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Existence of periodic solutions for Liénard-type p-Laplacian systems with variable coefficients

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Abstract. We study the existence of periodic solutions for Liénard-type p-Laplacian systems with variable coefficients by means of the topological degree theory. We present sufficient conditions for the existence of periodic solutions, improving some known results.

1. Introduction. In the past two decades, the p-Laplacian equation

$$(\phi_p(x'))' = f(t,x,x'), \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$ ($s \neq 0$) and $\phi_p(0) = 0$ for $p > 1$, has been extensively studied and applied to many scientific fields. For instance, it is used as the model of turbulent flow in a porous medium [8, 3], the model of animal and insect dispersion [11], and also the model of non-Newtonian liquid [7]. Recently, many important results have been established for the one-dimensional p-Laplacian equation (1.1) associated with two-point boundary conditions (see [1, 4, 5, 12, 10, 15] and references therein), with periodic boundary conditions (see [13, 16, 2]), as well as multi-point boundary conditions (see e.g. [6]). However, it seems that results on higher dimensional p-Laplacian equations are very few. It is worth mentioning that Manásevich and Mawhin [9] studied the existence of periodic solutions for the n-dimensional p-Laplacian system (1.1) by using extended continuation theorems.

In the present paper, we are concerned with the existence of T-periodic solutions for a Liénard-type p-Laplacian system with variable coefficients

$$(\phi_p(x'))' + F(t,x)x' + G(t,x) = E(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $F(t,x) = \operatorname{diag}(\beta_1(t)f_1(x_1), \ldots, \beta_n(t)f_n(x_n))$, $\beta_i \in C(\mathbb{R})$, $\beta_i(t+T) = \beta_i(t)$, $\beta_i' \in L^1[0,T]$, $f_i \in C(\mathbb{R})$ $(i = 1, \ldots, n)$, $G \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $G(t+T,x) = G(t,x)$, $E \in C(\mathbb{R}, \mathbb{R}^n)$, $E(t+T) = E(t)$, and $\phi_p : \mathbb{R}^n \to \mathbb{R}^n$ is

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defined by

$$\phi_p(x) = (\phi_p(x_1), \dots, \phi_p(x_n)), \qquad x = (x_1, \dots, x_n),$$
$$\phi_p(x_i) = |x_i|^{p-2} x_i, \qquad p > 1,\ i = 1, \dots, n.$$

Obviously, (1.2) is a classical non-autonomous n-dimensional Liénard equation when $p = 2$ and $F(t,x) = F(x) = \operatorname{diag}(f_1(x_1), \dots, f_n(x_n))$. In this case, $\int_0^T f_i(x_i(t))x_i'(t)\,dt = 0$ if $x(\cdot)$ is T-periodic. However, for the case of variable coefficients, since

$$\int_0^T \beta_i(t) f_i(x_i(t)) x_i'(t)\,dt \neq 0$$

and many methods and techniques cannot be applied, dealing with (1.2) is more difficult.

In addition, [14] studied the existence of periodic solutions for a scalar Duffing-type p-Laplacian equation

$$(\phi_p(x'))' + cx' + g(t,x) = e(t), \tag{1.3}$$

under the conditions

- (A$_1$) $xg(t,x) < 0$ for $|x| > 0$, $t \in \mathbb{R}$,
- (A$_2$) $2^{2-p}MT^p < 1$ and $g(t,x) \ge -M|x|^{p-1} - K$ for $x \ge 0$ and $t \in \mathbb{R}$.

Apparently, (1.3) is the same as (1.2) if $n = 1$ and $F(t,x) = C$; and our main results do not demand condition (A$_2$).

2. Main results. To state our results, we use standard notations: $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$; $|\cdot|$ denotes the Euclidean norm defined by

$$|x| = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n;$$

$|\cdot|_p$ denotes the norm in $L^p([0,T], \mathbb{R}^n)$ defined by

$$|x|_p = \Big(\sum_{i=1}^n \int_0^T |x_i(t)|^p\,dt\Big)^{1/p}.$$

Moreover $C_T^k(\mathbb{R}, \mathbb{R}^n) = \{x(\cdot) \in C^k(\mathbb{R}, \mathbb{R}^n) : x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$, $k = 0, 1$, and the norm in C_T is denoted by $|x|_\infty = \max_{t \in [0,T]} |x(t)|$. Finally, we set

$$\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt, \qquad \tilde{x}(t) = x(t) - \bar{x}, \qquad \text{for } x(\cdot) \in C(\mathbb{R}, \mathbb{R}^n).$$

