

Non-isotropic Hausdorff capacity of exceptional sets for pluri-Green potentials in the unit ball of \mathbb{C}^n

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Abstract. We study questions related to exceptional sets of pluri-Green potentials V_μ in the unit ball B of \mathbb{C}^n in terms of non-isotropic Hausdorff capacity. For suitable measures μ on the ball B , the pluri-Green potentials V_μ are defined by

$$V_\mu(z) = \int_B \log \frac{1}{|\phi_z(w)|} d\mu(w),$$

where for a fixed $z \in B$, ϕ_z denotes the holomorphic automorphism of B satisfying $\phi_z(0) = z$, $\phi_z(z) = 0$ and $(\phi_z \circ \phi_z)(w) = w$ for every $w \in B$. If $d\mu(w) = f(w)d\lambda(w)$, where f is a non-negative measurable function of B , and λ is the measure on B , invariant under all holomorphic automorphisms of B , then V_μ is denoted by V_f . The main result of this paper is as follows: Let f be a non-negative measurable function on B satisfying

$$\int_B (1 - |z|^2)^p f^p(z) d\lambda(z) < \infty$$

for some p with $1 < p < n/(n-1)$ and some α with $0 < \alpha < n + p - np$. Then for each τ with $1 \leq \tau \leq n/\alpha$, there exists a set $E_\tau \subseteq S$ with $H_{\alpha\tau}(E_\tau) = 0$ such that

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau,c}(\zeta)}} V_f(z) = 0$$

for all points $\zeta \in S \setminus E_\tau$. In the above, for $\alpha > 0$, H_α denotes the non-isotropic Hausdorff capacity on S , and for $\zeta \in S = \partial B$, $\tau \geq 1$, and $c > 0$, $\mathcal{T}_{\tau,c}(\zeta)$ are the regions defined by

$$\mathcal{T}_{\tau,c}(\zeta) = \{z \in B : |1 - \langle z, \zeta \rangle|^\tau < c(1 - |z|^2)\}.$$

1. Introduction. Exceptional sets of Blaschke products, holomorphic functions, or Green potentials in the unit disc of the complex plane are usually described in terms of Hausdorff capacity (measure). For example, Samuelsson [15] and Cargo [8] studied exceptional sets of Blaschke products

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in terms of Hausdorff capacity. In [7], Berman and Cohn prove several results concerning the existence of certain non-tangential and tangential limits of Blaschke products and their subproducts in terms of Hausdorff capacity. On the other hand, in higher dimension, the non-isotropic Hausdorff capacity is much more appropriate for the description of exceptional sets of holomorphic functions, or invariant Green potentials.

Many authors have investigated the non-isotropic Hausdorff capacity as a tool for the description of exceptional sets of some function classes. For example, in [17] and [18], Stoll investigated some questions related to the non-isotropic Hausdorff capacity of exceptional sets of invariant Green potentials in the unit ball B of \mathbb{C}^n . In [4], Ahern and Cohn, using the non-isotropic Hausdorff capacity, studied exceptional sets for Hardy–Sobolev functions in the unit ball in \mathbb{C}^n . Also, in [5], using L^p -estimates for maximal functions of invariant Poisson integrals Ahern and Cohn derived that the exceptional sets of the Poisson integrals of potentials are sets of zero Hausdorff measure. Ahern and Cascante [3] investigated exceptional sets for Poisson integrals of potentials on the unit sphere in \mathbb{C}^n . Some results for exceptional sets with zero non-isotropic Hausdorff measure for holomorphic Sobolev functions in the unit ball of \mathbb{C}^n are given in [2] by Ahern. In [11], Cohn gave a characterization of exceptional sets for a class of holomorphic functions in terms of non-isotropic Hausdorff capacities. Ahern and Nagel [6] studied, among many other things, the existence of tangential limits of Poisson integrals of L^p -potentials and functions in Besov spaces within tangential approach regions in terms of appropriate Hausdorff measure. Cascante and Ortega [9] characterized some closed tangential exceptional sets for Hardy–Sobolev spaces in the unit ball of \mathbb{C}^n in terms of non-isotropic Hausdorff measure. The main goal of Jamming and Roginskaya [12] and Roginskaya [13] is to describe the singular part of a measure on the unit sphere S in \mathbb{C}^n in terms of behavior of its harmonic and \mathcal{M} -harmonic extension near the boundary. Their description involves comparisons of the Hausdorff measure and the non-isotropic Hausdorff measure, associated to the isotropic Euclidean and non-isotropic (Korányi) distance on S , respectively.

Homogeneous spaces, in connection with Hausdorff measure and boundary behavior of some function classes have been studied by many authors. For example, Sjödin [16] investigates analogues of classical Riesz capacity and Hausdorff measure in a homogeneous space. Since the unit sphere S in \mathbb{C}^n , with the Korányi distance, is a space of homogeneous type, the results of his paper can be applied for this metric space. In [10], Cifuentes, Dorronsoro and Sueiro proved strong L^p estimates for weighted maximal functions of Poisson integrals on spaces of homogeneous type with respect to Hausdorff content. Sueiro [20] studies questions related to Fatou’s theorem and “fractional Cauchy kernel” for the unit ball B in \mathbb{C}^n . By studying

weighted maximal functions in metric spaces of homogeneous type he obtained estimates on fractional Cauchy kernel and a strong-type maximal estimate. He then used these estimates to obtain a result on the Hausdorff measure, defined by the Korányi balls, of the exceptional set of boundary limits of the fractional Cauchy integrals.

Let \mathcal{B} be the class of all Blaschke products on the open unit disc U of the complex plane \mathbb{C} , i.e., all functions of the form

$$\beta(z) = z^m \prod_{n=1}^{\infty} \frac{\bar{a}_n}{a_n} \cdot \frac{a_n - z}{1 - \bar{a}_n z},$$

where m is a nonnegative integer and $\{a_n\}$ is a sequence of non-zero complex numbers in U which satisfies the condition

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

The class of all non-decreasing, continuous functions h on $[0, \infty)$ such that $h(t) > 0$ if $t > 0$ and $h(t)/t$ is non-increasing on $(0, \infty)$ will be denoted by \mathcal{H} . Examples of such functions are $h(t) = t^\alpha$, $0 \leq \alpha \leq 1$. The subclass of \mathcal{H} , consisting of the functions $h \in \mathcal{H}$ satisfying the additional condition $h(0) = 0$, is denoted by \mathcal{H}_0 .

If $h \in \mathcal{H}_0$, then h is a determining function for a Hausdorff capacity (measure) H_h defined on the compact subsets K of the boundary ∂U of U by

$$H_h(K) = \inf \left\{ \sum_{j=1}^{\infty} h(r_j) \right\},$$

where the infimum is taken over all countable covers $\{D(\zeta_j, r_j) : j = 1, 2, \dots\}$ of K by discs $D(\zeta_j, r_j)$ centered at $\zeta_j \in \partial U$ and of radius r_j . For an arbitrary subset A of ∂U , the Hausdorff capacity H_h is defined by

$$H_h(A) = \sup\{H_h(K) : K \text{ a compact subset of } A\}.$$

Following Samuelsson's paper [15], for $\beta \in \mathcal{B}$, $\zeta \in \partial U$, and $t > 0$ let

$$\sigma(\beta, \zeta, t) = \sum_{|a_k - \zeta| \leq t} (1 - |a_k|).$$

Further, for $h \in \mathcal{H}$, $\beta \in \mathcal{B}$ we need the sets

$$\mathcal{L}(\beta, h) = \left\{ \zeta \in \partial U : \liminf_{r \rightarrow 1^-} \frac{1 - r}{h(1 - r)} \log \frac{1}{|\beta(r\zeta)|} = +\infty \right\},$$

$$\underline{\Sigma}(\beta, h) = \left\{ \zeta \in \partial U : \liminf_{t \rightarrow 0^+} \frac{\sigma(\beta, \zeta, t)}{h(t)} = +\infty \right\},$$

$$\begin{aligned}\bar{\Sigma}(\beta, h) &= \left\{ \zeta \in \partial U : \limsup_{t \rightarrow 0^+} \frac{\sigma(\beta, \zeta, t)}{h(t)} = +\infty \right\}, \\ \mathcal{L}_0(\beta, h) &= \left\{ \zeta \in \partial U : \liminf_{r \rightarrow 1^-} \frac{1-r}{h(1-r)} \log \frac{1}{\beta(r\zeta)} > 0 \right\}.\end{aligned}$$

