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On elliptic systems pertaining to the Schrödinger equation

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Abstract. We discuss the existence of solutions for a system of elliptic equations involving a coupling nonlinearity containing a critical and subcritical Sobolev exponent. We establish the existence of ground state solutions. The concentration of solutions is also established as a parameter λ becomes large.

1. Introduction. The aim of this paper is to establish the existence of ground state solutions to nonlinear systems of elliptic equations. We consider two types of problems, involving subcritical and critical growth. In the first part of the paper we examine a system containing a subcritical nonlinearity which couples the equations. The problem is a vector form of a scalar equation studied in [2]. Specifically we look at

$$-\Delta u_j + (\lambda a_j(x) + 1)u_j = f_j(U), \quad x \in \mathbb{R}^N,\ j = 1, \ldots, n, \tag{1.1}$$

where $U = (u_1, \ldots, u_n)$, $1 < q < p < 2^*$, $\lambda > 0$ and $a_j(x)$ satisfies certain assumptions. The nonlinearity $f_j(\cdot)$ is defined through the variational formulation. For $F(U) = (\sum_{j=1}^n |u_j|^q)^{p/q}$, we let $f_j(U) = \frac{1}{p}\frac{\partial F}{\partial u_j}$. The interesting feature is that the genuine vector solutions occur in the case $1 < q < 2$ (see Propositions 2.5–2.8).

The second part of the paper is devoted to the case $p = 2^*$. The particular problem introduces another coupling term in the equations, following the work in [1]. This problem is a vector form of a scalar equation presented in [6]:

$$-\Delta u_j + \lambda a_j(x)u_j = \sum_{k=1}^n a_{jk}u_k + f_j(U), \quad x \in \mathbb{R}^N,\ j = 1, \ldots, n. \tag{1.2}$$

We establish some existence results which are related to the best Sobolev constants.

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Solutions in both cases exhibit a similar behaviour when $\lambda \to \infty$ as they tend to concentrate to solutions of the Dirichlet problem in the set Ω where $a_i(x) = 0$.

We assume that the matrix $A = [a_{ij}]$ with constant coefficients is symmetric. The coefficients $a_j(x)$, $j = 1, \dots, n$, are nonnegative and continuous on $\mathbb{R}^N$. Throughout this work we make the assumption:

(A) $\Omega_j = \operatorname{int}(a_j^{-1}(0))$ are nonempty and bounded sets with smooth boundaries and $\overline{\Omega}_j = a_j^{-1}(0)$. Moreover, there exists some $M_0 > 0$ such that the sets

$$F_j = \{x \in \mathbb{R}^N : a_j(x) \le M_0\}$$

have finite Lebesgue measure.

Additional assumptions on a_j will be introduced and used in Section 4.

Throughout this paper we use standard notation and terminology. By $H^1(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$ we denote the usual Sobolev spaces equipped with the norms

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx \quad \text{and} \quad \|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$

respectively.

The corresponding Sobolev spaces of vector functions with n components are denoted by $H^1(\mathbb{R}^N, \mathbb{R}^n) = H^1(\mathbb{R}^N) \times \dots \times H^1(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N, \mathbb{R}^n) = D^{1,2}(\mathbb{R}^N) \times \dots \times D^{1,2}(\mathbb{R}^N)$ and equipped with the product norm. Analogous notation is used for the Lebesgue spaces $L^p(\mathbb{R}^N)$, with norm $|u|_p^p = \int_{\mathbb{R}^N} |u|^p\,dx$, and we denote the corresponding space of vector functions by $L^p(\mathbb{R}^N, \mathbb{R}^n) = L^p(\mathbb{R}^N) \times \dots \times L^p(\mathbb{R}^N)$.

In a given Banach space X, we denote weak convergence by "$\rightharpoonup$" and strong convergence by "$\to$". Let $F \in C^1(X, \mathbb{R})$. A sequence $\{u_m\} \subset X$ is said to be a *Palais-Smale sequence* for F at level c (a $(\mathrm{PS})_c$ *sequence* for short) if $F(u_m) \to c$ and $F'(u_m) \to 0$ in X^* as $m \to \infty$.

We say that F satisfies the *Palais–Smale* condition at level c (the $(\mathrm{PS})_c$ *condition* for short) if any $(\mathrm{PS})_c$ sequence is relatively compact in X.

For our purposes it will be convenient to use the weighted Sobolev spaces. Let $E_j = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a_j u^2\,dx < \infty\}$ and define the norm in E_j by

$$\|u\|_{E_j}^2 = \|u\|_{H^1}^2 + \int_{\mathbb{R}^N} a_j u^2\,dx.$$

We shall also use the norms

$$\|u\|_{E_{j,\lambda}} = \|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} a_j u^2\,dx, \qquad \lambda > 0,$$

which are equivalent to $\|\cdot\|_{E_j}$. Finally we introduce the weighted Sobolev spaces of vector functions:

$$E = E_1 \times \ldots \times E_n \quad \text{and} \quad E_\lambda = E_{1,\lambda} \times \ldots \times E_{n,\lambda}$$

with the norms

$$\|U\|_E^2 = \|u_1\|_{E_1}^2 + \ldots + \|u_n\|_{E_n}^2$$

and

$$\|U\|_{E_\lambda}^2 = \|u_1\|_{E_{1,\lambda}}^2 + \ldots + \|u_n\|_{E_{n,\lambda}}^2.$$

The associated scalar products in E and E_λ are denoted by $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_{E_\lambda}$ respectively.

Solutions of system (1.2) will be found as critical points of the functional $I_\lambda : E \to \mathbb{R}$ defined by

$$\begin{aligned} I_\lambda(U) = \frac{1}{2}\sum_{i=1}^n \Big(\int_{\mathbb{R}^N} |\nabla u_i|^2\,dx + \lambda \int_{\mathbb{R}^N} a_i u_i^2\,dx \Big) \\ - \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^n a_{ij} u_i u_j\,dx - \frac{1}{p}\int_{\mathbb{R}^N} F(U)\,dx. \end{aligned}$$

Since

$$\begin{aligned} \langle I_\lambda'(U), \Phi\rangle = \sum_{i=1}^n \int_{\mathbb{R}^N} \nabla u_i \nabla \phi_i\,dx + \lambda \sum_{i=1}^n \int_{\mathbb{R}^N} a_i u_i \phi_i\,dx \\ - \sum_{i,j=1}^n \int_{\mathbb{R}^N} a_{ij}\phi_i u_j\,dx - \sum_{i=1}^n \int_{\mathbb{R}^N} f_i(U)\phi_i\,dx \end{aligned}$$

for every $U, \Phi \in E_\lambda$, any critical point of I_λ is a weak solution of (1.2).

2. Subcritical case. In this section we consider the subcritical system (1.1). We assume that $F(U) = (\sum_{j=1}^n |u_j|^q)^{p/q}$, $2 < p < 2^*$, and we consider the cases $1 < q < 2$, $q = 2$ and $2 < q < p$.

The variational functional for (1.1) is given by

$$J_\lambda(U) = \frac{1}{2}\sum_{i=1}^n \Big(\int_{\mathbb{R}^N} (|\nabla u_i|^2 + (\lambda a_i + 1)u_i^2)\,dx \Big) - \frac{1}{p}\int_{\mathbb{R}^N} F(U)\,dx$$

for $U \in E_\lambda$. Solutions of (1.1) will be found by constrained minimisation:

$$M_\lambda = \inf\Big\{ \sum_{i=1}^n \int_{\mathbb{R}^N} (|\nabla u_i|^2 + (\lambda a_i + 1)u_i^2\Big)\,dx : \int_{\mathbb{R}^N} F(U)\,dx = 1,\ U \in E_\lambda \Big\}. \tag{2.3}$$

We commence with an observation with standard proof:

Lemma 2.1. *Let $\{U^m\}$ be a minimising sequence for (2.3). Then $W^m = M_\lambda^{1/(p-2)} U^m$ is a Palais–Smale sequence for J_λ, that is,*

$$J_\lambda(W^m) \to \left(\frac{1}{2} - \frac{1}{p}\right) M_\lambda^{p/(p-2)} \quad \text{and} \quad J_\lambda'(W^m) \to 0 \quad \text{in } E^*$$

as $m \to \infty$.