To prove our main results, we need two technical lemmas:

Lemma 2.1 ([9]). *Assume that Ω is an open bounded set in C_T^1 such that:*

(1) *For each $\lambda \in (0,1)$, the problem*

$$(\phi_p(x'))' = \lambda f(t,x,x'), \qquad x(0)=x(T),\ x'(0)=x'(T), \tag{2.1}$$

has no solution on $\partial\Omega$, where $f \in C(\mathbb{R}\times\mathbb{R}^{2n},\mathbb{R}^n)$, $x=(x_1,\dots,x_n) \in \mathbb{R}^n$.

(2) *The equation*

$$F(a) = \frac{1}{T}\int_0^T f(t,a,0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}^n$.

(3) *The Brouwer degree satisfies*

$$\deg_B(F,\Omega\cap\mathbb{R}^n,0) \neq 0.$$

Then problem (2.1) has a solution in $\bar{\Omega}$ when $\lambda = 1$.

In order to make use of Lemma 2.1 in the study of equation (1.2), let us consider the homotopy equation

$$(\phi_p(x'))' = \lambda[E(t) - F(t,x)x' - G(t,x)], \qquad 0 \le \lambda \le 1, \tag{2.2}$$

and establish the following lemma:

Lemma 2.2. *Suppose that:*

(1) *there exists a constant $d > 0$ such that*

$$\langle G(t,x), x\rangle \le 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ |x| > d;$$

(2) $\beta_i'(t)\int_0^z f_i(s)s\,ds \ge 0$, $t \in \mathbb{R}$, $z \in \mathbb{R}$, $i = 1,\dots,n$.

Then any T-periodic solution $x(\cdot)$ of equation (2.2) satisfies the inequality

$$|x'|_p \le \varepsilon|\bar{x}| + K(\varepsilon, |a_d|_1, |E|_1), \tag{2.3}$$

where $a_d(\cdot) \in L^1[0,T]$, $|G(t,x)| \le a_d(t)$ for $(t,x) \in [0,T]\times\mathbb{R}^n$, $|x| \le d$, ε is an arbitrary positive number, and $K(\cdot,\cdot,\cdot) > 0$ is a constant.

Proof. First, we define the function $r : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ by

$$r(t,x) = \begin{cases} G(t,x), & |x| > d, \\ G\left(t, d\dfrac{x}{|x|}\right)\dfrac{|x|}{d}, & 0 < |x| \le d, \\ 0, & x = 0, \end{cases}$$

and set

$$h(t,x) = G(t,x) - r(t,x).$$

Then, for any $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, we have

$$\langle r(t,x), x\rangle \le 0, \qquad |h(t,x)| \le 2a_d(t). \tag{2.4}$$

Rewrite (2.2) as

$$-(\phi_p(x'))' = \lambda\left[F(t,x)x' + r(t,x) + h(t,x) - E(t)\right]. \tag{2.5}$$

Taking the inner product with $x(t)$ on both sides of (2.5), integrating on $[0,T]$, and noting

$$\int_0^T \beta_i(t) f_i(x_i(t)) x_i(t) x_i'(t)\,dt = -\int_0^T \beta_i'(t) \int_0^{x_i(t)} f_i(s) s\,ds\,dt \le 0$$

with (2.4) we have

$$\begin{aligned}\int_0^T (|x_1'(t)|^p + \cdots + |x_n'(t)|^p)\,dt &= \lambda \sum_{i=1}^n \int_0^T \beta_i(t) f_i(x_i(t)) x_i(t) x_i'(t)\,dt \\ &\quad + \lambda \int_0^T \langle r(t,x) + h(t,x) - E(t), x\rangle\,dt \\ &\le \lambda \int_0^T \langle h(t,x) - E(t), x(t)\rangle\,dt \\ &= \lambda \int_0^T \langle h(t,x) - E(t), \tilde{x}(t) + \bar{x}\rangle\,dt \\ &\le \int_0^T (2|a_d(t)| + |E(t)|)(|\tilde{x}(t)| + |\bar{x}|)\,dt.\end{aligned} \tag{2.6}$$