In like manner, for $0 < \alpha < 1$ let

$$\mathcal{L}_\alpha(\beta, h) = \left\{ \zeta \in \partial U : \liminf_{\substack{z \rightarrow \zeta \\ z \in S_\alpha(\zeta)}} \frac{|z - \zeta|}{h(|z - \zeta|)} \log \frac{1}{|\beta(z)|} = +\infty \right\}$$

for all *Stolz domains* $S_\alpha(\zeta)$ defined by

$$S_\alpha(\zeta) = \{z \in U : |z - \zeta| \leq \sqrt{\alpha - \alpha^2}, |\arg(1 - \bar{\zeta}z)| \leq \arcsin \alpha\}.$$

In [15], Samuelsson proved the following results:

THEOREM A. *If β is a Blaschke product, then the set $\mathcal{L}_0(\beta, 1)$ is empty. If $h \in \mathcal{H}_0$, then*

$$H_h(\mathcal{L}_0(\beta, h)) = 0.$$

THEOREM B. *Let $h \in \mathcal{H}$ and let $\beta \in \mathcal{B}$. Then*

$$\underline{\Sigma}(\beta, h) \subseteq \mathcal{L}_\alpha(\beta, h) \subseteq \mathcal{L}(\beta, h) \subseteq \bar{\Sigma}(\beta, h).$$

This paper was motivated by the work of Stoll [18], concerning the *non-isotropic Hausdorff capacity* of invariant Green potentials G_μ on the unit ball B of \mathbb{C}^n . For $\zeta \in S = \partial B$, $\tau \geq 1$, and $c > 0$, let

$$\mathcal{T}_{\tau, c}(\zeta) = \{z \in B : |1 - \langle z, \zeta \rangle|^\tau < c(1 - |z|^2)\}.$$

The regions $\mathcal{T}_{\tau, c}(\zeta)$ have tangential contact with the boundary S of the ball B in all directions at the boundary point $\zeta \in S$.

When $\tau = 1$ and $2c = \alpha > 1$ the regions $\mathcal{T}_{\tau, c}(\zeta)$ reduce to the *admissible domains* of Korányi

$$D_\alpha(\zeta) = \left\{ z \in B : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

A function V on B is said to have a \mathcal{T}_τ -limit L at $\zeta \in S$ if

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau, c}(\zeta)}} V(z) = L$$

for every $c > 1$ ($c > 1/2$ when $\tau = 1$).

The *exceptional set* for a function f defined on the unit ball B is the set of all points $\zeta \in S$ such that f fails to have a $\mathcal{T}_\tau(\zeta)$ -limit at ζ .

In [18], Stoll proved the following result concerning tangential limits and exceptional sets:

THEOREM C. *Let f be a non-negative measurable function on B satisfying*

$$\int_B (1 - |z|^2)^\beta f^p(z) d\lambda(z) < \infty$$

for some $p > n$ and some β with $0 < \beta < n$. Then for each τ with $1 \leq \tau \leq n/\beta$, there exists a set $E_\tau \subseteq S$ with $H_{\beta\tau}(E_\tau) = 0$ such that G_f has $\mathcal{T}_\tau(\zeta)$ -limit 0 at all $\zeta \in S \setminus E_\tau$.

In the above, for $0 < \alpha < n$, H_α denotes the non-isotropic Hausdorff capacity on S , G_f denotes the Green potential of f , and λ denotes a measure on B invariant under the group of all holomorphic automorphisms of the ball B .

The methods used to prove Theorem C allowed Stoll to prove the following result:

THEOREM D. *Let $\{a_j\}$ be a sequence in B satisfying*

$$\sum_{j=1}^{\infty} (1 - |a_j|^2)^\beta < \infty$$

for some β with $0 < \beta < n$, and let μ be the measure given by $\mu = \sum \delta_{a_j}$, where δ_{a_j} is the unit point-mass measure at a_j . Then for each τ with $1 \leq \tau \leq n/\beta$, there exists a set $E_\tau \subseteq S$ with $H_{\beta\tau}(E_\tau) = 0$ such that G_μ has \mathcal{T}_τ -limit 0 at all $\zeta \in S \setminus E_\tau$.

In the above theorem, G_μ denotes the invariant Green potential of the measure μ on the unit ball B .

The purpose of our paper is to investigate analogues of Stoll's results (Theorems C and D) for pluri-Green potentials in the unit ball of \mathbb{C}^n . We also extend Samuelsson's results (Theorems A and B) by considering non-isotropic Hausdorff capacity of pluri-Green potentials.

The paper is organized as follows. In Section 2, we introduce the necessary notation and definitions. In Section 3, we give some preliminary results that will be used for the proofs of the main results which are given in Section 4.

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2. Notation. Throughout this paper we use the notation and terminology of Rudin's book [14] and most of the general results used in our paper can be found in that book. For $n \geq 1$, \mathbb{C}^n denotes the n -dimensional complex space, with the usual inner product $\langle z, w \rangle$ and norm $|z| = \sqrt{\langle z, z \rangle}$. B will denote the unit ball in \mathbb{C}^n with $d\nu$ the Lebesgue measure on B normalized

so that $\nu(B) = 1$, and $S = \partial B$ will be the boundary of B with $d\sigma$ the surface area measure on S , again normalized so that $\sigma(S) = 1$.

For $\delta > 0$ and $\zeta \in S$ let

$$Q(\zeta, \delta) = \{\eta \in S : |1 - \langle \eta, \zeta \rangle| < \delta\}.$$

REMARK. The set $Q(\zeta, \delta)$ is called a *Korányi ball*.

The following fact, whose proof can be found in [14, p. 66], will be used:

PROPOSITION K. *The function*

$$d(w, z) = \sqrt{|1 - \langle w, z \rangle|}, \quad w, z \in \overline{B},$$

satisfies the triangle inequality

$$d(w, z) \leq d(w, x) + d(x, z) \quad \text{for all } w, x, z \in \overline{B}.$$

Restricted to the sphere S , the function d is a metric, and the sets $Q(\zeta, \delta)$ are the corresponding open balls.

Let h be a non-decreasing function on $[0, \infty)$ vanishing at 0 and satisfying the condition

$$h(2x) \leq ch(x)$$

for some positive constant c . Such a function h will be referred to as *allowed*.

REMARK. It is easy to check that each $h \in \mathcal{H}_0$ is an allowed function.

For an allowed function h , the *non-isotropic Hausdorff capacity* H_h of a compact subset K of S is defined by

$$H_h(K) = \inf \left\{ \sum_{j=1}^{\infty} h(r_j) \right\},$$

where the infimum is taken over all countable covers $\{Q(\zeta_j, r_j) : j \in \mathbb{N}\}$ of K by Korányi balls $Q(\zeta_j, r_j)$ centered at $\zeta_j \in S$ and of radius r_j . For an arbitrary subset A of S , the non-isotropic Hausdorff capacity H_h is defined by

$$H_h(A) = \sup\{H_h(K) : K \text{ a compact subset of } A\}.$$

For basic definitions, background and more information on the non-isotropic Hausdorff capacity we refer to [12], [13] and [16].