Proof. We follow the argument from Theorem 2.1 in [7] (see also Lemma 8.2.1 in [4]). It is clear that

$$\begin{gathered} \|W^m\|_{E_\lambda}^2 = \|U^m\|_{E_\lambda}^2 M_\lambda^{2/(p-2)} = M_\lambda^{p/(p-2)} + o(1), \\ \int_{\mathbb{R}^N} F(W^m)\,dx = M_\lambda^{p/(p-2)} \int_{\mathbb{R}^N} F(U^m)\,dx = M_\lambda^{p/(p-2)} \\ J_\lambda(W^m) = \left(\frac{1}{2} - \frac{1}{p}\right) M_\lambda^{p/(p-2)} + o(1). \end{gathered} \tag{2.4}$$

For $\Phi \in H^1(\mathbb{R}^N, \mathbb{R}^n)$, we define the functional

$$J^m(\Phi) = \frac{1}{p} \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\partial F(W^m)}{\partial u_i}\, \phi_i\,dx,$$

where $\Phi = (\phi_1, \ldots, \phi_n)$. Since $\frac{\partial F}{\partial u_i} = p(\sum_{j=1}^n |u_j|^q)^{(p-q)/q} |u_i|^{q-1} \operatorname{sign}(u_i)$, applying the Hölder inequality we get

$$\begin{aligned} J^m(\Phi) &\le \int_{\mathbb{R}^N} \Big(\sum_{j=1}^n |w_j^m|^q\Big)^{(p-q)/q} \sum_{i=1}^n |w_i^m|^{q-2} |w_i^m|\,|\phi_i|\,dx \\ &\le \int_{\mathbb{R}^N} \Big(\sum_{j=1}^n |w_j^m|^q\Big)^{(p-q)/q} \Big(\sum_{i=1}^n |w_i^m|^q\Big)^{(q-1)/q} \Big(\sum_{i=1}^n |\phi_i|^q\Big)^{1/q} dx \\ &= \int_{\mathbb{R}^N} \Big(\sum_{j=1}^n |w_j^m|^q\Big)^{(p-1)/q} \Big(\sum_{i=1}^n |\phi_i|^q\Big)^{1/q} dx \\ &\le \Big(\int_{\mathbb{R}^N} \Big(\sum_{i=1}^n |w_i^m|^q\Big)^{p/q} dx\Big)^{(p-1)/p} \Big(\int_{\mathbb{R}^N} \Big(\sum_{i=1}^n |\phi_i|^q\Big)^{p/q} dx\Big)^{1/p} \\ &= \Big(\int_{\mathbb{R}^N} F(W^m)\,dx\Big)^{(p-1)/p} \Big(\int_{\mathbb{R}^N} F(\Phi)\,dx\Big)^{1/p}. \end{aligned}$$

It follows from (2.4) that

$$J^m(\Phi) \le M_\lambda^{(p-1)/(p-2)} \Big(\int_{\mathbb{R}^N} F(\Phi)\,dx\Big)^{1/p}.$$

If $\|\Phi\|_{E_\lambda} = 1$, then

$$M_\lambda \le \int_{\mathbb{R}^N} \left(\frac{|\nabla \Phi|^2}{(\int_{\mathbb{R}^N} F(\Phi)\,dx)^{2/p}} + \frac{1}{(\int_{\mathbb{R}^N} F(\Phi)\,dx)^{2/p}} \sum_{j=1}^n (\lambda a_j + 1)\phi_j^2 \right) dx.$$

Hence

$$\Big(\int_{\mathbb{R}^N} F(\Phi)\,dx \Big)^{2/p} \le M_\lambda^{-1} \|\Phi\|_{E_\lambda}^2 = M_\lambda^{-1}.$$

Therefore, for such Φ we have

$$J^m(\Phi) \le M_\lambda^{(p-1)/(p-2)} M_\lambda^{-1/2} = M_\lambda^{p/(2(p-2))}.$$

This yields

$$\|J^m\|_{H^{-1}(\mathbb{R}^N,\mathbb{R}^n)} \le M_\lambda^{p/(2(p-2))}.$$

We also have

$$\begin{aligned} J^m(W^m \|W^m\|_{E_\lambda}^{-1}) &= \frac{1}{p} \sum_{i=1}^n \frac{1}{\|W^m\|_{E_\lambda}} \int_{\mathbb{R}^N} \frac{\partial F(W^m)}{\partial u_i}\, w_i^m \,dx \\ &= \frac{1}{\|W^m\|_{E_\lambda}} \int_{\mathbb{R}^N} F(W^m)\,dx = \frac{M_\lambda^{p/(p-2)}}{M_\lambda^{p/(2(p-2))}} + o(1) = M_\lambda^{p/(2(p-2))} + o(1). \end{aligned}$$

By the Riesz representation theorem, there exists $V^m \in E_\lambda$ such that

$$J^m(\Phi) = (V^m, \Phi)_{E_\lambda} = \int_{\mathbb{R}^N} \Big(\nabla V^m \nabla \Phi + \sum_{i=1}^n (\lambda a_i + 1) v_i^m \phi_i \Big)\,dx$$

and

$$\|J^m\|_{H^{-1}} = \|V^m\|_{E_\lambda}.$$

From this, we deduce

$$(V^m, W^m \|W^m\|_{E_\lambda}^{-1})_{E_\lambda} = J^m(W^m \|W^m\|_{E_\lambda}^{-1}) = M_\lambda^{p/(2(p-2))} + o(1).$$

Hence

$$(V^m, W^m)_{E_\lambda} = M_\lambda^{p/(2(p-2))} \|W^m\|_{E_\lambda} + o(1) = M_\lambda^{p/(p-2)} + o(1).$$

We can now write

$$\begin{aligned} \|V^m - W^m\|_{E_\lambda}^2 &= \|V^m\|_{E_\lambda}^2 - 2(W^m, V^m)_{E_\lambda} + \|W^m\|_{E_\lambda}^2 \\ &= M_\lambda^{p/(p-2)} - 2M_\lambda^{p/(p-2)} + M_\lambda^{p/(p-2)} + o(1) = o(1). \end{aligned}$$

Since

$$\begin{aligned} \langle J_\lambda'(W^m), \Phi \rangle = \int_{\mathbb{R}^N} \Big(\nabla W^m \nabla \Phi + \sum_{i=1}^n \lambda(a_i + 1) w_i^m \phi_i \Big)\,dx \\ - \frac{1}{p} \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{\partial F(W^m)}{\partial u_i}\, \phi_i \,dx = (W^m, \Phi)_{E_\lambda} - (V^m, \Phi)_{E_\lambda}, \end{aligned}$$

we get $|\langle J_\lambda'(W^m), \Phi \rangle| \le \|V^m - W^m\|_{E_\lambda}$ and the result follows. ∎

This lemma shows that if U is a minimiser of (2.3) then $M_\lambda^{1/(p-2)}U$ is a solution of system (1.1).

Let $\{U^m\}$ be a minimising sequence for (2.3). Since U^m is bounded in E, we may assume that $U^m \rightharpoonup U$ in E. We now define the following two quantities:

$$\alpha_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|\geq R} F(U^m)\, dx,$$

$$\beta_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|\geq R} \Big(|\nabla U^m|^2 + \sum_{j=1}^{n} (\lambda a_j + 1)(u_j^m)^2\Big)\, dx,$$

which measure the loss of mass at infinity of a weakly convergent sequence U^m (see [4], [5]). It is clear that both α_∞ and β_∞ are finite.

We now note that the infimum M_λ, defined by (2.3), is bounded independently of $\lambda \geq 0$. Let

$$M_i = \inf\Big\{ \int_{\Omega_i} (|\nabla u|^2 + u^2)\, dx : \int_{\Omega_i} |u|^p\, dx = 1,\ u \in H_0^1(\Omega_i)\Big\}.$$

Testing M_λ with vector functions nonzero in the jth component, $\widetilde{U}_j = (0, \ldots, 0, u, 0, \ldots, 0)$, where $u \in H_0^1(\Omega)$, we derive the estimate

$$M_\lambda \leq \min_{j=1,\ldots,n} M_j. \tag{2.5}$$

In the proof of Theorem 2.2 below, we shall use only the second part of assumption (A), namely that the measures of the sets F_j are finite.

