ANNALES
POLONICI MATHEMATICI
109.2 (2013)

Asymptotics of solutions to the Dirichlet–Cauchy problem for parabolic equations in domains with edges

by Vu Trong Luong (Sonla), Nguyen Manh Hung (Hanoi) and Do Van Loi (Thanhhoa)

Abstract. This paper is concerned with the Dirichlet–Cauchy problem for second order parabolic equations in domains with edges. The asymptotic behaviour of the solution near the edge is studied.

1. Introduction. We are concerned with initial boundary value problems (IBVP) for parabolic equations or systems in non-smooth domains. Such problems in domains with conical points have been studied in [3, 4, 5]; we investigated the solvability and asymptotics of solutions in a neighbourhood of the conical point. Solonnikov [10] dealt with the Neumann problem in domains with edges for the classical heat equation. By using the Fourier transform to reduce the problem to an elliptic boundary value problem with parameter, he proved the unique solvability and obtained coercive estimates of the solution in a weighted Hölder norm. Frolova [2] extended the solvability results of [10] to the case of boundary conditions involving derivatives with respect to both space variables and time.

In the present paper, we consider the first initial boundary value problem for second order parabolic equations in domains with edges. We modify the approach suggested in [9, 3] to demonstrate the asymptotic representation of the generalized solution of the problem in a neighbourhood of the edge.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial\Omega$ consisting of two surfaces Γ_1, Γ_2 which intersect along a manifold l_0. Assume that in a neighbourhood of each point of l_0 the set $\overline{\Omega}$ is diffeomorphic to a dihedral angle. For any $P \in l_0$, two half-spaces $T_1(P)$ and $T_2(P)$ tangent to Ω, and a two-dimensional plane $\pi(P)$ normal to l_0, are defined. We denote by $\nu(P)$ the angle in the plane $\pi(P)$ (on the side of Ω) bounded by the rays $R_1 = T_1(P) \cap \pi(P)$ and $R_2 = T_2(P) \cap \pi(P)$, and by $\beta(P)$ the aperture

2010 *Mathematics Subject Classification*: 35K20, 35D30, 35B65, 35C20.
Key words and phrases: asymptotics, generalized solution, regularity.

DOI: 10.4064/ap109-2-2

of this angle. Set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ for each $T, 0 < T \leq \infty$. For each multi-index $p = (p_1, \dots, p_n) \in \mathbb{N}^n$, $|p| = p_1 + \dots + p_n$, the symbol $D^p u = \partial^{|p|} u / \partial x_1^{p_1} \cdots \partial x_n^{p_n}$ denotes the generalized derivative of order $|p|$ with respect to $x = (x_1, \dots, x_n)$; $u_{t^k} = \partial^k u / \partial t^k$ is the generalized derivative of order k with respect to t.

We denote by $H^l(\Omega), \overset{\circ}{H}{}^l(\Omega)$ the usual Sobolev spaces as in [1]. We denote by $H^l_\alpha(\Omega)$ $(\alpha \in \mathbb{R})$ the weighted Sobolev space of all functions u defined on Ω with the norm

$$\|u\|^2_{H^l_\alpha(\Omega)} = \sum_{0\leq|p|\leq l} \int_\Omega (r^{2(\alpha+|p|-l)} |D^p u|^2 + |u|^2)\, dx,$$

where $r^2 = x_1^2 + x_2^2$.

By $H^{l,k}(Q_T, \gamma), H^{l,k}_\alpha(Q_T, \gamma)$ $(\gamma \in \mathbb{R})$ we denote the weighted Sobolev spaces of functions u defined on Q_T with the norms

$$\|u\|^2_{H^{l,k}(Q_T,\gamma)} = \int_{Q_T} \Big(\sum_{0\leq|p|\leq l} |D^p u|^2 + \sum_{j=1}^k |u_{t^j}|^2 \Big) e^{-\gamma t}\, dx\, dt$$

and

$$\|u\|^2_{H^{l,k}_\alpha(Q_T,\gamma)} = \int_{Q_T} \Big(\sum_{0\leq|p|\leq l} r^{2(\alpha+|p|-l)} |D^p u|^2 + \sum_{j=0}^k |u_{t^j}|^2 \Big) e^{-\gamma t}\, dx\, dt.$$

The space $\overset{\circ}{H}{}^{l,k}(Q_T, \gamma)$ is the closure in $H^{l,k}(Q_T, \gamma)$ of the set of all infinitely differentiable functions on Q_T which vanish near S_T.