For each $z \in B$, ϕ_z denotes the holomorphic automorphism of B satisfying $\phi_z(0) = z$, $\phi_z(z) = 0$ and $\phi_z \circ \phi_z(w) = w$ for every $w \in B$. The following identity will be useful throughout:

$$(2.1) \quad 1 - |\phi_a(z)|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \langle z, a \rangle|^2}$$

for all $a \in B$ and $z \in \overline{B}$.

\mathcal{M} will denote the Möbius group (of all holomorphic automorphisms) of the ball B . By Cartan's theorem it follows that $\psi \in \mathcal{M}$ if and only if $\psi = u \circ \phi_a$ for a unique unitary transformation u on \mathbb{C}^n , where $a = \psi^{-1}(0)$.

For $r > 0$, let

$$B_r = \{z \in \mathbb{C}^n : |z| < r\},$$

and for $a \in B$, $a \neq 0$, let

$$E(a) = \phi_a(B_{1/2}).$$

Since ϕ_a is an involution, $z \in E(a)$ if and only if $|\phi_a(z)| < 1/2$.

We denote by λ the measure on B defined by

$$d\lambda(z) = \frac{d\nu(z)}{(1 - |z|^2)^{n+1}}.$$

This measure is \mathcal{M} -invariant, i.e.

$$\int_B f(z) d\lambda(z) = \int_B (f \circ \psi)(z) d\lambda(z)$$

for each $f \in L^1(d\lambda)$ and all $\psi \in \mathcal{M}$.

The Laplace–Beltrami operator (invariant Laplacian) $\tilde{\Delta}$ on B is given by

$$\tilde{\Delta}f(z) = \frac{1}{n+1} \Delta(f \circ \phi_z)(0) = \frac{4(1 - |z|^2)}{n+1} \sum_{i,j=1}^n (\delta_{i,j} - z_i \bar{z}_j) \frac{\partial^2 f}{\partial z_j \partial \bar{z}_i},$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

and Δ is the usual Laplacian in \mathbb{R}^{2n} .

The Green function for the Laplace–Beltrami operator $\tilde{\Delta}$ is given by

$$G(z, w) = g(\phi_z(w)),$$

where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1 - t^2)^{n-1} t^{-2n+1} dt.$$

REMARK. For $n = 1$, $g(z) = -\log(|z|)$, and the Green function for the Laplace–Beltrami operator $\tilde{\Delta}$ coincides with the usual Green function G on the unit disc U .

If μ is a non-negative regular Borel measure on B , then the function

$$G_\mu(z) = \int_B G(z, w) d\mu(w)$$

is called the *invariant Green potential* of μ on B , provided

$$\int_B G(z_0, w) d\mu(w) < \infty$$

for some $z_0 \in B$. The above condition is satisfied if and only if

$$\int_B (1 - |w|^2)^n d\mu(w) < \infty.$$

For each fixed $w \in B$, the function

$$z \mapsto \log \frac{1}{|\phi_z(w)|}$$

is called the *pluri-Green function* of B with pole at w . When $n = 1$,

$$\log \frac{1}{|\phi_a(z)|} = \log \left| \frac{1 - z\bar{a}}{z - a} \right|,$$

which is the usual Green function on U .

If μ is a non-negative regular Borel measure on B then the function

$$V_\mu(z) = \int_B \log \frac{1}{|\phi_z(w)|} d\mu(w)$$

is called the *invariant pluri-Green potential* of μ on B , provided

$$\int_B \log \frac{1}{|\phi_{z_0}(w)|} d\mu(w) < \infty,$$

for some $z_0 \in B$. In [1] it is shown that this is equivalent to

$$(2.2) \quad \int_B (1 - |w|^2) d\mu(w) < \infty.$$

For $\zeta \in S$ and $t > 0$ let

$$B_t(\zeta) = \{z \in B : |1 - \langle z, \zeta \rangle| \leq t\},$$

and for a non-negative regular Borel measure μ satisfying the growth condition (2.2) let

$$(2.3) \quad r(\mu, \zeta, t) = \int_{B_t(\zeta)} (1 - |w|^2) d\mu(w).$$

For a regular Borel measure μ on B satisfying the growth condition (2.2), and for $h \in \mathcal{H}$ and $0 < \alpha < 1$ we introduce the sets

$$\mathcal{L}(\mu, h) = \left\{ \zeta \in S : \liminf_{r \rightarrow 1^-} \frac{1 - r}{h(1 - r)} V_\mu(r\zeta) = +\infty \right\},$$

$$\mathcal{L}_\alpha(\mu, h) = \left\{ \zeta \in S : \liminf_{\substack{z \rightarrow \zeta \\ z \in D_\alpha(\zeta)}} \frac{|1 - \langle z, \zeta \rangle|}{h(|1 - \langle z, \zeta \rangle|)} V_\mu(z) = +\infty \right\},$$

$$\begin{aligned} \underline{\mathcal{R}}(\mu, h) &= \left\{ \zeta \in S : \liminf_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} = +\infty \right\}, \\ \overline{\mathcal{R}}(\mu, h) &= \left\{ \zeta \in S : \limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} = +\infty \right\}, \\ \mathcal{L}_0(\mu, h) &= \left\{ \zeta \in S : \liminf_{r \rightarrow 1^-} \frac{1-r}{h(1-r)} V_\mu(r\zeta) > 0 \right\}. \end{aligned}$$

A final remark on notation: throughout this paper we will use the same letter C or $C(a, b)$ to denote various absolute positive constants or positive constants which depend only on the indicated variables, but not necessarily the same on any two occurrences.

3. Preliminary results. In this section several preliminary results are given that will be used for the proof of the main theorems.

For $z \in B$, c a real number, and $\alpha > n$ consider

$$J_{c,\alpha}(z) = \int_B \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{\alpha+c}} d\lambda(w).$$

The following asymptotic estimate will play an important role in the proof of our main results; its proof can be found in [14, pp. 17–18].

PROPOSITION 3.1.

$$J_{c,\alpha}(z) \approx \begin{cases} (1 - |z|^2)^{-c}, & c > 0, \\ \log \frac{1}{1 - |z|^2}, & c = 0, \\ 1, & c < 0. \end{cases}$$

The notation $a(z) \approx b(z)$ means that the ratio $a(z)/b(z)$ has a finite limit as $|z| \rightarrow 0$.

The next, ‘‘Frostman type’’ theorem (Theorem 1 in [11], proved for $h(t) = t^m$) is the key to the proof of our main results. The extension to arbitrary allowed h (see [2]) poses no difficulty.

THEOREM 3.2. *Let h be an allowed function. For a compact subset K of S , $H_h(K) > 0$ if and only if K contains the support of a positive measure $\nu \neq 0$ satisfying*

$$(3.1) \quad \nu(Q(\zeta, \delta)) \leq Ch(\delta)$$

for all $\delta > 0$ and $\zeta \in S$ and an absolute constant C .

The following covering lemma (see [14, Lemma 5.2.3]) will also be needed:

LEMMA 3.3. *Suppose that E is the union of a finite collection $\{Q(\zeta_i, \delta_i)\}$ of Korányi balls. Then there exists a finite disjoint subcollection*

$\{Q(\zeta_{i_k}, \delta_{i_k})\}_{k=1}^m$ such that

$$E \subseteq \bigcup_{k=1}^m Q(\zeta_{i_k}, 9\delta_{i_k}) \quad \text{and} \quad \sigma(E) \leq C_n \sum_{k=1}^m \sigma(Q(\zeta_{i_k}, \delta_{i_k})),$$

where C_n is a constant depending only on n .

On several occasions in the paper we shall use the following elementary inequality:

$$(3.2) \quad \log x \leq x - 1 \quad \text{for } x > 0.$$

Now, we proceed to several other lemmas, analogous to those in [15].