Theorem 2.2. *There exists $\Lambda > 0$ such that problem* (1.1) *has a solution for $\lambda \geq \Lambda$.*

Proof. Let $\{U^m\}$ be a minimising sequence for M_λ. It is sufficient to prove that $\{U^m\}$ is convergent up to a subsequence in E. It follows from (2.5) that there exists a constant $K > 0$ such that

$$\int_{\mathbb{R}^N} \Big(|\nabla U^m|^2 + \sum_{j=1}^{n} (\lambda a_j + 1)(u_j^m)^2\Big)\, dx \leq K$$

for all m. We may assume that $U^m \rightharpoonup U$ in E. Then for each $1 \leq j \leq n$,

$$\int_{\{|x|\geq R,\, a_j(x)\geq M_0\}} (u_j^m)^2 dx \leq \frac{1}{\lambda M_0 + 1} \int_{\mathbb{R}^N} (\lambda a_j + 1)(u_j^m)^2 dx \leq \frac{K}{\lambda M_0 + 1}, \tag{2.6}$$

$$\begin{aligned}\int_{\{|x|\geq R,\, a_j(x)< M_0\}} (u_j^m)^2 dx &\leq \Big(\int_{\mathbb{R}^N} |u_j^m|^p\, dx\Big)^{2/p} \Big(\int_{\{|x|\geq R,\, a_j(x)<M_0\}} dx\Big)^{1-2/p} \\ &\leq C|\{x \geq R\} \cap \{a_j(x) < M_0\}|,\end{aligned} \tag{2.7}$$

where C is a constant depending only on K.

It follows from assumption (A) that

$$|\{|x| \geq R\} \cap \{a_j(x) < M_0\}| \to 0 \quad \text{as } R \to \infty. \tag{2.8}$$

Let $\theta = (2^* - p)/(2^* - 2)$. Applying the Hölder inequality we get

$$\begin{aligned}\int_{|x|\geq R} |u_j^m|^p \, dx &\leq \Big(\int_{|x|\geq R} |u_j^m|^2 \, dx\Big)^\theta \Big(\int_{|x|\geq R} |u_j^m|^{2^*} \, dx\Big)^{1-\theta} \\ &\leq C\Big(\int_{\{|x|\geq R,\, a_j(x)\geq M_0\}} (u_j^m)^2 \, dx + \int_{\{|x|\geq R,\, a_j(x)<M_0\}} (u_j^m)^2 \, dx\Big)^\theta.\end{aligned}$$

Using (2.6)–(2.8) we see that there exists $\Lambda > 0$ such that $\alpha_\infty < 1$ for $\lambda \geq \Lambda$. We now observe that

$$1 = \lim_{m\to\infty} \int_{\mathbb{R}^N} F(U^m) \, dx = \int_{\mathbb{R}^N} F(U) \, dx + \alpha_\infty.$$

To complete the proof, we need to show that $\alpha_\infty = 0$. Assume that $0 < \alpha_\infty < 1$. Let $\phi_R \in C^1(\mathbb{R}^N)$ be such that $\phi_R(x) = 1$ for $|x| > R + 1$, $\phi_R(x) = 0$ for $|x| \leq R$ and $0 \leq \phi_R(x) \leq 1$ on $\mathbb{R}^N$. By Lemma 2.1, we have $\langle J_\lambda'(U^m M^{1/(p-2)}), U^m M^{1/(p-2)} \phi_R^2\rangle \to 0$ as $m \to \infty$ uniformly for large R. From this we deduce that

$$\beta_\infty = M_\lambda \alpha_\infty. \tag{2.9}$$

On the other hand, by the Sobolev embedding theorem, we always have $M_\lambda \alpha_\infty^{2/p} \leq \beta_\infty$. Combined with (2.9), this implies that $\alpha_\infty \geq 1$, which is impossible, and this completes the proof. ∎

Theorem 2.2 can be extended to solve (1.1) in the case $\lambda = 0$.

PROPOSITION 2.3. *System* (1.1) *with* $\lambda = 0$ *has a solution attained as a minimiser of the variational problem* (2.3) *with* $\lambda = 0$.

Proof. We use the following fact known as the vanishing lemma (see [8]): if $\{u_m\}$ is a weakly convergent sequence in $H^1(\mathbb{R}^N)$ such that

$$\liminf_{m\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_m^2 \, dx = 0$$

for some $R > 0$ then $u_m \to 0$ in $L^s(\mathbb{R}^N)$ for all $2 < s < 2^*$.

Let $\{U^m\}$ be a minimising sequence. We may assume that $U^m \rightharpoonup U$ in $H^1(\mathbb{R}^N, \mathbb{R}^n)$.

If $\liminf_{m\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} (U^m)^2 \, dx = 0$, then by the above result, $u_m^i \to 0$ in $L^p(\mathbb{R}^N)$ for each $1 \leq i \leq n$. Since $\int_{\mathbb{R}^N} F(U^m) \, dx = 1$, this is impossible. Therefore, there exists a sequence $\{y_m\} \subset \mathbb{R}^N$ such that $\int_{B(0,R)} (U^m(x + y_m))^2 \, dx \geq \eta > 0$ for every m and some $\eta > 0$. Up to a subsequence, we have $U^m(\cdot + y_m) \rightharpoonup U \not\equiv 0$ in $H^1(\mathbb{R}^N, \mathbb{R}^n)$. Let

$$\beta_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|>R} (|\nabla U^m|^2 + (U^m)^2)\, dx$$

and let α_∞ be as in the proof of Theorem 2.2. We have

$$\int_{\mathbb{R}^N} F(U)\, dx + \alpha_\infty = 1.$$

To complete the proof, we need to show that $\alpha_\infty = 0$. In the contrary case, $0 < \alpha_\infty < 1$ since $U \not\equiv 0$. Repeating the final part of the proof of Theorem 2.2, we show that $\beta_\infty = M_0\alpha_\infty$. On the other hand, it follows from the definition of M_0 that $M_0\alpha_\infty^{2/p} \le \beta_0$.

Combining the last two inequalities we derive that $\alpha_\infty \ge 1$, which is a contradiction. ∎

Let

$$M_{\lambda,i} = \inf\Big\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda a_i + 1)u^2)\, dx : \int_{\mathbb{R}^N} |u|^p\, dx = 1,\ u \in H^1(\mathbb{R}^N)\Big\}, \quad i = 1, \ldots, n.$$

PROPOSITION 2.4. *Let $q \ge 2$. For every $\lambda \ge 0$, we have*

$$M_\lambda = \min_{j=1,\ldots,n} M_{\lambda,j}.$$

Proof. Let $U \in H^1(\mathbb{R}^N, \mathbb{R}^n)$. Then

$$\begin{aligned}\Big[\int_{\mathbb{R}^N} \Big(\sum_{j=1}^n |u_j|^q\Big)^{p/q} dx\Big]^{2/p} &= \Big\{\Big[\int_{\mathbb{R}^N} \Big(\sum_{i=1}^n |u_i|^q\Big)^{p/q} dx\Big]^{q/p}\Big\}^{2/q} \\ &\le \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^N} |u_i|^p\, dx\Big)^{q/p}\Big]^{2/q} \quad \text{(by Minkowski's inequality)} \\ &\le \sum_{i=1}^n \Big(\int_{\mathbb{R}^N} |u_i|^p\, dx\Big)^{2/p} \quad \text{(by Jensen's concave inequality)} \\ &\le \sum_{i=1}^n M_{\lambda,i}^{-1} \int_{\mathbb{R}^N} (|\nabla u_i|^2 + \lambda(a_i + 1)u_i^2)\, dx.\end{aligned}$$

From this, we deduce that

$$\min_{i=1,\ldots,n} M_{\lambda,i} \le M_\lambda.$$

The opposite inequality follows as before, by testing M_λ with vector functions of the form $\widetilde{U}_j = (0, \ldots, 0, u, 0, \ldots, 0)$, $u \in H^1(\mathbb{R}^N)$. ∎

Suppose that $\min_{i=1,\ldots,n} M_{\lambda,i} = M_{\lambda,j_0}$ for some j_0. This means that if w_{j_0} is a minimiser for M_{λ,j_0}, then $W_{j_0} = w_{j_0}e_{j_0}$ is a minimiser for M_λ, where

e_j is the vector $(0,\dots,0,1,0,\dots,0)\in\mathbb{R}^n$, with 1 as the jth component. In fact, in Proposition 2.5, we show that these are the only minimisers in the case $2<q<p$.