It follows that

$$\begin{aligned}\int_0^T (2|a_d(t)| + |E(t)|)|\tilde{x}(t)|\,dt &\le |\tilde{x}|_\infty (2|a_d|_1 + |E|_1) \\ &\le \frac{\mu^p}{p}|\tilde{x}|_\infty^p + \frac{1}{q\mu^q}(2|a_d|_1 + |E|_1)^q,\end{aligned} \tag{2.7}$$

where μ is an arbitrary positive constant, and $p, q > 1$ with $1/p + 1/q = 1$. Noting $\int_0^T \tilde{x}_i(t)\,dt = 0$ where $\tilde{x}_i(t)$ is the component of $\tilde{x}(t)$, there exists $t_i \in [0,T]$ such that $\tilde{x}_i(t_i) = 0$ $(i = 1, \ldots, n)$. It is easy to check from

$$\tilde{x}_i(t) = \int_{t_i}^t \tilde{x}_i'(s)\,ds = \int_{t_i}^t x_i'(s)\,ds,$$

that

$$|\tilde{x}_i(t)| \le \int_0^T |\tilde{x}_i'(s)|\,ds,$$

$$|\tilde{x}_i(t)|^p \le \Big(\int_0^T |\tilde{x}_i'(s)|\,ds\Big)^p \le T^{p/q} \int_0^T |\tilde{x}_i'(s)|^p\,ds,$$

$$\sum_{i=1}^{n} |\tilde{x}_i(t)|^p \le T^{p/q} \sum_{i=1}^{n} \int_0^T |\tilde{x}_i'(s)|^p \, ds.$$

Thus

$$|\tilde{x}|_\infty^p \le T^{p/q} |x'|_p^p. \tag{2.8}$$

From (2.8), inequality (2.7) can be rewritten as

$$\int_0^T (2|a_d(t)| + |E(t)|)|\tilde{x}(t)| \, dt \le \frac{\mu^p}{p} T^{p/q} |x'|_p^p + \frac{1}{q\mu^q}(2|a_d|_1 + |E|_1)^q. \tag{2.9}$$

On the other hand, we know that

$$\begin{aligned} \int_0^T (2|a_d(t)| + |E(t)|)|\bar{x}| \, dt &= |\bar{x}|(2|a_d|_1 + |E|_1), \\ &\le \frac{\eta^p}{p}|\bar{x}|^p + \frac{1}{q\eta^q}(2|a_d|_1 + |E|_1)^q, \end{aligned} \tag{2.10}$$

where η is an arbitrary positive number. Choosing μ such that $1-(\mu^p/p)T^{p/q} > 0$, (2.6) together with (2.9) and (2.10) gives

$$|x'|_p^p \le \frac{\eta^p}{c^2 p}|\bar{x}|^p + \frac{1}{c^2 q}\left(\frac{1}{\mu^q} + \frac{1}{\eta^q}\right)(2|a_d|_1 + |E|_1)^q$$

where $c^2 = 1 - (\mu^p/p)T^{p/q} > 0$. Letting

$$\varepsilon^p := \frac{\eta^p}{c^2 p}, \qquad K^p := \frac{1}{c^2 q}\left(\frac{1}{\mu^q} + \frac{1}{\eta^q}\right)(2|a_d|_1 + |E|_1)^q,$$

we obtain (2.3). ∎

Remark. From the proof of the lemma, we see that if the functional $G(t,x)$ satisfies Carathéodory's condition, the result of the lemma is still true.

Next, we state one of our main results:

Theorem 2.3. *Suppose that*

(H$_1$) *there exists a constant $d > 0$ such that, for any $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $|x_i| > d$,*

$$G_i(t,x)x_i < 0, \qquad i = 1, \ldots, n, \tag{2.11}$$

where $G_i(t,x)$ is the component of $G(t,x)$;

(H$_2$) $\beta_i'(t)\int_0^z f_i(s)s\,ds \ge 0$ *and* $\beta_i'(t)z\int_0^z f_i(s)\,ds \ge 0$, *for all $t, z \in \mathbb{R}$ and $i = 1, \ldots, n$;*

(H$_3$) $\int_0^T E(t)dt = 0$.

Then equation (1.2) has a T-periodic solution.