LEMMA 3.4. *Let t be a fixed number such that $0 < t < 1/3$ and let $I_t = [1 - 3t, 1 - 2t]$. For $\zeta \in S$ and $z \in B$ let*

$$G_{\zeta,t}(z) = \int_{I_t} \log \frac{1}{|\phi_z(r\zeta)|} dr.$$

Then there exist positive constants C_1 and C_2 , independent of ζ , z and t , such that

$$(3.3) \quad G_{\zeta,t}(z) \leq C_1(1 - |z|^2)$$

and

$$(3.4) \quad G_{\zeta,t}(z) \leq C_2 t^2 \frac{1 - |z|}{|1 - \langle z, \zeta \rangle|^2}$$

for all $z \in B$.

Proof. First, suppose that $\zeta = e_1 = (1, \dots, 0)$. For $z \in B$, by the identity (2.1) we have

$$\begin{aligned} G_{e_1,t}(z) &= \frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{(1-r^2)(1-|z|^2)}{|1 - \langle z, re_1 \rangle|^2 - (1-r^2)(1-|z|^2)} \right\} dr \\ &\leq \frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{(1-r^2)(1-|z|^2)}{||z| - r|^2} \right\} dr. \end{aligned}$$

CASE 1. If $\min\{||z| - r| : r \in I_t\} \geq t$, then by (3.2) we have

$$\frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{12t(1-|z|)}{t^2} \right\} dr \leq 6 \frac{1-|z|}{t} \int_{I_t} dr \leq 6(1-|z|),$$

which proves (3.3) for $\zeta = e_1$ in this case.

CASE 2. If $\min\{||z| - r| : r \in I_t\} \leq t$, then $1 - |z| = 1 - r - (|z| - r) > 1 - r - t \geq t$. Therefore in this case we have

$$\begin{aligned}
 G_{e_1,t}(z) &\leq \frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{6t(1-|z|^2)}{||z|-r|^2} \right\} dr \leq \frac{1}{2} \int_{I_t} \log \left\{ 1 + \left(\frac{4(1-|z|)}{||z|-r|} \right)^2 \right\} dr \\
 &\leq \int_{|z|}^{\infty} \log \left\{ 1 + \left(\frac{4(1-|z|)}{||z|-r|} \right)^2 \right\} dr \\
 &\leq 4(1-|z|) \int_0^{\infty} \frac{1}{x^2} \log(1+x^2) dx \leq C(1-|z|^2),
 \end{aligned}$$

which proves (3.3) for $\zeta = e_1$.

To prove (3.4) for $\zeta = e_1$, first notice that for $z = (z_1, \dots, z_n) \in B$ we have

$$(3.5) \quad |1 - \langle z, re_1 \rangle|^2 - (1-r^2)(1-|z|^2) \geq |z_1 - r|^2.$$

The last inequality is easily verified. Indeed, using the obvious inequality $|z| \geq |z_1|$ we have

$$\begin{aligned}
 |1 - \langle z, re_1 \rangle|^2 - (1-r^2)(1-|z|^2) &= |1 - rz_1|^2 - (1-r^2)(1-|z|^2) \\
 &\geq |1 - rz_1|^2 - (1-r^2)(1-|z_1|^2) \\
 &= |r - z_1|^2.
 \end{aligned}$$

Now let $z = (z_1, \dots, z_n) \in B$ be such that $|1 - z_1| > 4t$. Then for $r \in I_t$,

$$\begin{aligned}
 |z_1 - r| &\geq |1 - z_1| - (1-r) \geq |1 - z_1| - 3t \\
 &> |1 - z_1| - \frac{3}{4}|1 - z_1| = \frac{1}{4}|1 - z_1|.
 \end{aligned}$$

Therefore from (3.5) and (3.2) it follows that

$$\begin{aligned}
 G_{e_1,t}(z) &= \frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{(1-r^2)(1-|z|^2)}{|1 - \langle z, re_1 \rangle|^2 - (1-r^2)(1-|z|^2)} \right\} dr \\
 &\leq \frac{1}{2} \int_{I_t} \log \left\{ 1 + \frac{12t(1-|z|)}{|z_1 - r|^2} \right\} dr \leq \frac{1}{2} \int_{I_t} \frac{(12t)(1-|z|)}{|z_1 - r|^2} dr \\
 &\leq \frac{1}{2} \frac{(12t)(16t)(1-|z|)}{|1 - z_1|^2} = 96t^2 \frac{1-|z|}{|1 - \langle z, e_1 \rangle|^2}.
 \end{aligned}$$

If $|1 - z_1| < 4t$, then by (3.3) for $\zeta = e_1$, we have

$$\begin{aligned}
 G_{e_1,t}(z) &\leq C_1(1-|z|^2) = C_1(1-|z|^2) \frac{|1 - z_1|^2}{|1 - z_1|^2} \\
 &\leq 2C_1 16t^2 \frac{1-|z|}{|1 - \langle z, e_1 \rangle|^2} = C_2 t^2 \frac{1-|z|}{|1 - \langle z, \zeta \rangle|^2}.
 \end{aligned}$$

This completes the proof of (3.4) for $\zeta = e_1$. Now let $\zeta \in S$ be arbitrary, and let ψ be the unitary transformation on \mathbb{C}^n such that $\psi(e_1) = \zeta$. From the identity (2.1) we have $|\phi_z(\psi(w))| = |\phi_{\psi^{-1}(z)}(w)|$, and so the proof of (3.3) and (3.4) for arbitrary ζ reduces to the one for $\zeta = e_1$. ■

LEMMA 3.5. *Let t and I_t be as in Lemma 3.4 and let $\zeta \in S$ and $h \in \mathcal{H}$ be such that*

$$r(\mu, \zeta, x) \leq h(x)$$

for all $x > 0$ and every non-negative regular Borel measure μ on B satisfying the growth condition (2.2). Then there exists an absolute constant C which does not depend on t such that

$$\inf_{r \in I_t} \{V_\mu(r\zeta)\} \leq C \frac{h(t)}{t}.$$

Proof. From the definition of $G_{\zeta,t}$ and Fubini's theorem we have

$$\begin{aligned} \int_B G_{\zeta,t}(z) d\mu(z) &= \int_{I_t} \left(\int_B \log \frac{1}{|\phi_z(r\zeta)|} d\mu(z) \right) dr \\ &= \int_{I_t} \left(\int_B \log \frac{1}{|\phi_{r\zeta}(z)|} d\mu(z) \right) dr \\ &= \int_{I_t} V_\mu(r\zeta) dr \geq \int_{I_t} (\inf_{s \in I_t} \{V_\mu(s\zeta)\}) dr = t \inf_{s \in I_t} V_\mu(s\zeta). \end{aligned}$$

Thus

$$\inf_{r \in I_t} \{V_\mu(r\zeta)\} \leq \frac{1}{t} \int_B G_{\zeta,t}(z) d\mu(z)$$

for every t with $0 < t < 1/3$. Therefore from (3.2) and (3.3) it follows that

$$\begin{aligned} \inf_{r \in I_t} \{V_\mu(r\zeta)\} &\leq \frac{1}{t} \int_{B_t(\zeta)} G_{\zeta,t}(z) d\mu(z) + \frac{1}{t} \int_{B \setminus B_t(\zeta)} G_{\zeta,t}(z) d\mu(z) \\ &\leq \frac{C_1}{t} \int_{B_t(\zeta)} (1 - |z|^2) d\mu(z) + C_2 t \int_{B \setminus B_t(\zeta)} \frac{1 - |z|^2}{|1 - \langle z, \zeta \rangle|^2} d\mu(z). \end{aligned}$$