PROPOSITION 2.5. *Let $2\le q<p<2^*$. If $q=2$, assume additionally that $M_{\lambda,j_0}<M_{\lambda,j}$ for each $j\ne j_0$. Then the minimiser for M_λ has the form $U=w_{j_0}e_{j_0}$, where w_{j_0} is a minimiser for M_{λ,j_0}.*

Proof. Let $U=(u_1,\dots,u_n)$ be a minimiser for M_λ. Let $u_{j_1},\dots,u_{j_k}$ be the nonzero components of U and suppose that $k\ge 2$. Then we have

$$\sum_{s=1}^{k} M_{\lambda,j_s}\Big(\int_{\mathbb{R}^N}|u_{j_s}|^p\,dx\Big)^{2/p}\le\int_{\mathbb{R}^N}\Big(|\nabla U|^2+\sum_{s=1}^{k}(\lambda a_{j_s}+1)u_{j_s}^2\Big)\,dx=M_{\lambda_{j_0}}.$$

This yields

$$\sum_{s=1}^{k}\Big(\int_{\mathbb{R}^N}|u_{j_s}|^p\,dx\Big)^{2/p}\le 1,$$

and the inequality is strict if $q=2$ by the assumption of the proposition.

On the other hand, we have

$$1=\Big[\int_{\mathbb{R}^N}\Big(\sum_{s=1}^{k}|u_{j_s}|^q\Big)^{p/q}dx\Big]^{q/p}\le\sum_{s=1}^{k}\Big(\int_{\mathbb{R}^N}|u_{j_s}|^p\,dx\Big)^{q/p}.$$

Since $q\ge 2$, we get a contradiction. So it follows that one component of U must be nonzero. Since $M_\lambda=M_{\lambda,j_0}$ we must have $U=w_{j_0}e_{j_0}$. ∎

If $a_1=\dots=a_n$ and $q=2$, we have n minimisers of the form e_jw. However, we obtain other minimisers with the form $(\alpha_1w,\dots,\alpha_nw)$.

PROPOSITION 2.6. *Suppose that $q=2$ and $a_1=\dots=a_n$. Then $U=(\alpha_1w,\dots,\alpha_nw)$ with $\alpha_1^2+\dots+\alpha_n^2=1$ are the only minimisers for M_λ, where w is a minimiser of $M_{\lambda,i}$.*

Proof. Following the proof of Proposition 2.4, the chain of inequalities must be equalities. According to Minkowski's inequality, equality can only hold if each component is a multiple of a common term. In order that $\int_{\mathbb{R}^N}F(U)\,dx=1$, we require $\sum_{i=1}^n\alpha_i^2=1$. ∎

We now consider the case $1<q<2$. As before, we set $M_{\lambda,j_0}=\min_{i=1,\dots,n}M_{\lambda,i}$. Let

$$A(x)=\frac{a_1(x)+\dots+a_n(x)}{n},$$
$$M_\lambda^{[A]}=\inf\Big\{\int_{\mathbb{R}^N}(|\nabla u|^2+(\lambda A+1)u^2)\,dx:\int_{\mathbb{R}^N}|u|^p\,dx=1\Big\}.$$

Proposition 2.7. *Let $1 < q < 2$. Then*

$$n^{(q-2)/q} M_{\lambda,j_0} \le M_\lambda \le n^{(q-2)/q} M_\lambda^{[A]}.$$

In particular, if $a_1 = \ldots = a_n$ then $M_\lambda = n^{(q-2)/q} M_{\lambda,j_0}$.

Proof. By the Minkowski inequality and the weighted mean inequalities (see e.g. [9]), we have

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}\Big(\sum_{i=1}^n |u_i|^q\Big)^{p/q} dx\Big)^{2/p} &\le \Big[\sum_{i=1}^n \Big(\int_{\mathbb{R}^N} |u_i|^p\, dx\Big)^{q/p}\Big]^{2/q} \\ &\le \Big[\sum_{i=1}^n \Big(M_{\lambda,i}^{-1} \int_{\mathbb{R}^N} (|\nabla u_i|^2 + (\lambda a_i + 1)u_i^2)\, dx\Big)^{q/2}\Big]^{2/q} \\ &\le \frac{n^{2/q}}{n} \sum_{i=1}^n M_{\lambda,i}^{-1} \int_{\mathbb{R}^N} (|\nabla u_i|^2 + (\lambda a_i + 1)u_i^2)\, dx \\ &\le n^{(2-q)/q} M_{\lambda,j_0}^{-1} \int_{\mathbb{R}^N} \sum_{i=1}^n (|\nabla u_i|^2 + (\lambda a_i + 1)u_i^2)\, dx,\end{aligned}$$

and this gives the estimate $n^{(q-2)/q} M_{\lambda,j_0} \le M_\lambda$. The other inequality follows by testing M_λ with $U = (n^{-1/q} w_A, \ldots, n^{-1/q} w_A)$ where w_A is a ground state for $M_\lambda^{[A]}$. ■

We now examine the form of minimisers for M_λ when $1 < q < 2$. We aim to show that we cannot have minimisers of the form we_{j_0} or $(\alpha_1 w, \ldots, \alpha_n w)$ for some $w \in H^1(\mathbb{R}^N)$ and constants α_i.

We commence with the observation that $M_{\lambda,i}$ depends continuously on a_i in the sense of $L^{p/(p-2)}(\mathbb{R}^N)$ convergence. Let $\eta \in C(\mathbb{R}^N)$ with $0 \le \eta \le 1$ and $\operatorname{supp}(\eta) \subset B(0,1)$. We set

$$M_\lambda^{[a]} = \inf\Big\{\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda a + 1)u^2)\, dx : \int_{\mathbb{R}^N} |u|^p\, dx = 1,\ u \in H^1(\mathbb{R}^N)\Big\},$$

$$M_\lambda^{[a+\varepsilon\eta]} = \inf\Big\{\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda a + \lambda\varepsilon\eta + 1)u^2)\, dx : \int_{\mathbb{R}^N} |u|^p\, dx = 1,\ u \in H^1(\mathbb{R}^N)\Big\},$$

where $a(\cdot)$ satisfies assumption (A) and $\varepsilon > 0$ is a constant. Then for every $u \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \eta u^2\, dx \le \Big(\int_{\mathbb{R}^N} \eta^{p/(p-2)}\, dx\Big)^{(p-2)/p} \Big(\int_{\mathbb{R}^N} |u|^p\, dx\Big)^{2/p}$$

and consequently

$$M_\lambda^{[a]} \le M_\lambda^{[a+\varepsilon\eta]} \le M_\lambda^{[a]} + \lambda\varepsilon\Big[\int_{\mathbb{R}^N} \eta^{p/(p-2)}\,dx\Big]^{(p-2)/p}.$$

We restrict ourselves to the case $n = 2$.

PROPOSITION 2.8. (i) *Let $a_1 \le a_2$. Then there are no solutions of the form $(0, u_2)$.*

(ii) *Let $a_1 \le a_2$ be sufficiently close to each other in $L^{p/(p-2)}(\mathbb{R}^N)$ so that $2^{(q-2)/2} < M_{\lambda,1}/M_{\lambda,2} \le 1$. Then there are no solutions minimising M_λ of the form $(u_1, 0)$.*

(iii) *If $a_1 \ne a_2$, then there are no solutions of the form $(\alpha w, \beta w)$ with $w \in H^1(\mathbb{R}^N)$ and constants α, β.*

Proof. (i) If $U = (0, u_2)$ is a minimiser for M_λ then

$$M_{\lambda,2} \le \int_{\mathbb{R}^N} (|\nabla u_2|^2 + (\lambda a_2 + 1)u^2)\,dx = M_\lambda \le 2^{(q-2)/2} M_\lambda^{[A]} \le 2^{(q-2)/2} M_{\lambda,2},$$

which is impossible.

(ii) If $U = (u_1, 0)$ is a minimiser for M_λ then

$$M_{\lambda,1} \le \int_{\mathbb{R}^N} (|\nabla u_1|^2 + (\lambda a_1 + 1)u_1^2)\,dx = M_\lambda \le 2^{(q-2)/q} M_\lambda^{[A]} \le 2^{(q-2)/q} M_{\lambda,2}.$$

This yields $M_{\lambda,1}/M_{\lambda,2} \le 2^{(q-2)/2}$, which is impossible.

(iii) If $U = (\alpha w, \beta w)$ is a minimiser with $\alpha \ne 0$, $\beta \ne 0$ and $w > 0$, then $u_1 = \alpha v$ and $u_2 = \beta v$ with $v = M^{1/(p-2)} w$ satisfy

$$-\Delta u_1 + (\lambda a_1 + 1)u_1 = (u_1^q + u_2^q)^{(p-q)/q} u_1^{q-1},$$
$$-\Delta u_2 + (\lambda a_2 + 1)u_2 = (u_1^q + u_2^q)^{(p-q)/q} u_2^{q-1}.$$

From this we derive

$$\lambda v(a_1 - a_2) = (\alpha^q + \beta^q)^{(p-q)/q} (\alpha^{q-2} - \beta^{q-2}) v^{p-1},$$

which is impossible. ■

We remark that if a_1 is very close to, but slightly smaller than a_2, then all of (i), (ii) and (iii) can be satisfied, and the solution is a genuine vector function. Thus, $q = 2$ appears to be an important threshold inducing transitions in the vector nature of solutions.