Denote by $L_2(Q_T, \gamma), H^l_\alpha(Q_T, \gamma)$ the spaces of functions $u(x,t)$ defined on Q_T with the norms

$$\|u\|^2_{L_2(Q_T,\gamma)} = \int_{Q_T} |u|^2 e^{-\gamma t} dx dt,$$

$$\|u\|^2_{H^l_\alpha(Q_T,\gamma)} = \sum_{0\leq|p|+k\leq l} \int_{Q_T} (r^{2(\alpha+|p|+k-l)} |D^p u_{t^k}|^2 + |u|^2) e^{-\gamma t}\, dx\, dt.$$

Notice that if $T < \infty$, then we can omit the weight γ.

Let

$$L(x,t,\partial)u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t) \frac{\partial u}{\partial x_j} \Big) + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u$$

be a second order partial differential operator, where $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ are real-valued functions on Q_T belonging to $C^\infty(\overline{Q_T})$. Moreover, suppose that $a_{ij} = a_{ji}$, $i,j = 1, \dots, n$, are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0, T)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0|\xi|^2 \tag{1.1}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $(x,t) \in Q_T$, where $\mu_0 = \text{const} > 0$. We consider the problem

$$u_t + L(x,t,\partial)u = f \quad \text{in } Q_T, \tag{1.2}$$

with the initial condition

$$u|_{t=0} = 0 \quad \text{on } \Omega, \tag{1.3}$$

and the boundary condition

$$u|_{S_T} = 0. \tag{1.4}$$

Let us denote

$$B(u,v;t) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx + \sum_{i=1}^{n}\int_{\Omega} b_i(x,t)\frac{\partial u}{\partial x_i}v\,dx + \int_{\Omega} c(x,t)uv\,dx,$$

a time-dependent bilinear form. Applying condition (1.1) and similar arguments to the proof of Gårding's inequality it follows that

$$B(u,u;t) \geq \mu_0\|u\|^2_{H^1(\Omega)} - \lambda_0\|u\|^2_{L_2(\Omega)}, \quad \text{a.e. } t \in [0,T), \tag{1.5}$$

for all $u \in \mathring{H}^{1,1}(Q_T,\gamma)$, where $\mu_0 = \text{const} > 0$ and $\lambda_0 = \text{const} \geq 0$. Without loss of generality, we shall deal explicitly with the case when $\lambda_0 = 0$, since by the substitution $v = e^{\lambda_0 t}u$, problem (1.2)–(1.4) can be transformed to a problem with $\lambda_0 = 0$.

We denote by $(\cdot,\cdot)$ the inner product in $L_2(\Omega)$. A function $u(x,t)$ is called a *generalized solution* in $\mathring{H}^{1,1}(Q_T,\gamma)$ of problem (1.2)–(1.4) if $u(x,t) \in \mathring{H}^{1,1}(Q_T,\gamma)$, $u(x,0) = 0$, and the equality

$$(u_t, v) + B(u,v;t) = (f,v), \quad \text{a.e. } t \in [0,T), \tag{1.6}$$

holds for all $v \in \mathring{H}^1(\Omega)$.

2. Preliminaries. In this section, we will present some results on the well-posedness of the problem in weighted Sobolev spaces and the regularity in the time variable.

THEOREM 2.1. *Let $f \in H^0_\alpha(Q_T,\gamma_0)$, $\gamma_0 > 0$, $\alpha \in [0,1]$, and suppose the coefficients of the operator L satisfy*

$$\sup\{|a_{ij}|, |a_{ijt}|, |b_i|, |c| : i,j = 1,\dots,n;\ (x,t) \in \overline{Q}_T\} \leq \mu.$$

Then for each $\gamma \geq \gamma_0$, problem (1.2)–(1.4) has a unique generalized solution u in $\mathring{H}^{1,1}(Q_T,\gamma)$, and the following estimate holds:

$$\|u\|^2_{H^{1,1}(Q_T,\gamma)} \leq C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)} \tag{2.1}$$

where C is a constant independent of u and f. This solution depends continuously on f.

Proof. Firstly, we will prove the existence by Galerkin's approximation method. Let $\{\omega_k\}_{k=1}^{\infty}$ be an orthogonal basis of $\overset{\circ}{H}{}^1(\Omega)$ which is orthonormal in $L_2(\Omega)$. Put

$$u^N(x,t) = \sum_{k=1}^{N} C_k^N(t)\omega_k(x)$$

where $C_k^N(t)$, $t \in [0,T)$, $k = 1,\dots,N$, is the solution of the following system of ordinary differential equations:

$$(u_t^N, \omega_k) + B(u^N, \omega_k; t) = (f, \omega_k), \quad t \in [0,T),\ k = 1,\dots,N, \tag{2.2}$$

with the initial conditions

$$C_k^N(0) = 0, \quad k = 1,\dots,N. \tag{2.3}$$

Multiplying (2.2) by $C_k^N(t)$, then summing over k from 1 to N, we arrive at

$$(u_t^N, u^N) + B(u^N, u^N; t) = (f, u^N), \quad t \in [0,T).$$

This can be rewritten in the form

$$\frac{d}{dt}(\|u^N\|^2_{L_2(\Omega)}) + 2B(u^N, u^N; t) = 2(f, u^N). \tag{2.4}$$