If for $j = 1, 2, \dots$ we set

$$A_{j,t}(\zeta) = \{z \in B : 2^{j-1}t < |1 - \langle z, \zeta \rangle| \leq 2^j t\},$$

then from the last estimate and the definition (2.3) of $r(\mu, \zeta, t)$ we have

$$\begin{aligned} \inf_{r \in I_t} \{V_\mu(r\zeta)\} &\leq \frac{C_1}{t} r(\mu, \zeta, t) + C_2 t \sum_{j=1}^{\infty} \int_{A_{j,t}(\zeta)} \frac{1 - |z|^2}{|1 - \langle z, \zeta \rangle|^2} d\mu(z) \\ &\leq \frac{C_1}{t} r(\mu, \zeta, t) + C_2 t \sum_{j=1}^{\infty} \int_{A_{j,t}(\zeta)} \frac{1 - |z|^2}{t^2 2^{2j-2}} d\mu(z) \\ &\leq \frac{C_1}{t} r(\mu, \zeta, t) + \frac{C_2}{t} \sum_{j=1}^{\infty} 2^{2-2j} r(\mu, \zeta, 2^j t) = \frac{C}{t} h(t). \end{aligned}$$

By hypothesis, $r(\mu, \zeta, x) \leq h(x)$ for every $x > 0$, and therefore from the above it follows that

$$\begin{aligned} \inf_{r \in I_t} \{V_\mu(r\zeta)\} &\leq \frac{C_1}{t} h(t) + \frac{C_2}{t} \sum_{j=1}^{\infty} 2^{-2j+2} h(2^j t) \\ &\leq \frac{C_1}{t} h(t) + \frac{C_2}{t} \sum_{j=1}^{\infty} 2^{-2j+2} 2^j h(t) = \frac{C_1 + C_2'}{t} h(t). \end{aligned}$$

Above we used the fact that $h(kx) \leq kh(x)$ for every $k \geq 1$ and every $x > 0$. This follows easily from the hypothesis that the function $h(t)/t$ is non-increasing on the interval $(0, \infty)$. ■

LEMMA 3.6. *Let $\alpha > 1/2$ be fixed. Then there exist two positive constants C_1 and C_2 such that*

$$(a) \quad C_1 \liminf_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} \leq \liminf_{\substack{z \rightarrow \zeta \\ z \in D_\alpha(\zeta)}} \frac{|1 - \langle z, \zeta \rangle|}{h(|1 - \langle z, \zeta \rangle|)} V_\mu(z),$$

and

$$(b) \quad \liminf_{r \rightarrow 1^-} \frac{1-r}{h(1-r)} V_\mu(r\zeta) \leq C_2 \limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)}$$

for all $\zeta \in S$, $h \in \mathcal{H}$, and all non-negative regular Borel measures μ satisfying the growth condition (2.2).

Proof. From the identity (2.1) and the inequality (3.2) we have

$$\begin{aligned} V_\mu(z) &= -\frac{1}{2} \int_B \log \left\{ 1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} \right\} d\mu(w) \\ &\geq \frac{1}{2} \int_B \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} d\mu(w) \\ &\geq \frac{1}{2} \int_{B_t(\zeta)} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} d\mu(w), \end{aligned}$$

where $t = |1 - \langle z, \zeta \rangle|$. Therefore, for all $z \in D_\alpha(\zeta)$ we have

$$(3.6) \quad V_\mu(z) \geq \frac{|1 - \langle z, \zeta \rangle|}{\alpha} \int_{B_t(\zeta)} \frac{(1 - |w|)}{|1 - \langle z, w \rangle|^2} d\mu(w).$$

Since $\sqrt{|1 - \langle z, w \rangle|}$ satisfies the triangle inequality on $\bar{B} \times \bar{B}$ (Theorem K), for all $w \in B_t(\zeta)$ we have

$$|1 - \langle z, w \rangle| \leq 4|1 - \langle z, \zeta \rangle|.$$

Therefore, for all $z \in D_\alpha(\zeta)$, from (3.6) we have

$$\begin{aligned} V_\mu(z) &\geq \frac{|1 - \langle z, \zeta \rangle|}{\alpha} \frac{1}{16|1 - \langle z, \zeta \rangle|^2} \int_{B_t(\zeta)} (1 - |w|) d\mu(w) \\ &\geq \frac{r(\mu, \zeta, |1 - \langle z, \zeta \rangle|)}{32\alpha|1 - \langle z, \zeta \rangle|}. \end{aligned}$$

Hence, for all $z \in D_\alpha(\zeta)$,

$$\frac{|1 - \langle z, \zeta \rangle|}{h(|1 - \langle z, \zeta \rangle|)} V_\mu(z) \geq C(\alpha) \frac{r(\mu, \zeta, |1 - \langle z, \zeta \rangle|)}{h(|1 - \langle z, \zeta \rangle|)},$$

from which our estimate (a) easily follows.

To prove (b), we may assume that

$$\limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} = l, \quad 0 \leq l < \infty.$$

Let $\varepsilon > 0$ be given. Then there exists $t_0 > 0$ such that $r(\mu, \zeta, t) < (l + \varepsilon)h(t)$ for all $0 < t \leq t_0$. Let μ_1 be the restriction of μ to $B_{t_0}(\zeta)$, and let

$$\begin{aligned} V_1(z) &= \int_{B_{t_0}(\zeta)} \log \frac{1}{|\phi_z(w)|} d\mu(w) = \int_B \log \frac{1}{|\phi_z(w)|} d\mu_1(w), \\ V_2(z) &= \int_{B \setminus B_{t_0}(\zeta)} \log \frac{1}{|\phi_z(w)|} d\mu(w). \end{aligned}$$

It is obvious that $V_2(r\zeta) \rightarrow 0$ as $r \rightarrow 1^-$. Now consider V_1 . Let $x > 0$. If $x \leq t_0$, then

$$r(\mu_1, \zeta, x) = r(\mu, \zeta, x) < (l + \varepsilon)h(x).$$

If $x > t_0$, then

$$r(\mu_1, \zeta, x) = r(\mu, \zeta, t_0) < (l + \varepsilon)h(t_0) \leq (l + \varepsilon)h(x).$$

Therefore, for all $x > 0$ we have

$$r(\mu_1, \zeta, x) \leq (l + \varepsilon)h(x).$$

Applying now Lemma 3.5 to μ_1 and $(l + \varepsilon)h(x)$ we obtain

$$\begin{aligned} \inf \left\{ \frac{(1-r)}{h(1-r)} V_{\mu_1}(r\zeta) : r \in I_t \right\} &\leq \frac{3t}{h(3t)} \inf \{ V_{\mu_1}(r\zeta) : r \in I_t \} \\ &\leq \frac{3t}{h(3t)} C_1(l + \varepsilon) \frac{h(t)}{t} = 3C_1 \frac{h(t)}{h(3t)} (l + \varepsilon) \leq 3C_1(l + \varepsilon). \end{aligned}$$

In the last step above we have used the fact that $h(t) \leq h(3t)$.