3. Palais–Smale sequences for critical nonlinearities. Henceforth, we consider problem (1.2) with $p = 2^* = 2N/(N-2)$ for $N \ge 4$. The best Sobolev constant for the nonlinearity F is defined by

$$S_F = \inf\Big\{\sum_{i=1}^n \int_{\mathbb{R}^N} |\nabla u_i|^2\,dx : U \in D^{1,2}(\mathbb{R}^N, \mathbb{R}^n),\ \int_{\mathbb{R}^N} F(U)\,dx = 1\Big\}.$$

According to Theorem 1.1 in [1], the constant S_F is attained by a function $U \in D^{1,2}(\mathbb{R}^N, \mathbb{R}^n)$.

We commence by examining a sequence U^m whose norm $\|U^m\|_{E_{\lambda_m}}$ with $\lambda_m \to \infty$ is bounded independently of m.

Lemma 3.1. *Let $\|U^m\|_{E_{\lambda_m}} \le K$ for $\lambda_m \to \infty$ and some constant K independent of m. Then there exists $U \in H_0^1(\Omega_1) \times \ldots \times H_0^1(\Omega_n)$ such that $U^m \rightharpoonup U$ in E and $U^m \to U$ in $L^2(\mathbb{R}^N, \mathbb{R}^n)$.*

Proof. We follow some ideas from [6]. We may assume that $\lambda_m \ge 1$ for all m. We have $\|U^m\|_E \le \|U^m\|_{E_{\lambda_m}} \le K$. Therefore up to a subsequence, $U^m \rightharpoonup U$ in E and $U^m \to U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^n)$. Since for every $\delta > 0$ and $j = 1, \ldots, n$,

$$\delta \int_{\{a_j(x) \ge \delta\}} (u_j^m)^2 \, dx \le \int_{\{a_j(x) \ge \delta\}} a_j (u_j^m)^2 \, dx \le \frac{K}{\lambda_m}$$

we see that $u_j(x) = 0$ almost everywhere on $\mathbb{R}^N \setminus \Omega_j$. Since $\partial\Omega_j$ are smooth, it follows that $u_j \in H_0^1(\Omega_j)$, $j = 1, \ldots, n$. It remains to show that $U^m \to U$ in $L^2(\mathbb{R}^N, \mathbb{R}^n)$. First we observe that

$$\int_{F_j^c} (u_j^m)^2 \, dx \le \frac{1}{\lambda_m M_0} \int_{F_j^c} \lambda_m a_j (u_j^m)^2 \, dx \le \frac{K}{\lambda_m M_0} \to 0 \quad \text{as } m \to \infty,$$

where $F_j^c = \mathbb{R}^N \setminus F_j$. For $B^c(0,R) = \mathbb{R}^N \setminus B(0,R)$ we have

$$\int_{B^c(0,R) \cap F_j} (u_j^m - u_j)^2 \, dx \le C \|u_j^m - u_j\|^2_{H^1(\mathbb{R}^N)} |B^c(0,R) \cap F_j|^{2/N} \to 0$$

as $R \to \infty$ uniformly in m. On the other hand, for each $R > 0$ we have

$$\int_{B(0,R)} (u_j^m - u_j)^2 \, dx \to 0,$$

and the result follows. ∎

To proceed further we denote by $\lambda_1(\Omega_j)$, $j = 1, \ldots, n$, the first eigenvalues of the operator $-\Delta$ on Ω_j with Dirichlet boundary conditions.

Let $A_\lambda^j = -\Delta + \lambda a_j$ be self-adjoint operators on $L^2(\mathbb{R}^N)$. Denote by $(\cdot,\cdot)$ the scalar product on $L^2(\mathbb{R}^N)$. We set

$$(A_\lambda^j u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda a_j uv) \, dx$$

for $u, v \in E$. By $\|A\|$ we denote the norm of the linear mapping in $\mathbb{R}^n$ with matrix $A = [a_{ij}]$.

PROPOSITION 3.2. *Suppose that* $\|A\| < \min_j \lambda_1(\Omega_j)$. *Then for every* μ *satisfying* $\|A\| \le \mu < \min_j \lambda_1(\Omega_j)$ *there exists* $\Lambda(\mu)$ *such that*

$$\sum_{j=1}^{n}(A_\lambda^j u_j, u_j) - \sum_{j,k=1}^{n} \int_{\mathbb{R}^N} a_{jk} u_j u_k \, dx \ge \min_j \alpha_\mu^j \|U\|_{E_\lambda}^2 + (\mu - \|A\|)|U|_2^2 \tag{3.10}$$

for $u \in E$ *and* $\lambda \ge \Lambda(\mu)$, *where*

$$\alpha_\mu^j = \frac{\lambda_1(\Omega_j) - \mu}{\lambda_1(\Omega_j) + 2 + \mu}.$$

Proof. We follow some ideas from [6]. We commence by showing that for every μ satisfying $0 < \mu < \lambda_1(\Omega_j)$ there exists $\Lambda_j(\mu) > 0$ such that

$$(A_\lambda^j u, u) - \mu(u, u) \ge \alpha_\mu^j \|u\|_{E_{j,\lambda}}^2 \quad \text{for } \lambda \ge \Lambda_j(\mu). \tag{3.11}$$

We set

$$a_\lambda^j = \inf\{(A_\lambda^j u, u) : u \in E_j,\ |u|_2 = 1\}.$$

We claim that for $\mu \in (0, \lambda_1(\Omega_j))$, there exists $\Lambda_j(\mu) > 0$ such that

$$a_\lambda^j \ge \frac{\mu + \lambda_1(\Omega_j)}{2} \quad \text{for } \lambda \ge \Lambda_j(\mu).$$

Arguing by contradiction, we can find a sequence $\lambda_m \to \infty$ such that $a_{\lambda_m}^j \to \alpha_j \le (\mu + \lambda_1(\Omega_j))/2$ as $m \to \infty$. Let $u_m \in E_j$ be such that $|u_m|_2 = 1$ and $((A_{\lambda_m}^j - a_{\lambda_m}^j)u_m, u_m) \to 0$ as $m \to \infty$. Then

$$\begin{aligned}
\|u_m\|_{E_{j,\lambda_m}}^2 &= \int_{\mathbb{R}^N} (|\nabla u_m|^2 + (1 + \lambda_m a_j)u_m^2) \, dx \\
&= ((A_{\lambda_m}^j - a_{\lambda_m}^j)u_m, u_m) + (1 + a_{\lambda_m}^j)|u_m|_2^2 \\
&\le 1 + \frac{\mu + \lambda_1(\Omega_j)}{2}
\end{aligned}$$

for large m and $j = 1, \ldots, n$. It follows from Lemma 3.1 that there exists $U \in H_0^1(\Omega_1) \times \ldots \times H_0^1(\Omega_n)$ such that $U_m \rightharpoonup U$ in E and $U_m \to U$ in $L^2(\mathbb{R}^N, \mathbb{R}^n)$. Thus $|u_j|_2 = 1$ for $j = 1, \ldots, n$. Moreover, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla u_j|^2 - \alpha_j u_j^2) \, dx &\le \liminf_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla u_m^j|^2 - a_{\lambda_m}^j (u_m^j)^2) \, dx \\
&\le \liminf_{m\to\infty} ((A_{\lambda_m}^j - a_{\lambda_m}^j)u_m^j, u_m^j) = 0.
\end{aligned}$$

Hence

$$\int_{\Omega_j} |\nabla u_j|^2 \, dx \le \alpha_j \le \frac{\mu + \lambda_1(\Omega_j)}{2} < \lambda_1(\Omega_j).$$

However, this is impossible since $\int_{\Omega_j} u_j^2 \, dx = 1$ and $\lambda_1(\Omega_j)$ is the first eigenvalue of $-\Delta$ on Ω_j.