By the Cauchy inequality and the Hardy inequality, for all $\alpha \in [0,1]$ we have

$$\begin{aligned} |(f,u^N)| &\le \|r^\alpha f\|_{L_2(\Omega)}\|r^{-\alpha}u^N\|_{L_2(\Omega)} \le C\|f\|_{H^0_\alpha(\Omega)}\|r^{-1}u^N\|_{L_2(\Omega)} \\ &\le C\|f\|_{H^0_\alpha(\Omega)}\|u^N\|_{H^1(\Omega)} \le C(\varepsilon)\|f\|^2_{H^0_\alpha(\Omega)} + \varepsilon\|u^N\|^2_{H^1(\Omega)} \end{aligned} \tag{2.5}$$

for any small ε, where $C = C(\varepsilon)$ is a constant independent of N, f, t. Combining the estimate above and (1.5), we deduce from (2.4) that

$$\frac{d}{dt}(\|u^N(\cdot,t)\|^2_{L_2(\Omega)}) + 2(\mu_0 - \varepsilon)\|u^N(\cdot,t)\|^2_{H^1(\Omega)} \le C\|f(\cdot,t)\|^2_{H^0_\alpha(\Omega)} \tag{2.6}$$

for a.e. $t \in [0,T)$. Multiplying (2.6) by $e^{-\gamma t}$, then integrating with respect to t from 0 to τ, $\tau \in (0,T)$, we obtain

$$\int_0^\tau e^{-\gamma t}\Big(\frac{d}{dt}\|u^N\|^2_{L_2(\Omega)}\Big)\,dt + 2(\mu_0 - \varepsilon)\int_0^\tau e^{-\gamma t}\|u^N\|^2_{H^1(\Omega)}\,dt \le C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)}.$$

Notice that

$$\begin{aligned} \int_0^\tau e^{-\gamma t}\Big(\frac{d}{dt}\|u^N\|^2_{L_2(\Omega)}\Big)\,dt &= \int_0^\tau \frac{d}{dt}(e^{-\gamma t}\|u^N\|^2_{L_2(\Omega)})\,dt + \gamma\int_0^\tau e^{-\gamma t}\|u^N\|^2_{L_2(\Omega)}\,dt \\ &= e^{-\gamma\tau}\|u^N(x,\tau)\|^2_{L_2(\Omega)} + \gamma\int_0^\tau e^{-\gamma t}\|u^N\|^2_{L_2(\Omega)}\,dt \ge 0. \end{aligned}$$

The inequalities above yield

$$\int_0^\tau e^{-\gamma t}\|u^N\|^2_{H^1(\Omega)}dt \le C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)}, \qquad \forall \tau\in(0,T). \tag{2.7}$$

Since the right-hand side of (2.7) is independent of τ, we get

$$\|u^N\|^2_{H^{1,0}(Q_T,\gamma)} \le C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)} \tag{2.8}$$

where C is a constant independent of u, f and N.

Multiplying (2.2) by $e^{-\gamma t}dC_k^N/dt$, then summing over k from 1 to N, we obtain

$$e^{-\gamma t}(u_t^N,u_t^N) + e^{-\gamma t}B(u^N,u_t^N;t) = e^{-\gamma t}(f,u_t^N) \tag{2.9}$$

for a.e. $0\le t<T$. To simplify notation, write $u_{x_i}=\partial u/\partial x_i$; then

$$\begin{aligned} e^{-\gamma t}B(u^N,u_t^N;t) &= \int_\Omega \sum_{i,j=1}^n e^{-\gamma t}a_{ij}u^N_{x_j}u^N_{x_it}\,dx \\ &\quad + \Big(\int_\Omega\sum_{i=1}^n b_iu^N_{x_i}u^N_t\,dx + \int_\Omega cu^Nu^N_t\,dx\Big)e^{-\gamma t} \\ &=: I + IIe^{-\gamma t}. \end{aligned} \tag{2.10}$$

It is easily seen that

$$I = \frac12\int_\Omega\sum_{i,j=1}^n \frac{\partial}{\partial t}[e^{-\gamma t}a_{ij}u^N_{x_j}u^N_{x_i}]\,dx - \frac12\int_\Omega\sum_{i,j=1}^n \frac{\partial e^{-\gamma t}a_{ij}}{\partial t}u^N_{x_j}u^N_{x_i}\,dx. \tag{2.11}$$