Therefore

$$\liminf_{r \rightarrow 1^-} \frac{1-r}{h(1-r)} V_{\mu_1}(r\zeta) \leq 3C_1(l + \varepsilon),$$

and since $\varepsilon > 0$ was arbitrary it follows that

$$\liminf_{r \rightarrow 1^-} \frac{1-r}{h(1-r)} V_\mu(r\zeta) \leq 3C_1 l = 3C_1 \limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)},$$

as required. ■

LEMMA 3.7. *Let $h \in \mathcal{H}_0$. For a non-negative regular Borel measure μ on B which satisfies the growth condition (2.2) let*

$$\overline{\mathcal{R}}_0(\mu, h) = \left\{ \zeta \in S : \limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} > 0 \right\}.$$

Then

$$H_h(\overline{\mathcal{R}}_0(\mu, h)) = 0.$$

Proof. Since H_h is subadditive it is enough to show that $H_h(\overline{\mathcal{R}}_a(\mu, h)) = 0$ for all $a > 0$, where for $a > 0$,

$$\overline{\mathcal{R}}_a(\mu, h) = \left\{ \zeta \in S : \limsup_{t \rightarrow 0^+} \frac{r(\mu, \zeta, t)}{h(t)} > a \right\}.$$

Let $\varepsilon > 0$ be given. From the growth condition (2.2) and the regularity of the measure μ it follows that there exists a compact set $K \subseteq B$ such that

$$(3.7) \quad \int_{B \setminus K} (1 - |w|^2) d\mu(w) < \varepsilon.$$

Let F be a compact subset of $\overline{\mathcal{R}}_a$, and let

$$\mathcal{F}_a = \left\{ Q(\zeta, 2t) : \frac{r(\mu, \zeta, t)}{h(t)} > a, \zeta \in F, 0 < t < \varrho(K) \right\},$$

where $\varrho(K) = \inf\{1 - |z| : z \in K\}$. Notice that \mathcal{F}_a is a covering of F by Korányi balls. Since F is a compact set, there exist points $\zeta_1, \dots, \zeta_m \in S$ and positive numbers t_1, \dots, t_m such that

$$F \subseteq \bigcup_{j=1}^m Q(\zeta_j, 2t_j)$$

and

$$r(\mu, \zeta_j, t_j) > ah(t_j), \quad 0 < t_j < \varrho(K).$$

By the covering Lemma 3.3, there exists a finite disjoint subcollection of \mathcal{F}_a , which for convenience we denote by $\{Q(\eta_j, 2t_j) : j = 1, \dots, k\}$, such that

$$(3.8) \quad F \subseteq \bigcup_{j=1}^k Q(\eta_j, 18t_j),$$

$$r(\mu, \eta_j, t_j) > ah(t_j), \quad 0 < t_j < \varrho(K), \quad j = 1, \dots, k.$$

Now let ν be any positive measure on S satisfying the condition (3.1) in Theorem 3.2:

$$\nu(Q(\zeta, \delta)) \leq Ch(\delta)$$

for all $\zeta \in S$ and every $\delta > 0$. Then from (3.9) we have

$$\begin{aligned} (3.9) \quad \nu(F) &\leq \sum_{j=1}^k \nu(Q(\eta_j, 18t_j)) \leq C \sum_{j=1}^k h(18t_j) \leq C \sum_{j=1}^k h(t_j) \\ &\leq \frac{C}{a} \sum_{j=1}^k r(\mu, \eta_j, t_j) = \frac{C}{a} \sum_{j=1}^k \int_{B_{t_j}(\eta_j)} (1 - |w|^2) d\mu(w). \end{aligned}$$

If $z \in B$ is such that $|1 - \langle z, \eta_j \rangle| \leq t_j$ for some $j = 1, \dots, k$, then

$$1 - |z| \leq |1 - \langle z, \eta_j \rangle| \leq t_j < \varrho(K).$$

Therefore

$$(3.10) \quad \bigcup_{j=1}^k B_{t_j}(\eta_j) \subseteq \bigcup_{j=1}^k \{z \in B : |1 - \langle z, \eta_j \rangle| \leq t_j\} \subseteq B \setminus K.$$

From the trivial inequality $|1 - \langle z, \zeta \rangle| \geq \frac{1}{2} |1 - \langle z/|z|, \zeta \rangle|$ for $z \in B$ and $\zeta \in S$ we have

$$\{z \in B : |1 - \langle z, \eta_j \rangle| \leq t_j\} \subseteq \{z \in B : |1 - \langle z/|z|, \eta_j \rangle| \leq 2t_j\}.$$

Since $\{Q(\eta_j, 2t_j) : j = 1, \dots, k\}$ is a pairwise disjoint family of Korányi balls, from the last inclusion it follows that the family $\{B_{t_j}(\eta_j) : j = 1, \dots, k\}$ is also pairwise disjoint. Therefore, from (3.9) and (3.10) we have

$$\begin{aligned} \nu(F) &\leq C \sum_{j=1}^k \int_{B_{t_j}(\eta_j)} (1 - |z|^2) d\mu(z) = C \int_{\bigcup_{j=1}^k B_{t_j}(\eta_j)} (1 - |z|^2) d\mu(z) \\ &\leq C \int_{B \setminus K} (1 - |z|^2) d\mu(z) < C\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we have $\nu(F) = 0$ and hence Theorem 3.2 implies the result. ■

4. Main results. As immediate consequences of Lemmas 3.6 and 3.7 we have the following results:

THEOREM 4.1. *If μ is a non-negative Borel regular measure on B satisfying the usual growth condition*

$$\int_B (1 - |w|^2) d\mu(w) < \infty,$$

then the set $\mathcal{L}_0(\mu, 1)$ is empty. If $h \in \mathcal{H}_0$, then

$$H_h(\mathcal{L}_0(\mu, h)) = 0.$$

THEOREM 4.2. *Let $h \in \mathcal{H}$, $0 < \alpha < 1$ and let μ be a non-negative Borel regular measure on B satisfying the growth condition*

$$\int_B (1 - |w|^2) d\mu(w) < \infty.$$

Then

$$\underline{\mathcal{R}}(\mu, h) \subseteq \mathcal{L}_\alpha(\mu, h) \subseteq \mathcal{L}(\mu, h) \subseteq \overline{\mathcal{R}}(\mu, h).$$

For a non-negative regular Borel measure μ on B satisfying the growth condition (2.2) let F_μ be the function on B defined by

$$(4.1) \quad F_\mu(z) = (1 - |z|^2) \int_B \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} d\mu(w).$$

If $d\mu(w) = f(w)d\lambda(w)$, where f is a non-negative measurable function on B , then we denote F_μ by F_f .

The next two propositions will be needed for the proofs of our next results. They provide sufficient conditions for the existence of the \mathcal{T}_τ -limit of F_μ and F_f at a point $\zeta \in S$. Although their proofs are almost identical to those of Propositions 1 and 2 in [18], they are included here only for reasons of completeness.

PROPOSITION 4.3. *Let μ be a non-negative regular Borel measure on B satisfying the growth condition (2.2). If $\zeta \in S$ is such that*

$$(4.2) \quad \int_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\tau\alpha}} d\mu(w) < \infty$$

and

$$(4.3) \quad \int_{B \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\tau\gamma}} d\mu(w) < \infty$$

for some $c > 0$, $\alpha > 0$, $\gamma < 1$, and $\tau \geq 1$, then

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau,c}(\zeta)}} F_\mu(z) = 0.$$

Proof. For $0 < R < 1$, let $A_R = \{z : R < |z| < 1\}$. Then

$$(4.4) \quad \begin{aligned} F_\mu(z) &\leq C(R)(1 - |z|^2) + (1 - |z|^2) \int_{A_R} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} d\mu(w) \\ &= C(R)(1 - |z|^2) + (1 - |z|^2) \int_{A_R \cap \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha (1 - |w|^2)^{1-\alpha}}{|1 - \langle z, w \rangle|^{2-\alpha} |1 - \langle z, w \rangle|^\alpha} d\mu(w) \end{aligned}$$

$$\begin{aligned}
& + (1 - |z|^2) \int_{A_R \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma (1 - |w|^2)^{1-\gamma}}{|1 - \langle z, w \rangle|^{2-\gamma} |1 - \langle z, w \rangle|^\gamma} d\mu(w) \\
& \leq C(R)(1 - |z|^2) + C \int_{A_R \cap \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^\alpha} d\mu(w) \\
& + C \int_{A_R \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^\gamma} d\mu(w).
\end{aligned}$$

Now let $z \in \mathcal{T}_{\tau,c}(\zeta)$ and let $w \in B$. Then

$$|1 - \langle z, \zeta \rangle|^\tau \leq c(1 - |z|^2) \leq (2c) |1 - \langle z, w \rangle|.$$