Proof. In order to use Lemma 2.1, we first consider equation (2.2) and find an a priori estimate for its T-periodic solutions. Suppose that $x(\cdot)$ is a T-periodic solution of (2.2); then $x(\cdot)$ satisfies inequality (2.3), i.e.,

$$|x'|_p \le \varepsilon|\bar{x}| + K(\varepsilon, |a_d|_1, |E|_1).$$

Integrating both sides of (2.2) on $[0,T]$, and using the conditions $x(0) = x(T)$ and $x'(0) = x'(T)$, we have

$$\int_0^T \beta_i(t) f_i(x_i(t)) x_i'(t)\,dt + \int_0^T G_i(t, x(t))\,dt = 0, \qquad i = 1, \dots, n.$$

Integration by parts yields

$$-\int_0^T \beta_i'(t) \int_0^{x_i(t)} f_i(s)\,ds\,dt + \int_0^T G_i(t, x(t))\,dt = 0, \qquad i = 1, \dots, n.$$

By conditions (H_1), (H_2), and since $x(t) = \bar{x} + \tilde{x}(t)$, we have

$$\bar{x}_i - \max_{t\in[0,T]} |\tilde{x}_i(t)| \le \min_{t\in[0,T]} x_i(t) < d,$$
$$\bar{x}_i + \max_{t\in[0,T]} |\tilde{x}_i(t)| \ge \max_{t\in[0,T]} x_i(t) > -d, \qquad i = 1, \dots, n.$$

Thus, we obtain

$$|\bar{x}_i| \le d + \max_{t\in[0,T]} |\tilde{x}_i(t)| = d + |\tilde{x}_i|_\infty \le d + |\tilde{x}|_\infty, \qquad i = 1, \dots, n,$$

and

$$|\bar{x}| = \Big(\sum_{i=1}^n |\bar{x}_i|^p\Big)^{1/p} \le n^{1/p}(d + |\tilde{x}|_\infty). \tag{2.12}$$

From (2.8) and (2.12), it is easy to derive that

$$|\bar{x}| \le n^{1/p} d + n^{1/p} T^{1/q} |x'|_p. \tag{2.13}$$

Combining (2.3) with (2.13), and choosing $\varepsilon > 0$ such that $1 - \varepsilon n^{1/p} T^{1/q} > 0$, we have

$$|x'|_p \le \frac{n^{1/p} d\varepsilon + K(\varepsilon, |a_d|_1, |E|_1)}{1 - \varepsilon n^{1/p} T^{1/q}} =: d_1. \tag{2.14}$$

It follows from (2.8) and (2.13) that

$$|\tilde{x}|_\infty \le T^{1/q} d_1, \qquad |\bar{x}| \le n^{1/p} d + n^{1/p} T^{1/q} d_1,$$

which directly leads to

$$|x|_\infty \le |\bar{x}| + |\tilde{x}|_\infty \le T^{1/q} d_1 + n^{1/p} d + n^{1/p} T^{1/q} d_1 =: d_2.$$

Since $x_i(0) = x_i(T)$, there exists $t_i \in (0,T)$ such that $x_i'(t_i) = 0$. Integrating (2.2) from t_i to t, we get

$$\phi_p(x_i'(t)) + \lambda \int_{t_i}^{t} \beta_i(s) f_i(x_i(s)) x_i'(s)\, ds + \lambda \int_{t_i}^{t} G_i(s, x(s))\, ds = \lambda \int_{t_i}^{t} E_i(s)\, ds,$$

$i = 1, \ldots, n$. This yields

$$\begin{aligned} |x_i'(t)|^{p-1} &\le M_1 \int_0^T |x_i'(t)|\, dt + M_2 T + M_3 T \\ &\le M_1 T^{1/q} \Big(\int_0^T |x_i'(t)|^p\, dt \Big)^{1/p} + M_2 T + M_3 T \\ &\le M_1 T^{1/q} d_1 + (M_2 + M_3) T =: M_4^{p-1}, \end{aligned}$$

where $M_1 = \max_{0 \le t \le T,\, |x| \le d_2} |\beta_i(t) f_i(x_i)|$, $M_2 = \max_{0 \le t \le T,\, |x| \le d_2} |G(t,x)|$, $M_3 = \max_{0 \le t \le T} |E(t)|$. Thus,

$$|x'(t)| \le n^{1/p} M_4 =: d_3, \qquad t \in [0,T].$$

Define

$$\Omega = \{x \in C_T^1 : |x|_\infty < d_2 + 1,\ |x'|_\infty < d_3 + 1\},$$
$$F(\cdot) = \int_0^T G(t, \cdot)\, dt : \mathbb{R}^n \to \mathbb{R}^n.$$