Furthermore,

$$|II| \le C(\epsilon)\|u^N\|^2_{H^1(\Omega)} + \epsilon\|u^N_t\|^2_{L_2(\Omega)},$$

$$|(f,u^N_t)| \le C(\epsilon)\|f\|^2_{L_2(\Omega)} + \epsilon\|u^N_t\|^2_{L_2(\Omega)}.$$

Combining the above inequalities and (2.9)–(2.11), we deduce

$$\begin{aligned} e^{-\gamma t}\|u^N_t\|^2_{L_2(\Omega)} &+ \frac12\int_\Omega\sum_{i,j=1}^n\frac{\partial}{\partial t}[e^{-\gamma t}a_{ij}u^N_{x_j}u^N_{x_i}]\,dx \\ &\le C(\epsilon)e^{-\gamma t}[\|u^N\|^2_{H^1(\Omega)} + \|f\|^2_{L_2(\Omega)}] + 2\epsilon e^{-\gamma t}\|u^N_t\|^2_{L_2(\Omega)} \\ &\quad + \frac12\int_\Omega\sum_{i,j=1}^n\frac{\partial e^{-\gamma t}a_{ij}}{\partial t}u^N_{x_j}u^N_{x_i}\,dx. \end{aligned}$$

Since $a_{ij}, \partial a_{ij}/\partial t, e^{-\gamma t}$ are bounded, using Cauchy's inequality, we get

$$(2.12)\qquad e^{-\gamma t}\|u_t^N\|^2_{L_2(\Omega)} + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial t}[e^{-\gamma t}a_{ij}u^N_{x_j}u^N_{x_i}]\,dx \le C_1(\epsilon)e^{-\gamma t}[\|u^N\|^2_{H^1(\Omega)} + \|f\|^2_{L_2(\Omega)}] + 2\epsilon e^{-\gamma t}\|u_t^N\|^2_{L_2(\Omega)}.$$

Choosing $\epsilon = 1/4$, then integrating (2.12) with respect to t from 0 to τ $(0 < \tau < T)$, we find

$$(2.13)\qquad \|u_t^N\|^2_{L_2(Q_\tau,\gamma)} + \int_\Omega \sum_{i,j=1}^n \int_0^\tau \frac{\partial}{\partial t}[e^{-\gamma t}a_{ij}u^N_{x_j}u^N_{x_i}]\,d\tau\,dx \le C[\|u^N\|^2_{H^{1,0}(Q_\tau,\gamma)} + \|f\|^2_{L_2(Q_\tau,\gamma)}].$$

By a simple calculation using (1.1), we obtain

$$\|u_t^N\|^2_{L_2(Q_\tau,\gamma)} \le C[\|u^N\|^2_{H^{1,0}(Q_\tau,\gamma)} + \|f\|^2_{L_2(Q_\tau,\gamma)}].$$

Letting $\tau \to T$ and using (2.8) we find that

$$(2.14)\qquad \|u_t^N\|^2_{L_2(Q_T,\gamma)} \le C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)}.$$

It follows readily from (2.8) and (2.14) that

$$(2.15)\qquad \|u^N\|^2_{H^{1,1}(Q_T,\gamma)} \le C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)},$$

where C is a constant independent of u, f and N.

According to (2.15), by standard weak convergence arguments, the sequence $\{u^N\}_{N=1}^\infty$ has a subsequence convergent to a function $u \in \mathring{H}^{1,1}(Q_T,\gamma)$, which is a generalized solution of problem (1.2)–(1.4). Moreover, it follows from (2.15) that inequality (2.1) holds.

Finally, we will prove the uniqueness of the generalized solution. It suffices to check that the only generalized solution of problem (1.2)–(1.4) with $f \equiv 0$ is $u \equiv 0$. By setting $v = u(\cdot,t)$ in (1.6) (for $f \equiv 0$), we get

$$\frac{d}{dt}(\|u(\cdot,t)\|^2) + 2B(u,u;t) = 0.$$

By (1.5), we have

$$\frac{d}{dt}(\|u\|^2_{L_2(\Omega)}) + 2\mu_0\|u\|^2_{H^1(\Omega)} \le 0 \quad \text{for a.e. } t \in [0,T).$$

Since $u|_{t=0} = 0$, it follows that $u = 0$ on Q_T. By (2.15), we also see that the solution u depends continuously on f. ■

By the same arguments used in the proof of Theorem 2.1 together with inductive arguments (cf. [3]), we obtain the following theorem:

THEOREM 2.2. *Let $h \in \mathbb{N}^*$, and assume that*

(i) $\sup\{|a_{ijt^{k+1}}|, |b_{it^k}|, |c_{t^k}| : i,j = 1,\dots,n;\ (x,t) \in \overline{Q}_T,\ k \le h\} \le \mu$,

(ii) $f_{t^k} \in H^0_\alpha(Q_T,\gamma_0)$ *for all* $k \le h$; $f_{t^k}(x,0) = 0$ *for all* $0 \le k \le h-1$.