Using this and the fact that the function $\sqrt{|1 - \langle z, w \rangle|}$ satisfies the triangle inequality on $\bar{B} \times \bar{B}$ (Theorem K) we see that for all $z \in \mathcal{T}_{\tau,c}(\zeta)$ and all $w \in B$,

$$\begin{aligned}
|1 - \langle w, \zeta \rangle| & \leq 2(|1 - \langle w, z \rangle| + |1 - \langle z, \zeta \rangle|) \\
& \leq 2|1 - \langle w, z \rangle| + 2(2c)^{1/\tau} |1 - \langle w, z \rangle|^{1/\tau} \\
& \leq C(c, \tau) |1 - \langle w, z \rangle|^{1/\tau}.
\end{aligned}$$

Thus

$$|1 - \langle w, \zeta \rangle|^\tau \leq C(c, \tau) |1 - \langle w, z \rangle|$$

for all $z \in \mathcal{T}_{\tau,c}(\zeta)$ and $w \in B$, and therefore by (4.4),

$$\begin{aligned}
(4.5) \quad F_\mu(z) & \leq C(R)(1 - |z|^2) + C(c, \tau) \int_{A_R \cap \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\alpha\tau}} d\mu(w) \\
& + C(c, \tau) \int_{A_R \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\gamma\tau}} d\mu(w)
\end{aligned}$$

for all $z \in \mathcal{T}_{\tau,c}(\zeta)$.

As a consequence of hypotheses (4.2) and (4.3) we have

$$\lim_{R \rightarrow 1} \int_{A_R \cap \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\alpha\tau}} d\mu(w) = \lim_{R \rightarrow 1} \int_{A_R \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\gamma\tau}} d\mu(w) = 0.$$

Therefore, for any $\varepsilon > 0$, we can find R with $0 < R < 1$ such that

$$\begin{aligned}
C(c, \tau) \int_{A_R \cap \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\alpha\tau}} d\mu(w) \\
+ C(c, \tau) \int_{A_R \setminus \mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\gamma\tau}} d\mu(w) < \varepsilon.
\end{aligned}$$

Hence, for this R , from (4.5) we have

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau, c}(\zeta)}} F_{\mu}(z) < \varepsilon,$$

which proves the result. ■

PROPOSITION 4.4. *Let f be a non-negative measurable function on B . If $\zeta \in S$ and $c > 0$, $\tau \geq 1$ are such that*

$$(4.6) \quad \int_{\mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^{\alpha}}{|1 - \langle w, \zeta \rangle|^{\tau \alpha}} f^p(w) d\lambda(w) < \infty$$

and

$$(4.7) \quad \int_{B \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^{\gamma}}{|1 - \langle w, \zeta \rangle|^{\tau \gamma}} f^p(w) d\lambda(w) < \infty$$

for some p with $1 < p < n/(n-1)$, and some α and γ with $0 < \alpha < n+p-np$ and $0 < \gamma < n+p-np$, then

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau, c}(\zeta)}} F_f(z) = 0.$$

Proof. As in Proposition 4.2, for $z \in B$ and $0 < R < 1$ we have

$$\begin{aligned} F_f(z) &\leq C(R)(1 - |z|^2) + (1 - |z|^2) \int_{A_R} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} f(w) d\lambda(w) \\ &= C(R)(1 - |z|^2) + (1 - |z|^2) \int_{A_R \cap \mathcal{T}_{\tau, c}(\zeta)} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} f(w) d\lambda(w) \\ &\quad + (1 - |z|^2) \int_{A_R \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} f(w) d\lambda(w). \end{aligned}$$

If $q = p/(p-1)$ is the conjugate exponent of p , then by Hölder's inequality,

$$\begin{aligned} F_f(z) &\leq C(R)(1 - |z|^2) + (1 - |z|^2) \left\{ \int_{A_R} \frac{(1 - |w|^2)^{(1-\alpha/p)q}}{|1 - \langle z, w \rangle|^{(2-\alpha/p)q}} d\lambda(w) \right\}^{1/q} \\ &\quad \times \left\{ \int_{A_R \cap \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^{\alpha}}{|1 - \langle z, w \rangle|^{\alpha}} f^p(w) d\lambda(w) \right\}^{1/p} \\ &\quad + (1 - |z|^2) \left\{ \int_{A_R} \frac{(1 - |w|^2)^{(1-\gamma/p)q}}{|1 - \langle z, w \rangle|^{(2-\gamma/p)q}} d\lambda(w) \right\}^{1/q} \\ &\quad \times \left\{ \int_{A_R \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^{\gamma}}{|1 - \langle z, w \rangle|^{\gamma}} f^p(w) d\lambda(w) \right\}^{1/p}. \end{aligned}$$

If we apply to this estimate the asymptotic estimate from Proposition 3.1,

$$\int_{A_R} \frac{(1 - |w|^2)^{(1-k/p)q}}{|1 - \langle z, w \rangle|^{(2-k/p)q}} d\lambda(w) \leq C(1 - |z|^2)^{-q}, \quad k = \alpha \quad \text{or} \quad k = \gamma,$$

provided $(1 - k/p)q > n$, i.e., $k < n + p - np$, we have

$$(4.8) \quad F_f(z) \leq C(R)(1 - |z|^2) + C \left\{ \int_{A_R \cap \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^\alpha} f^p(w) d\lambda(w) \right\}^{1/p} \\ + C \left\{ \int_{A_R \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^\gamma} f^p(w) d\lambda(w) \right\}^{1/p}.$$

As in Proposition 4.3 we have

$$|1 - \langle w, z \rangle| \geq C(c, \tau)|1 - \langle w, \zeta \rangle|^\tau$$

for all $z \in \mathcal{T}_{\tau, c}(\zeta)$ and all $w \in B$. Therefore, for $z \in \mathcal{T}_{\tau, c}(\zeta)$ from (4.8) it follows that

$$F_f(z) \leq C(R)(1 - |z|^2) \\ + C(c, \tau) \left\{ \int_{A_R \cap \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\tau\alpha}} f^p(w) d\lambda(w) \right\}^{1/p} \\ + C(c, \tau) \left\{ \int_{A_R \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\tau\gamma}} f^p(w) d\lambda(w) \right\}^{1/p},$$

and so hypotheses (4.6) and (4.7) give the proof as in Proposition 4.3. ■

The proof of the following proposition is omitted since it closely parallels the proof of a similar result, Proposition 3, given in [18, pp. 148–151].

PROPOSITION 4.5. *Let μ be a non-negative regular Borel measure on B satisfying*

$$\int_B (1 - |w|^2)^\beta d\mu(w) < \infty$$

for some β with $0 < \beta < 1$, and let $1 \leq \tau \leq n/\beta$. Then

(a) for any γ with $\beta < \gamma \leq 1$,

$$H_{\beta\tau} \left(\left\{ \zeta \in S : \int_{B \setminus \mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\gamma}{|1 - \langle w, \zeta \rangle|^{\tau\gamma}} d\mu(w) = \infty \right\} \right) = 0,$$

(b) for any α with $0 < \alpha < \beta$,

$$H_{\beta\tau} \left(\left\{ \zeta \in S : \int_{\mathcal{T}_{\tau, c}(\zeta)} \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, \zeta \rangle|^{\tau\alpha}} d\mu(w) = \infty \right\} \right) = 0.$$

From the above propositions we have the following

COROLLARY 4.6. *If μ is a non-negative regular Borel measure on B satisfying*

$$\int_B (1 - |w|^2)^\beta d\mu(w) < \infty$$

for some β with $0 < \beta < 1$, then for each τ with $1 \leq \tau \leq n/\beta$ there exists a set $E_\tau \subseteq S$ with $H_{\beta\tau}(E_\tau) = 0$ such that F_μ has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_\tau$.