Note that $F(a) = 0$ has no solution on $\partial\Omega \cap \mathbb{R}^n$ from the condition (2.11) and $d_2 > d$. Now we may construct a homotopy $H(\cdot, \lambda) : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by

$$H(a, \lambda) = \lambda a - (1 - \lambda) F(a) =: H_\lambda(a).$$

From (H_1) it is easy to verify that

$$\langle H(a,\lambda), a \rangle > 0 \quad \text{ on } \partial\Omega \cap \mathbb{R}^n,\ 0 \le \lambda \le 1.$$

Thus, we have

$$\deg_B(H_\lambda, \Omega \cap \mathbb{R}^n, 0) = \deg_B(-F, \Omega \cap \mathbb{R}^n, 0) = \deg_B(I, \Omega \cap \mathbb{R}^n, 0) = 1.$$

By Lemma 2.1, we conclude that (1.2) has a T-periodic solution. ∎

Applying Theorem 2.3, we immediately get

COROLLARY 2.4. *Assume conditions* (H_1) *and* (H_3) *in Theorem* 2.3 *hold. Then the Liénard-type p-Laplacian system*

$$(\phi_p(x'(t)))' + \operatorname{diag}(c_1 f_1(x_1), \ldots, c_n f_n(x_n)) x' + G(t,x) = E(t),$$

where c_i $(i = 1, \ldots, n)$ are constants, has a T-periodic solution.

In addition, if condition (H_1) is weakened to condition (1) of Lemma 2.2, then the following result can be obtained.

Theorem 2.5. *Assume that conditions* (1), (2) *of Lemma* 2.2 *and*

$$(\mathrm{C}_1)\qquad b(\cdot)=\limsup_{|x|\to\infty}\frac{\langle G(\cdot,x),x\rangle}{|x|}\in L^1[0,T],\ b(t)\le 0,\qquad \int_0^T b(t)\,dt<0$$

hold. Then equation (1.2) *has a* T*-periodic solution.*

Proof. We first look for an a priori estimate for T-periodic solutions of (2.2). Suppose that $x(\cdot)$ is such a solution; then it satisfies (2.3), i.e.,

$$|x'|_p\le\varepsilon|\bar{x}|_p+K(\varepsilon,|a_d|_1,|E|_1).$$

In order to estimate $|\bar{x}|$, set $\varepsilon\le 1/(2T^{1/q})$ in (2.3). Then (2.8) gives

$$|\tilde{x}|_\infty\le T^{1/q}|x'|_p\le\tfrac12|\bar{x}|+c_1,$$

where $c_1>0$ is independent of λ. We deduce from $x(t)=\tilde{x}(t)+\bar{x}$ that

$$(2.15)\qquad |x(t)|\ge|\bar{x}|-|\tilde{x}|_\infty\ge\tfrac12|\bar{x}|-c_1,\qquad t\in[0,T].$$

Now, we claim that there exists a constant $c_2>0$ independent of λ such that

$$(2.16)\qquad |\bar{x}|\le c_2.$$

Otherwise there exists $\lambda_n\in(0,1]$ such that $x_n(\cdot)$, a T-periodic solution of (2.2) (when $\lambda=\lambda_n$), has the property $|\bar{x}_n|\to\infty$ as $n\to\infty$. Together with (2.15) this leads to

$$(2.17)\qquad |x_n(t)|\to\infty\qquad\text{as } n\to\infty,$$

uniformly on $[0,T]$. Taking the inner product with $x_n(t)$ on both sides of (2.2) (when $\lambda=\lambda_n$) and integrating over $[0,T]$, we obtain

$$-\int_0^T\langle\phi_p(x_n'(t)),x_n'(t)\rangle\,dt+\lambda_n\int_0^T\langle G(t,x_n(t)),x_n(t)\rangle\,dt$$
$$-\lambda_n\sum_{i=1}^n\int_0^T\beta_i'(t)\int_0^{x_{ni}(t)}f_i(s)s\,ds\,dt=\lambda_n\int_0^T\langle E(t),x_n(t)\rangle\,dt,$$

where $x_{ni}(t)$ is the ith component of $x_n(t)$.