Then for each $\gamma \geq \gamma_0$*, the generalized solution* $u \in \mathring{H}^{1,1}(Q_T, \gamma)$ *of problem* (1.2)–(1.4) *has derivatives with respect to* t *up to order* h*, and*

$$\|u_{t^h}\|^2_{H^{1,1}(Q_T,\gamma)} \leq C \sum_{j=0}^{h} \|f_{t^j}\|^2_{H^0_\alpha(Q_T,\gamma_0)} \tag{2.16}$$

where C *is a constant independent of* u *and* f*.*

3. Regularity of the generalized solution. We reduce the operator with coefficients at $P \in l_0$, $t \in [0, T)$,

$$L_0^{(2)} := -\sum_{i,j=1}^{2} a_{ij}(P,t) \frac{\partial^2}{\partial x_i \partial x_j},$$

to its canonical form. After a linear transformation of coordinates that realizes this reduction, T_1 and T_2 go over into hyperplanes T_1' and T_2', respectively, the angle between which is denoted by $\omega(P,t)$. It is easy to see that $\omega(P,t)$ does not depend on the method by which $L_0^{(2)}$ is reduced to its canonical form. The function $\omega(P,t)$ is infinitely differentiable, and $\omega(P,t) > 0$.

Theorem 3.1. *Let the assumptions of Theorem* 2.2 *be satisfied for a given positive integer* h*. Furthermore, let* $\alpha \in [0,1]$, $1 - \alpha < \pi/\omega$*. Then the generalized solution* $u \in \mathring{H}^{1,1}(Q_T, \gamma)$ *of problem* (1.2)–(1.4) *has derivatives with respect to* t *up to order* h, $u_{t^h} \in H^{2,0}_\alpha(Q_T, \gamma)$ *and*

$$\|u_{t^h}\|^2_{H^{2,0}_\alpha(Q_T,\gamma)} \leq C \sum_{k=0}^{h} \|f_{t^k}\|^2_{H^0_\alpha(Q_T,\gamma_0)},$$

where C *is a constant independent of* u, f*.*

Proof. We use induction on h. Firstly, we consider the case $h = 0$. It is easy to see that $u(\cdot, t_0)$, $t_0 \in (0, T)$, is the generalized solution of the problem

$$L(x, t_0, \partial)u = F(x, t_0) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0,$$

where $F(x,t_0) = f(x,t_0) - u_t(x,t_0) \in H^0_\alpha(\Omega)$. From [8, Thm. 2], we get $u(\cdot, t_0) \in H^2_\alpha(\Omega)$ and

$$\begin{aligned} \|u(\cdot,t_0)\|^2_{H^2_\alpha(\Omega)} &\leq C[\|F(\cdot,t_0)\|^2_{H^0_\alpha(\Omega)} + \|u\|^2_{L_2(\Omega)}] \\ &\leq C[\|f\|_{H^0_\alpha(\Omega)} + \|u_t\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)}]. \end{aligned} \tag{3.1}$$

Multiplying the above inequality with $e^{-t_0\gamma}$, then integrating with respect to t_0 from 0 to T and using the estimates from Theorem 2.2, we obtain

$$\|u\|^2_{H^{2,0}_\alpha(Q_T,\gamma)} \leq C\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)}.$$

Thus, the assertion is valid for $h = 0$.

Next, suppose that the assertion is true for $h-1$; we will prove it for $k=h$. Differentiating (1.2) h times with respect to t, we find

$$L_{u_{t^h}} = f_{t^h} - u_{t^{h+1}} - \sum_{k=0}^{h-1} \binom{k}{h} L_{t^{h-k}} u_{t^k} =: F. \tag{3.2}$$