Proof. Fix α and γ satisfying $0 < \alpha < \beta < \gamma \leq 1$. By Proposition 4.5, there exists a set $E_\tau \subseteq S$ with $H_{\beta\tau}(E_\tau) = 0$ such that the integrals in (a) and (b) of Proposition 4.5 are finite for all $\zeta \in S \setminus E_\tau$. For those ζ , now apply Proposition 4.3. ■

COROLLARY 4.7. *If μ is a non-negative regular Borel measure on B satisfying the condition*

$$\int_B (1 - |w|^2)^\beta d\mu(w) < \infty,$$

for some β with $0 < \beta < 1$, then for each τ with $1 \leq \tau \leq n/\beta$,

$$H_{\beta\tau}(\{\zeta \in S : \mu(\mathcal{T}_{\tau,c}(\zeta)) = \infty\}) = 0.$$

Proof. Since $|1 - \langle w, \zeta \rangle|^\tau < c(1 - |w|^2)$ for all $w \in \mathcal{T}_{\tau,c}(\zeta)$,

$$\int_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\beta}{|1 - \langle w, \zeta \rangle|^{\tau\beta}} d\mu(w) \geq C\mu(\mathcal{T}_{\tau,c}(\zeta)).$$

The result is now an immediate consequence of Proposition 4.4(b). ■

Now we are ready for our next main theorems.

THEOREM 4.8. *Let f be a non-negative measurable function on B satisfying*

$$(4.9) \quad \int_B (1 - |z|^2)^\alpha f^p(z) d\lambda(z) < \infty$$

for some p with $1 < p < n/(n - 1)$ and some α with $0 < \alpha < n + p - np$. Then for each τ with $1 \leq \tau \leq n/\alpha$ there exists a set $E_\tau \subseteq S$ with $H_{\alpha\tau}(E_\tau) = 0$ such that V_f has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_\tau$.

Proof. For a function f which satisfies (4.9) let

$$V_1(z) = \int_{E(z)} \log \frac{1}{|\phi_z(w)|} f(w) d\lambda(w),$$

$$V_2(z) = \int_{B \setminus E(z)} \log \frac{1}{|\phi_z(w)|} f(w) d\lambda(w).$$

Recall that $E(z) = \{w \in B : |\phi_z(w)| < 1/2\}$.

From the inequality (3.2) it follows that

$$\log \frac{1}{|z|} \leq C(1 - |z|^2) \quad \text{for all } z \in B \text{ with } |z| \geq 1/2.$$

Therefore from the identity (2.1) we have

$$\log \frac{1}{|\phi_z(w)|} \leq C \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z \rangle|^2}$$

for all $w \in B \setminus E(z)$. Thus,

$$V_2(z) \leq C(1 - |z|^2) \int_B \frac{1 - |w|^2}{|1 - \langle w, z \rangle|^2} f(w) d\lambda(w) = CF_f(z)$$

for all $z \in B$.

If we take $d\mu(w) = f^p(w)d\lambda(w)$, then from Propositions 4.4 and 4.5 it follows that there exists a set $E_2 \subseteq S$ such that $H_{\alpha\tau}(E_2) = 0$ and F_f has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_2$. Therefore there exists a set $E_2 \subseteq S$ such that $H_{\alpha\tau}(E_2) = 0$ and V_2 has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_2$.

Now consider the function V_1 . Taking $d\mu(w) = f^p(w)d\lambda(w)$ in Corollary 4.5, it follows that there exists a set $E_1 \subseteq S$ with $H_{\alpha\tau}(E_1) = 0$ such that

$$(4.10) \quad \int_{\mathcal{T}_{\tau,c'}(\zeta)} f^p(w) d\lambda(w) < \infty$$

for all $\zeta \in S \setminus E_1$ and any $c' > 0$. Suppose that $c > 0$ and $z \in \mathcal{T}_{\tau,c}(\zeta)$. Then by [19, Lemma 8.17] we have

$$E(z) \subseteq \mathcal{T}_{\tau,c'}(\zeta) \cap A_r \quad \text{for every } c' > c2^{\tau+1} \text{ and } r \geq 2|z|^2 - 1.$$

Let q be the conjugate exponent of p . From the above inclusion, the \mathcal{M} -invariance of λ and Hölder's inequality we have

$$\begin{aligned} V_1(z) &\leq \left\{ \int_{E(z)} \left(\log \frac{1}{|\phi_z(w)|} \right)^q d\lambda(w) \right\}^{1/q} \left\{ \int_{E(z)} f^p(w) d\lambda(w) \right\}^{1/p} \\ &= \left\{ \int_{\{w: |w| < 1/2\}} \left(\log \frac{1}{|\phi_z(w)|} \right)^q d\lambda(w) \right\}^{1/q} \left\{ \int_{E(z)} f^p(w) d\lambda(w) \right\}^{1/p} \\ &\leq C \left\{ \int_{\mathcal{T}_{\tau,c'}(\zeta) \cap A_r} f^p(w) d\lambda(w) \right\}^{1/p}, \end{aligned}$$

where C is a constant which is independent of z tending to the boundary of B .

Therefore, for all $z \in B$ we have

$$V_1(z) \leq C \left\{ \int_{\mathcal{T}_{\tau,c'}(\zeta) \cap A_r} f^p(w) d\lambda(w) \right\}^{1/p}.$$

Now, as a consequence of (4.10) the above integral goes to 0 as $r \rightarrow 1$. Thus V_1 has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_1$. If we take $E_\tau = E_1 \cup E_2$, then $H_{\beta\tau}(E_\tau) = 0$ and V_f has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_\tau$. ■

THEOREM 4.9. *Let $\{z_j : j = 1, 2, \dots\}$ be a sequence in B satisfying*

$$\sum_{j=1}^{\infty} (1 - |z_j|^2)^\alpha < \infty$$

for some $0 < \alpha < 1$, and let μ be the measure given by $\mu = \sum \delta_{z_j}$, where δ_{z_j} is the unit point-mass measure at z_j . Then for each τ with $1 \leq \tau \leq n/\alpha$, there exists a set $E_\tau \subseteq S$ with $H_{\alpha\tau}(E_\tau) = 0$ such that V_μ has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_\tau$.

Proof. As in Theorem 4.8, for the measure μ , let

$$V_1(z) = \int_{E(z)} \log \frac{1}{|\phi_z(w)|} d\mu(w), \quad V_2(z) = \int_{B \setminus E(z)} \log \frac{1}{|\phi_z(w)|} d\mu(w).$$

As in Theorem 4.8, for the function F_μ defined by (4.1), we have

$$V_2(z) \leq C F_\mu(z)$$

for all $z \in B$, where C is a constant independent on z .

By Corollary 4.5 there exists a set $E_1 \subseteq S$ such that $H_{\beta\tau}(E_1) = 0$ and F_μ has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_1$. Therefore V_2 has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_1$. By Corollary 4.6 there exists a set $E_2 \subseteq S$ such that $H_{\alpha\tau}(E_2) = 0$ and

$$\mu(\mathcal{T}_{\tau,c}(\zeta)) < \infty \quad \text{for all } \zeta \in S \setminus E_2.$$

Now, it is obvious that $\mu(\mathcal{T}_{\tau,c}(\zeta)) < \infty$ if and only if $\mathcal{T}_{\tau,c}(\zeta)$ contains only a finite number of terms of the sequence $\{z_k\}$. But for those ζ for which $\mu(\mathcal{T}_{\tau,c}(\zeta)) < \infty$ we obviously have

$$\lim_{\substack{z \rightarrow \zeta \\ z \in \mathcal{T}_{\tau,c}(\zeta)}} V_1(z) = 0,$$

which means that V_1 has \mathcal{T}_τ -limit 0 at all points $\zeta \in S \setminus E_2$. Now take $E_\tau = E_1 \cup E_2$. ■

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