Since

$$\int_0^T\langle\phi_p(x_n'(t)),x_n'(t)\rangle\,dt=\int_0^T|x_n'(t)|^p\,dt\ge 0,$$

and

$$\int_0^T\beta_i'(t)\int_0^{x_{ni}(t)}f_i(s)s\,ds\,dt\ge 0,$$

we have

$$0 \le \int_0^T \langle G(t, x_n(t)), x_n(t)\rangle\, dt - \int_0^T \langle E(t), \tilde{x}_n(t)\rangle\, dt. \tag{2.18}$$

From condition (C_1) and (2.17), we know that for any given $\varepsilon > 0$, there exists a constant $N > 0$ such that when $n \ge N$,

$$\langle G(t, x_n(t)), x_n(t)\rangle \le [b(t) + \varepsilon]|x_n(t)|, \qquad t \in [0, T]. \tag{2.19}$$

Combining (2.18) and (2.19), we deduce

$$0 \le \min_{t\in[0,T]} |x_n(t)| \int_0^T b(t)\, dt + T\varepsilon |x_n|_\infty + |E|_1 |\tilde{x}_n|_\infty, \qquad n \ge N. \tag{2.20}$$

From (2.3) and (2.8) we see that

$$|\tilde{x}_n|_\infty \le T^{1/q}\varepsilon |\bar{x}_n| + c_3, \tag{2.21}$$

where c_3 is a number independent of λ_n. Thus, we have

$$|x_n|_\infty \le |\tilde{x}_n|_\infty + |\bar{x}_n| \le [1 + T^{1/q}\varepsilon]|\bar{x}_n| + c_3. \tag{2.22}$$

In addition, we know from (2.15) that

$$\min_{t\in[0,T]} |x_n(t)| \ge \tfrac{1}{2}|\bar{x}_n| - c_1. \tag{2.23}$$

Noting $\int_0^T b(t)\, dt < 0$, (2.20)–(2.23) yield

$$0 \le \tfrac{1}{2}|\bar{x}_n| \int_0^T b(t)\, dt + \varepsilon[T(1 + T^{1/q}\varepsilon) + |E|_1 T^{1/q}]|\bar{x}_n| + c_4, \tag{2.24}$$

where c_4 is a constant independent of λ_n. Choosing $\varepsilon > 0$ such that

$$\tfrac{1}{2}\int_0^T b(t)\, dt + \varepsilon[T(1 + T^{1/q}\varepsilon) + |E|_1 T^{1/q}] < 0,$$

we see from (2.17) that (2.24) is a contradiction as $n \to \infty$. Consequently, the claim (2.16) is true, and moreover,

$$|\tilde{x}|_\infty \le \tfrac{1}{2}c_2 + c_1.$$

Thus

$$|x|_\infty \le |\tilde{x}|_\infty + |\bar{x}| \le \tfrac{1}{2}c_2 + c_1 + c_2 =: c_5.$$

Using a similar argument to the proof of Theorem 2.3, we find that there exists a constant $d_6 > 0$ independent of λ such that $|x'|_\infty \le d_6$. Using Lemma 2.1 again, we can immediately conclude that (1.2) has a T-periodic solution. ∎

3. Example. To illustrate the applications of the above theorems,we give an example. Consider the system ($T = 2\pi$)

$$\begin{pmatrix}\phi_p(x_1')\\ \phi_p(x_2')\end{pmatrix}' + \begin{pmatrix}\beta(t)x_1^2 & 0\\ 0 & \beta(t)x_2^4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}' - \begin{pmatrix}(\sin t)^2 x_1^3 x_2^2\\ (\cos t)^2 x_1^4 x_2^5\end{pmatrix} = \begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

where

$$\beta(t) = \begin{cases}\arctan(\tan t), & k\pi - \pi/2 < t < k\pi + \pi/2,\\ \pm\pi/2, & t = k\pi \pm \pi/2,\ k = \pm 1, \pm 2, \ldots.\end{cases}$$

It is easy to check that the above system satisfies the conditions of Theorem 2.3. So we know that it has a T-periodic solution.

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