We now write

$$(3.12)\qquad (A^j_\lambda u,u) - \mu \int_{\mathbb{R}^N} u^2\,dx$$
$$= \frac{\lambda_1(\Omega_j)-\mu}{\lambda_1(\Omega_j)+2+\mu}\,(A^j_\lambda u,u) + \frac{2\mu+2}{\lambda_1(\Omega_j)+2+\mu}\,(A^j_\lambda u,u) - \mu\int_{\mathbb{R}^N} u^2\,dx$$
$$\geq \frac{\lambda_1(\Omega_j)-\mu}{\lambda_1(\Omega_j)+2+\mu}\,(A^j_\lambda u,u) + \frac{(\mu+1)(\lambda_1(\Omega_j)+\mu)}{\lambda_1(\Omega_j)+2+\mu}\int_{\mathbb{R}^N} u^2\,dx - \mu\int_{\mathbb{R}^N} u^2\,dx$$
$$= \frac{\lambda_1(\Omega_j)-\mu}{\lambda_1(\Omega_j)+2+\mu}\,(A^j_\lambda u,u) + \frac{\lambda_1(\Omega_j)-\mu}{\lambda_1(\Omega_j)+2+\mu}\int_{\mathbb{R}^N} u^2\,dx$$
$$= \frac{\lambda_1(\Omega_j)-\mu}{\lambda_1(\Omega_j)+2+\mu}\,\|u\|^2_{E_{j,\lambda}}.$$

From (3.12) we derive the estimate

$$\sum_{j=1}^n (A^j_\lambda u_j,u_j) - \sum_{j,k=1}^n a_{jk}\int_{\mathbb{R}^N} u_k u_j\,dx$$
$$\geq \min_j \alpha^j_\mu \|U\|^2_{E,\lambda} - \sum_{j,k=1}^n a_{jk}\int_{\mathbb{R}^N} u_k u_j\,dx + \mu\sum_{j=1}^n\int_{\mathbb{R}^N} u_j^2\,dx$$
$$\geq \min_j \alpha^j_\mu \|U\|^2_{E,\lambda} + (\mu - \|A\|)|U|_2^2,$$

and the result follows. ∎

Proposition 3.3. *Let $I_\lambda(U^m)\to c < S_F^{N/2}/N$ and $I'_\lambda(U^m)\to 0$ in E^*_λ. Then $\{U^m\}$ is relatively compact in E_λ.*

Proof. First we show that $\{U^m\}$ is bounded in E_λ. To show this we use Proposition 3.2. Indeed, for $\|A\| < \mu < \min_j \lambda_1(\Omega_j)$, we know that

$$I_\lambda(U^m) - \frac{1}{2^*}\langle I'_\lambda(U^m),U^m\rangle = \frac{1}{N}\Big(\sum_{j=1}^n (A^j_\lambda u^m_j,u^m_j) - \sum_{j,k=1}^n a_{jk}\int_{\mathbb{R}^N} u^m_j u^m_k\,dx\Big)$$
$$\geq \min_j \alpha^j_\mu \|U^m\|^2_{E_\lambda} + (\mu-\|A\|)|U^m|_2^2.$$

Hence $\{U^m\}$ is bounded in E_λ and we may assume that $U^m \rightharpoonup U$ in E_λ. It is easy to show that U is a weak solution of system (1.2). Hence

$$(3.13)\qquad \sum_{j=1}^n (A^j_\lambda u_j,u_j) - \sum_{j,k=1}^n a_{jk}\int_{\mathbb{R}^N} u_j u_k\,dx$$
$$= \frac{1}{2^*}\sum_{i=1}^n u_i\,\frac{\partial F(U)}{\partial u_i} = \int_{\mathbb{R}^N}\sum_{i=1}^n u_i f_i(U) = \int_{\mathbb{R}^N} F(U)\,dx.$$

We set $w_j^m = u_j^m - u_j$ and $W^m = (w_1^m, \dots, w_n^m)$. Applying Brézis–Lieb's lemma, we get

$$\int_{\mathbb{R}^N} F(U^m)\,dx = \int_{\mathbb{R}^N} F(U)\,dx + \int_{\mathbb{R}^N} F(W^m)\,dx + o(1),$$

and from the weak convergence of U^m to U in E_λ we derive

$$(A_\lambda^j u_j^m, u_j^m) = (A_\lambda^j u_j, u_j) + (A_\lambda^j w_j^m, w_j^m) + o(1)$$

and

$$\sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} u_j^m u_k^m\,dx = \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} w_j^m w_k^m\,dx + \sum_{j,k=1}^n \int_{\mathbb{R}^N} a_{jk} u_j u_k\,dx + o(1).$$

Therefore we can write

$$\begin{aligned}\langle I_\lambda'(U^m), U^m\rangle = {} & \sum_{j=1}^n (A_\lambda^j w_j^m, w_j^m) + \sum_{j=1}^n (A_\lambda^j u_j, u_j) \\ & - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} w_j^m w_k^m\,dx - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} u_j u_k\,dx \\ & - \int_{\mathbb{R}^N} F(U)\,dx - \int_{\mathbb{R}^N} F(W^m)\,dx + o(1).\end{aligned}$$

It then follows from (3.13) that

$$\sum_{j=1}^n (A_\lambda^j w_j^m, w_j^m) - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} w_j^m w_k^m\,dx = \int_{\mathbb{R}^N} F(W^m)\,dx + o(1).$$

Since $\{U^m\}$ is bounded, we may also assume that

$$\sum_{j=1}^n (A_\lambda^j w_j^m, w_j^m) - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} w_j^m w_k^m\,dx \to b$$

and

$$\int_{\mathbb{R}^N} F(W^m)\,dx \to b.$$

Similarly we have

$$c + o(1) = I_\lambda(U^m) = I_\lambda(U) + I_\lambda(W^m) = I_\lambda(U) + \left(\frac{1}{2} - \frac{1}{2^*}\right) b + o(1).$$

It follows from (3.13) and Proposition 3.2 that $I_\lambda(U) \ge 0$. Hence

$$c \ge \frac{1}{N}\, b. \tag{3.14}$$

Taking $\Lambda(\mu)$ larger if necessary, we may assume that $M_0\Lambda \ge \mu > \|A\|$. Let $\lambda > \Lambda(\mu)$. Since $|F_j| < \infty$, we have $\int_{F_j} (w_j^m)^2\,dx \to 0$ as $m \to \infty$. It then follows from the Sobolev inequality that

$$S_F\Big(\int_{\mathbb{R}^N} F(W^m)\,dx\Big)^{2/2^*} \le \sum_{j=1}^n \int_{\mathbb{R}^N} |\nabla w_j^m|^2\,dx$$

$$\le \sum_{j=1}^n \int_{\mathbb{R}^N} |\nabla w_j^m|^2\,dx + \sum_{j=1}^n \int_{F_j^c} (\lambda a_j - \mu)(w_j^m)^2\,dx$$

$$\le \sum_{j=1}^n \int_{\mathbb{R}^N} |\nabla w_j^m|^2\,dx + \sum_{j=1}^n \int_{\mathbb{R}^N} (\lambda a_j - \mu)(w_j^m)^2\,dx + o(1)$$

$$\le \sum_{j=1}^n \int_{\mathbb{R}^N} |\nabla w_j^m|^2\,dx + \sum_{j=1}^n \int_{\mathbb{R}^N} \lambda a_j (w_j^m)^2\,dx - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} w_j^m w_k^m\,dx + o(1).$$

Letting $m \to \infty$, we get

$$S_F b^{2/2^*} \le b.$$

If $b > 0$, this yields $S_F^{N/2} \le b$. This combined with (3.14) gives $c \ge N^{-1} S_F^{N/2}$, which is impossible. Consequently, $b = 0$ and by Proposition 3.2, $\|W^m\|_{E_\lambda} \to 0$ as $m \to \infty$, and this completes the proof. ∎

4. Existence of solutions. At various points we use the following assumptions on the coefficients $a_j(\cdot)$:

(A1) All coefficients a_j vanish on a common set D, that is,

$$a_j^{-1}(0) = D \ne \emptyset, \quad j = 1, \ldots, n,$$

where D is a bounded domain in $\mathbb{R}^N$ with a smooth boundary.

(A2) $\operatorname{int} \bigcap_{j=1}^n a_j^{-1}(0) \ne \emptyset$.