By the assumptions of the theorem and the induction assumption, this implies that $f_{t^h} \in H^0_\alpha(\Omega)$, $u_{t^{h+1}} \in L_2(\Omega) \subset H^0_\alpha(\Omega)$, $\alpha \in [0,1]$, and $u_{t^k} \in H^0_\alpha(\Omega)$, $k \le h-1$. Therefore, $F(\cdot, t_0) \in H^0_\alpha(\Omega)$ for a.e. $t_0 \in (0,T)$. By using again [8, Thm. 2], we get $u_{t^h} \in H^2_\alpha(\Omega)$ for a.e. $t_0 \in (0,T)$ and

$$\begin{aligned} \|u_{t^h}\|^2_{H^2_\alpha(\Omega)} &\le C\|F\|^2_{H^0_\alpha(\Omega)} \\ &\le C\Big[\|f_{t^h}\|^2_{H^0_\alpha(\Omega)} + \|u_{t^{h+1}}\|^2_{L_2(\Omega)} + \sum_{k=0}^{h-1} \|u_{t^k}\|^2_{L_2(\Omega)}\Big]. \end{aligned} \tag{3.3}$$

Multiplying (3.3) with $e^{-t_0\gamma}$, then integrating with respect to t_0 from 0 to T and using again the estimates from Theorem 2.2, we obtain

$$\|u_{t^h}\|^2_{H^{2,0}_\alpha(Q_T,\gamma)} \le C \sum_{k=0}^{h} \|f_{t^k}\|^2_{H^0_\alpha(Q_T,\gamma_0)}.$$

This means that the assertion of the theorem is valid for $k=h$. ∎

THEOREM 3.2. *Assume that* $f, f_t \in H^h_\alpha(Q_T,\gamma_0)$, $f_{t^k}(x,0) = 0$ *for all* $k \le h-1$, *and*

$$h + 1 - \alpha < \pi/\omega, \qquad \alpha \in [0,1].$$

Then the generalized solution u *of problem* (1.2)–(1.4) *is in* $H^{2+h}_\alpha(Q_T,\gamma)$. *Moreover,*

$$\|u_{t^h}\|^2_{H^{2+h}_\alpha(Q_T,\gamma)} \le C(\|f\|^2_{H^h_\alpha(Q_T,\gamma_0)} + \|f_t\|^2_{H^h_\alpha(Q_T,\gamma_0)}), \tag{3.4}$$

where C *is a constant independent of* u, f.

Proof. We have

$$\begin{aligned} \|u\|^2_{H^2_\alpha(Q_T,\gamma)} &= \sum_{|p|+k\le 2} \int_{Q_T} (r^{2(\alpha+|p|+k-2)}|D^p u_{t^k}|^2 + |u|^2)e^{-\gamma t}\,dx\,dt \\ &= \sum_{|p|\le 2} \int_{Q_T} (r^{2(\alpha+|p|-2)}|D^p u|^2 + |u|^2)e^{-\gamma t}\,dx\,dt \\ &\quad + \sum_{|p|\le 1} \int_{Q_T} (r^{2(\alpha+|p|-1)}|D^p u_t|^2)e^{-\gamma t}\,dx\,dt + \int_{Q_T} r^{2\alpha}|u_{tt}|^2 e^{-\gamma t}\,dx\,dt \\ &= \|u\|^2_{H^{2,0}_\alpha(Q_T,\gamma)} + \|u_t\|^2_{H^{1,0}_\alpha(Q_T,\gamma)} + \|u_{tt}\|^2_{H^0_\alpha(Q_T,\gamma)} \\ &= \sum_{k=0}^{2} \|u_{t^k}\|^2_{H^{2-k,0}_\alpha(Q_T,\gamma)}. \end{aligned}$$

Therefore, $u \in H^2_\alpha(Q_T, \gamma)$ by Theorem 3.1. Moreover, we have

$$\|u\|^2_{H^2_\alpha(Q_T,\gamma)} = \sum_{k=0}^{2} \|u_{t^k}\|^2_{H^{2-k,0}_\alpha(Q_T,\gamma)} \le C(\|f\|^2_{H^0_\alpha(Q_T,\gamma_0)} + \|f_t\|^2_{H^0_\alpha(Q_T,\gamma_0)}).$$

Thus, the assertion is valid for $h = 0$. Suppose it is true for $h-1$. It is easy to see that

$$\|u\|^2_{H^{2+h}_\alpha(Q_T,\gamma)} = \sum_{k=0}^{h+2} \|u_{t^k}\|^2_{H^{h+2-k,0}_\alpha(Q_T,\gamma)}. \tag{3.5}$$

We will prove that

$$u_{t^k} \in H^{h+2-k,0}_\alpha(Q_T, \gamma), \quad k = 0, \dots, h, \tag{3.6}$$

and

$$\|u_{t^k}\|^2_{H^{h+2-k,0}_\alpha(Q_T,\gamma)} \le C \sum_{s=0}^{k} \|f_{t^s}\|^2_{H^{h-k,0}_\alpha(Q_T,\gamma_0)}, \quad k \le h. \tag{3.7}$$

