Let $q \in (0,1)$ and let $\varphi, \psi : I \to \mathbb{R}$ be given by* (26). *A function $F : I^2 \to \mathbb{R}$ is a solution of the equation*

$$F(A_q^{[\varphi]}(x,y), A_r^{[\psi]}(x,y)) = F(x,y), \tag{29}$$

continuous at every point of the diagonal $\{(x,y) \in I^2 : x = y\}$, if, and only if, there is a continuous function $f : I \to \mathbb{R}$ such that

$$F(x,y) = f\left(\frac{x+y}{2}\right), \qquad x, y \in I.$$

Proof. Assume that $F : I^2 \to \mathbb{R}$ satisfies (29) and is continuous on the diagonal. By induction we get

$$F(x,y) = F((A_q^{[\varphi]}, A_r^{[\psi]})^n(x,y)), \qquad x, y \in I,\ n \in \mathbb{N}.$$

Letting $n \to \infty$ and making use of (28) and of the continuity of F at the diagonal we obtain

$$F(x,y) = F(A(x,y), A(x,y)), \qquad x, y \in I.$$

Putting

$$f(u) := F(u,u), \qquad u \in I,$$

we get the desired form of F. In view of (27) the converse is clear. ∎

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