In what follows we denote by Ω a set equal to D if (A1) holds and equal to $\operatorname{int} \bigcap_{j=1}^n a_j^{-1}(0)$ if $(A2)$ holds. We shall also use the notation $\Omega_j = a_j^{-1}(0)$ introduced in Section 1. We set

$$I_\Omega(U) = \frac{1}{2} \sum_{j=1}^n \int_\Omega |\nabla u_j|^2\,dx - \frac{1}{2} \sum_{j,k=1}^n a_{jk} \int_\Omega u_j u_k\,dx - \frac{1}{2^*} \int_\Omega F(U)\,dx$$

for $U \in H_0^1(\Omega, \mathbb{R}^n)$. A critical point U of I_Ω is a solution of the Dirichlet problem for (1.2) in Ω, that is,

$$(4.15)\qquad \begin{aligned} -\Delta u_j &= \sum_{k=1}^n a_{jk} u_k + f_j(U) && \text{in } \Omega,\\ u_j(x) &= 0 && \text{on } \partial\Omega,\ j = 1, \ldots, n. \end{aligned}$$

The solvability of problem (4.15) has been investigated in the paper [1] when f is slightly more general. Let $\lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ in Ω.

Define

$$S_{F,\Omega} = \inf\Big\{ \sum_{j=1}^{n} \int_{\Omega} |\nabla u_j|^2\,dx - \sum_{j,k=1}^{n} a_{jk} \int_{\Omega} u_j u_k\,dx : \\ U \in H_0^1(\Omega, \mathbb{R}^n),\ \int_{\Omega} F(U)\,dx = 1\Big\}.$$

Assuming that A is symmetric and $\|A\| < \lambda_1(\Omega)$, by Theorem 1.1 of [1] we know that if $S_{F,\Omega} < S_F$, then problem (4.15) admits a solution. This result will be used to derive the existence result for system (1.2) through application of the mountain pass lemma. The above assumption will be maintained throughout this section. It is easy to check that the functional I_λ defined in Section 3 has the mountain pass geometry: there exist $\alpha > 0$ and $\varrho > 0$ such that $I_\lambda(U) \geq \alpha$ for $\|U\|_E = \varrho$. We can also find $W \in E$ such that $\|W\|_E \geq \varrho$ and $I_\lambda(W) < 0$. The mountain pass level is defined by

$$c_* = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = W\}$. It follows from the definition of c_* that

$$\begin{aligned} c_* &\leq \inf_{U \in E \setminus \{0\}} \max_{0 \leq t < \infty} I_\lambda(tU) \\ &\leq \frac{1}{N} \inf_{U \in E \setminus \{0\}} \frac{(\sum_{j=1}^{n} \int_{\mathbb{R}^N} (|\nabla u_j|^2 + \lambda a_j u_j^2)\,dx - \sum_{j,k=1}^{n} a_{jk} \int_{\mathbb{R}^N} u_j u_k\,dx)^{N/2}}{(\int_{\mathbb{R}^N} F(U)\,dx)^{(N-2)/2}}. \end{aligned}$$

Since $a_j(x) = 0$ on Ω, $j = 1, \ldots, n$, we deduce from the above inequality that

$$c_* \leq \frac{1}{N} \inf_{U \in H_0^1(\Omega, \mathbb{R}^n) \setminus \{0\}} \frac{(\sum_{j=1}^{n} \int_{\Omega} |\nabla u_j|^2\,dx - \sum_{j,k=1}^{n} a_{jk} \int_{\Omega} u_j u_k\,dx)^{N/2}}{(\int_{\Omega} F(U)\,dx)^{(N-2)/2}}.$$

Thus, if $S_{F,\Omega} < S_F$, then $c_* < S_F^{N/2}/N$. Applying Proposition 3.3, we get the following existence result:

THEOREM 4.1. *Suppose that*

$$S_{F,\Omega} < S_F. \tag{4.16}$$

Then problem (1.2) *admits a solution.*

Let

$$w(x) = \frac{(N(N-2))^{(N-2)/4}}{(1 + |x|^2)^{(N-2)/2}}.$$

The function w, called an *instanton*, solves the equation

$$-\Delta w = |w|^{2^*-2} w \quad \text{in } \mathbb{R}^N.$$

It is well known that the best Sobolev constant S, defined by

$$S = \inf_{u\in D^{1,2}(\Omega)\setminus\{0\}} \int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

is attained by w. Every positive minimiser for S has the form $T_{\varepsilon,x_0} = \varepsilon^{(2-N)/2} w\big(\frac{x-x_0}{\varepsilon}\big)$ for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$.

We have the following minimisers for S_F:

Proposition 4.2. *If $q \ge 2$ then $S_F = S$ is achieved only by $U = T_{\varepsilon,x_0} e_j$, for any $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$, $1 \le j \le n$.*

If $q < 2$ then $S_F = n^{(q-2)/q} S$ is achieved only by $U = (n^{-1/q} T_{x_0,\varepsilon}, \ldots, n^{-1/q} T_{x_0,\varepsilon})$, for any $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$.

If $q = 2$, then $S_F = S$ is achieved by $(\alpha_1 T_{x_0,\varepsilon}, \ldots, \alpha_n T_{x_0,\varepsilon})$, where $\sum_{i=1}^n \alpha_i^2 = 1$, $0 \le \alpha_i \le 1$, for any $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$.

Proof. Lemma 3 of [1] implies that $S_F = S$ if $q \ge 2$. Following a method of proof similar to that of Proposition 2.4, we note that Jensen's inequality for concave functions is an equality if all elements in the sum are zero, apart from one element.

The proofs of the remaining parts follow the methods of Propositions 2.4 and 2.7. For $q < 2$, we note that the weighted mean inequality in the proof of Proposition 2.7 is only an equality if all elements in the sum are identical. If $q = 2$, we note that equality holds for Minkowski's inequality in the proof of Proposition 2.4 only if each element in the sum is a multiple of a common term. ∎

We have the following result:

Corollary 4.3. *Suppose that $F(U) = (\sum_{i=1}^n |u_i|^q)^{2^*/q}$. If $q \ge 2$, assumption* (A) *holds and $a_{ii} > 0$ for some i, then problem* (1.2) *admits a solution. If $q < 2$, assumption* (A2) *holds and $\sum_{i,j=1}^n a_{ij} > 0$, then problem* (1.2) *admits a solution.*

Proof. Since the mountain pass geometry has been confirmed, we only need to verify (4.16).

If $q \ge 2$, then

$$\begin{aligned}\inf\Big\{ \sum_{j=1}^n \int_{\mathbb{R}^N} (|\nabla u_j|^2 + \lambda a_j u_j^2)\,dx - \sum_{j,k=1}^n a_{jk} \int_{\mathbb{R}^N} u_j u_k\,dx :\\ u \in H^1(\mathbb{R}^N, \mathbb{R}^n),\ \int_{\mathbb{R}^N} F(U)\,dx = 1\Big\}\\ < \inf_{u\in H_0^1(\Omega_i)} \frac{\int_{\Omega_i} |\nabla u|^2\,dx - a_{ii}\int_{\Omega_i} u^2\,dx}{(\int_{\Omega_i} |u|^{2^*}\,dx)^{2/2^*}} < S.\end{aligned}$$

The last part of this estimate follows from [3].

If $q < 2$, let $x_0 \in \bigcap_j \Omega_j$, and let r be sufficiently small that $B(x_0, r) \subset \bigcap_j \Omega_j$. Define $u_\varepsilon = \phi_{r,x_0} T_{\varepsilon,x_0}$, where ϕ_{r,x_0} is a smooth function which is zero on $\mathbb{R}^N \setminus B(x_0, r)$ and one on $B(x_0, r/2)$. Let $U = (n^{-1/q} u_\varepsilon, \ldots, n^{-1/q} u_\varepsilon)$. Then the Brézis–Nirenberg estimates give

$$\sum_{j=1}^{n} \int_{\mathbb{R}^N} |\nabla u_j|^2\, dx = n^{1-2/q} S^{N/2} + O(\varepsilon^{N-2}),$$

$$\sum_{j,k=1}^{n} a_{jk} \int_{\mathbb{R}^N} u_j u_k\, dx = \begin{cases} n^{-2/q} \sum_{i,j=1}^{n} a_{ij} \varepsilon^2 \log(\varepsilon) + O(\varepsilon^2), & N = 4, \\ n^{-2/q} \sum_{i,j=1}^{n} a_{ij} \varepsilon^2 + O(\varepsilon^{N-2}), & N \geq 5, \end{cases}$$

$$\Big(\int_{\mathbb{R}^N} \sum_{j=1}^{n} |u_j|^q\, dx \Big)^{p/q} = S^{N/2} + O(\varepsilon^N),$$

giving $S_{F,\Omega} < S_F = n^{(q-2)/q} S$. ∎

We remark that if A is a matrix consisting not only of diagonal elements, then the solutions possess a genuine vector structure. Suppose we seek solutions of the form $u e_j$, $u \in H^1$. Suppose that $a_{kj} \neq 0$ for some $k \neq j$. Then the kth component of the elliptic system is

$$0 = a_{k1} \times 0 + \ldots + a_{kj} u_k + \ldots + a_{kn} \times 0,$$

giving $u_j = 0$ and yielding only the trivial solution.