By using Theorem 3.1, this holds for $k = h$. Suppose that it holds for $k = h, h-1, \dots, j+1$; we will prove it for $k = j$. Returning once more to (3.2) ($h = j$), we get

$$Lu_{t_j} = f_{t_j} - u_{t^{j+1}} - \sum_{k=0}^{j-1} \binom{j}{k} Lu_{t^{j-k}} u_{t^k} =: F_1.$$

Notice that $f_{t^j} \in H^h_\alpha(\Omega) \subset H^{h-j}_\alpha(\Omega)$ for a.e. $t \in (0,T)$ (by the assumptions of the theorem), $u_{t^{j+1}} \in H^{h-j+1}_\alpha(\Omega) \subset H^{h-j}_\alpha(\Omega)$ for a.e. $t \in (0,T)$ (by (3.6) which holds for $k = j+1$), $u_{t^k} \in H^{h+1-k}_\alpha(\Omega) \subset H^{h-j}_\alpha(\Omega)$, $k = 0, \dots, j-1$ (by the induction assumption for $k = h-1$).

This implies that $F_1(\cdot, t) \in H^{h-j}_\alpha(\Omega)$ for a.e. $t \in (0,T)$. From [8, Thm. 2], we obtain

$$u_{t^j} \in H^{h+2-j}_\alpha(\Omega) \quad \text{for a.e. } t \in (0,T)$$

and

$$\begin{aligned} \|u_{t^j}\|^2_{H^{h+2-j}_\alpha(\Omega)} &\le C\|F_1\|_{H^{h-j}_\alpha(\Omega)} \\ &\le C\Big[\|f_{t^j}\|^2_{H^{h-j}_\alpha(\Omega)} + \|u_{t^{j+1}}\|^2_{H^{h-j}_\alpha(\Omega)} + \sum_{k=0}^{j-1} \|u_{t^k}\|^2_{H^{h-j}_\alpha(\Omega)}\Big]. \end{aligned} \tag{3.8}$$

Multiplying (3.8) with $e^{-\gamma t}$, then integrating with respect to t from 0 to T, we arrive at

$$\|u_{t^j}\|^2_{H^{h+2-j,0}_\alpha(Q_T,\gamma)} \le C \sum_{k=0}^{j} \|f_{t^k}\|^2_{H^{h-j,0}_\alpha(Q_T,\gamma_0)}.$$

This means that (3.6) and (3.7) are true for $k = j$, so they hold for all $k = 0, 1, \dots, h$. From (3.5), we get

$$\|u\|^2_{H^{h+2}_\alpha(Q_T,\gamma)} \le C \sum_{k=0}^{h+1} \|f_{t^k}\|^2_{H^{h-k,0}_\alpha(Q_T,\gamma_0)} = C(\|f\|^2_{H^h_\alpha(Q_T,\gamma_0)} + \|f_t\|^2_{H^h_\alpha(Q_T,\gamma_0)}).$$

The proof is complete. ∎

4. Asymptotics of the solution in a neighbourhood of the edge. In the previous section, we have seen that if $k+1-\alpha < \pi/\omega$, $\alpha \in [0,1]$ and $f, f_t \in H^k_\alpha(Q_T,\gamma_0)$, then the solution u is in $H^{2+k}_\alpha(Q_T,\gamma_0)$. Now we study the solution in the case $\pi/\omega < k+1-\alpha$. In this case we can obtain for u an asymptotic representation in a neighbourhood of $l_0 : x_1 = x_2 = 0$. To start, we denote $y_1 = x_1$, $y_2 = x_2$, $y = (y_1, y_2)$, $z_i = x_{i+2}$, $z = (z_1, \dots, z_{n-2})$, $r = x_1^2 + x_2^2$; (r, φ) are the polar coordinates of $y = (y_1, y_2) \in \Omega_z = \Omega \cap \{z = \text{const}\}$. Set $Q_{z,T} = \Omega_z \times (0,T)$.

LEMMA 4.1. *Suppose that the following hypotheses are satisfied:*

(i) $f_{t^s} \in H^{k,0}_\alpha(Q_T,\gamma_0)$ *for all* $s \le h$; $f_{t^s}(x,0) = 0$ *for all* $s \le h-1$.
(ii) $k - \alpha < \pi/\omega < k+1-\alpha < 2\pi/\omega$, $\alpha \in [0,1]$.

Let u be the solution of (1.2)–(1.4) *with $u \equiv 0$ outside some neighbourhood of l_0. Then*

$$u(y,z,t) = c(z,t) r^{\pi/\omega} \Phi(z,\varphi,t) + u_1(y,z,t)$$

where $c_{t^s} \in L_2(Q_T,\gamma)$, $\Phi \in C^\infty$ and $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q_{z,T},\gamma)$ for all $s \le h$.