If $q < 2$, and A consists only of diagonal elements, then the mountain pass solution possesses a genuine vector structure.

THEOREM 4.4. *Let* $1 < q < 2$. *Suppose that for each* $1 \leq i \leq n$, $a_{ii} |\Omega_i|^{2/N} < S(1 - n^{(q-2)/q})$. *Then there exists* $\lambda_0 > 0$ *so that for each* $\lambda > \lambda_0$, *the ground state* (*mountain pass*) *solution has a genuine vector form*, *that is*, *it does not take the form* $u e_j$.

Proof. By Proposition 4.2, the bound for the mountain pass level is

$$c^* < \frac{1}{N} S_F^{N/2} = \frac{1}{N} n^{\frac{q-2}{q} \frac{N}{2}} S^{N/2}.$$

If $\overline{U}$ is a solution, then $(I'_\lambda(\overline{U}), \overline{U}) = 0$. Suppose that $\overline{U} = u e_j$. Then

$$\begin{aligned} I_\lambda(\overline{U}) &= \frac{1}{N} \bigg(\frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda a_j u^2 - a_{jj} u^2)}{(\int_{\mathbb{R}^N} |u|^{2^*}\, dx)^{2/2^*}} \bigg)^{N/2} \\ &\geq \frac{1}{N} \bigg(\inf_{u \in H^1} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\big)^{2/2^*}} + \inf_{u \in H^1} \frac{\int_{\mathbb{R}^N} (\lambda a_j - a_{jj}) u^2\, dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\, dx)^{2/2^*}} \bigg)^{N/2} \end{aligned}$$

$$\geq \frac{1}{N}\bigg(S - \sup_{u\in H^1} \frac{\int_{\mathbb{R}^N}(a_{jj}-\lambda a_j)u^2\,dx}{(\int_{\mathbb{R}^N}|u|^{2^*}\,dx)^{2/2^*}}\bigg)^{N/2}.$$

Let $\Gamma_j = \{x \in \mathbb{R}^N : a_{jj} > \lambda a_j(x)\}$. For λ sufficiently large, this set is of finite measure by assumption (A). Furthermore, $|\Gamma_j| \to |\Omega_j|$ as $\lambda \to \infty$. Now,

$$\sup_{u\in H^1} \frac{\int_{\mathbb{R}^N}(a_{jj}-\lambda a_j)u^2\,dx}{(\int_{\mathbb{R}^N}|u|^{2^*}\,dx)^{2/2^*}} \leq \sup_{u\in H^1} \frac{\int_{\Gamma_j}(a_{jj}-\lambda a_j)u^2\,dx}{(\int_{\Gamma_j}|u|^{2^*}\,dx)^{2/2^*}}$$
$$\leq a_{jj}\sup_{u\in H^1} \frac{\int_{\Gamma_j}u^2\,dx}{(\int_{\Gamma_j}|u|^{2^*}\,dx)^{2/2^*}} \leq a_{jj}|\Gamma_j|^{2/N},$$

where the last step follows by Hölder's inequality.

Thus, $I_\lambda(\overline{U}) \geq N^{-1}(S - a_{jj}|\Gamma_j|^{2/N})^{N/2}$. For λ sufficiently large, $|\Gamma_j|$ is arbitrarily close to $|\Omega_j|$. Thus a contradiction arises as $I_\lambda(\overline{U})$ exceeds the mountain pass energy for any element $\overline{U}$ of the Nehari manifold with only one nonzero component. ∎

We now examine the behaviour of solutions to (1.2) as $\lambda \to \infty$.

THEOREM 4.5. *Suppose that the assumptions of Corollary* 4.3 *hold. Let* $\lambda_m \to \infty$ *and let* $\{U^m\}$ *be a sequence of corresponding mountain pass solutions of* (1.2), *such that* $I_{\lambda_m}(U^m) \to c < S^{N/2}/N$. *Then, up to a subsequence,* $U^m \to U$ *in* E *and* $U \in H^1_0(\Omega_1)\times\ldots\times H^1_0(\Omega_n)$. *If* (A1) *holds, then* U *is a solution of* (4.15). *In particular, if* $a_j^{-1}(0) = \Omega_j$ *and* Ω_j *are pairwise disjoint then* u_j (*meaning the* j*th component of* U) *satisfies*:

$$\begin{cases} -\Delta u_j = a_{jj}u + \widetilde{f}_j(u_j) & \text{in } \Omega_j,\\ u_j = 0 & \text{on } \partial\Omega_j,\end{cases} \tag{4.17}$$

where $\widetilde{f}_j(u_j) = f_j(0,\ldots,0,u_j,0,\ldots,0)$.

Proof. Since $\|U^m\|_{E,\lambda_m} \leq K$ for some $K > 0$ and all m, by Lemma 3.1 we may assume that $U^m \rightharpoonup U$ in E and $U^m \to U$ in $L^2(\mathbb{R}^N,\mathbb{R}^n)$, with $U \in H^1_0(\Omega,\mathbb{R}^n)$. Let $w_j^m = u_j^m - u_j$, $W^m = (w_1^m,\ldots,w_n^m)$. From the proof of Proposition 3.3 we have

$$\sum_{j=1}^n (A^j_{\lambda_m} w_j^m, w_j^m) = \int_{\mathbb{R}^N} F(W^m)\,dx + o(1).$$

We may assume that

$$\lim_{m\to\infty}\sum_{j=1}^n (A^j_{\lambda_m} w_j^m, w_j^m) = \lim_{m\to\infty}\int_{\mathbb{R}^N} F(W^m)\,dx = b.$$

We claim that $b = 0$. Assuming that $b > 0$, it follows from the Sobolev inequality that

$$S\Big(\int_{\mathbb{R}^N} F(W^m)\,dx\Big)^{2/2^*} \le \sum_{j=1}^n \int_{\mathbb{R}^N} |\nabla w_j^m|^2\,dx \le \sum_{j=1}^n (A_{\lambda_m}^j w_j^m, w_j^m).$$

Letting $m \to 0$, we deduce from this that

$$b \ge S^{N/2}. \tag{4.18}$$

On the other hand,

$$\begin{aligned} I_{\lambda_m}(U^m) = I_{\lambda_m}(W^m) + I_{\lambda_m}(U) &= \frac{1}{N}\sum_{j=1}^n (A_{\lambda_m}^j w_j^m, w_j^m) + I_{\lambda_m}(U) + o(1) \\ &= \frac{1}{N}\sum_{j=1}^n (A_{\lambda_m}^j w_j^m, w_j^m) + I_{\lambda_m}(U), \end{aligned}$$

yielding $b < Nc$. Combining this with (4.18), we have a contradiction. Since $b = 0$, we deduce from Proposition 3.2 that $U^m \to U$ in E.

It is clear that if $a_j^{-1}(0) = D$ for each j (that is, if (A1) holds) then U satisfies (4.15). If $a_j^{-1}(0) = \Omega_j$ are pairwise disjoint, then if $x \in \Omega_j$, we have $f_j(U) = f_j(0, \ldots, 0, u_j, 0, \ldots, 0)$, which means that u_j is a solution of the Dirichlet problem (4.17). ∎

Finally, we make the following observation about the mountain pass levels for I_λ and I_Ω. Let us denote these by c_λ and c_Ω, respectively. By Theorem 4.5, we have $\lim_{\lambda\to\infty} c_\lambda \le c \le c_\Omega$. On the other hand, by Corollary 4.3, for each large λ, there is a solution. So by Theorem 4.5, c is achieved by I_Ω. Thus $c \ge c_\Omega$ with, necessarily, $\lim_{\lambda\to\infty} c_\lambda = c_\Omega$.

REMARK 4.6. A similar result can be derived for solutions of (1.1) obtained through the constrained minimisation (2.3). In this case, the limiting Dirichlet problem has the form

$$\begin{aligned} -\Delta u + u = |u|^p &\quad \text{in } \Omega_i, \\ u = 0 &\quad \text{on } \partial\Omega_i. \end{aligned}$$

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