Proof. Using (i), we deduce from Theorem 3.2 that $u_{t^s} \in H^{k+1,0}_\alpha(Q_T,\gamma)$ for all $s \le h$, in particular, $u_z \in H^k_\alpha(\Omega)$ and $u_{tz} \in H^k_\alpha(\Omega)$ for a.e. $t \in (0,T)$. On the other hand, we have

$$Lu_z = f_z - u_{tz} - L_z u =: f_1$$

where

$$L_z = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ijz}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^n b_{iz}\frac{\partial}{\partial x_i} + c_z$$

and $f_1 \in H^{k-1}_\alpha(\Omega)$ for a.e. $t \in (0,T)$. Using Theorem 3.1, we obtain $u_z \in H^{k+1}_\alpha(\Omega)$ for a.e. $t \in (0,T)$. Therefore, equality (1.2) can be rewritten in the form

$$L_0^{(2)} u = F \tag{4.1}$$

where $F \in H^k_\alpha(\Omega)$ for a.e. $t \in (0,T)$. Now we can apply Theorem 1′ of [9] to get

$$u(y,z,t) = c(z,t) r^{\pi/\omega} \Phi(z,\varphi,t) + u_1(y,z,t) \tag{4.2}$$

where $\Phi \in C^\infty$, $u_1 \in H^{k+2}_\alpha(\Omega_z)$ and

$$|c(z,t)|^2 \le C(\|F\|^2_{H^k_\alpha(\Omega_z)} + \|u\|^2_{L_2(\Omega_z)}),$$
$$\|u_1\|^2_{H^{k+2}_\alpha(\Omega_z)} \le C(\|F\|^2_{H^k_\alpha(\Omega_z)} + \|u\|^2_{L_2(\Omega_z)}), \qquad z \in l_0,\ t \in (0,T).$$

Hence, $c \in L_2(Q_T, \gamma)$, $u_1 \in H^{k+2,0}_\alpha(Q_{z,T}, \gamma)$. This implies that the conclusion holds for $h = 0$. Suppose it is true for $h-1$; we will prove it for $k = h$. Denoting $v = u_{t^h}$, and differentiating (4.1) h times with respect to t, we find

$$L_0^{(2)} v = F_{t^h} - \sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} u_{t^{h-j}}. \tag{4.3}$$

Setting $S_0 = r^{\pi/\omega}\Phi$, we have

$$\sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} u_{t^{h-j}} = \sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} (cS_0)_{t^{h-j}} + \sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} (u_1)_{t^{h-j}}. \tag{4.4}$$

The first term of the right-hand side of (4.4) can be rewritten in the following form:

$$\begin{aligned}
\sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} (cS_0)_{t^{h-j}} &= \sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} \Big(\sum_{i=0}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} S_{0t^i} \Big) \\
&= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=0}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L^{(2)}_{0t^j} S_{0t^i} \\
&= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L^{(2)}_{0t^j} S_{0t^i} \\
&\quad + \sum_{j=1}^{h} \binom{h}{j} c_{t^{h-j}} L^{(2)}_{0t^j} S_0 \\
&= \sum_{j=0}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L^{(2)}_{0t^j} S_{0t^i} \\
&\quad + \sum_{j=1}^{h} \binom{h}{j} c_{t^{h-j}} L^{(2)}_{0t^j} S_0 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L^{(2)}_0 S_{0t^i} \\
&= F_1 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L^{(2)}_0 S_{0t^i}.
\end{aligned}$$

From the assumptions of the lemma and the inductive assumptions, this implies that $F_1 \in H^k_\alpha(\Omega_z)$. Hence, from (4.4) we obtain

$$\sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} u_{t^{h-j}} = F_2 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L^{(2)}_0 S_{0t^i}$$

where $F_2 \in H^k_\alpha(\Omega_z)$. Employing the equality above, we infer from (4.3) that

$$L^{(2)}_0 v = F_3 + \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L^{(2)}_0 S_{0t^i}. \tag{4.5}$$

Thus,

$$L^{(2)}_0 \Big(v - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} \Big) = F_3$$

where $F_3 \in H^k_\alpha(\Omega_z)$. Analogously to the case $h = 0$, we get

$$v - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} = d(z,t) S_0 + u_2(y,z,t).$$

Therefore,

$$u_{t^h} = \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} + d(z,t) S_0 + u_2(y,z,t) \tag{4.6}$$

where $d \in L_2(Q_T,\gamma_0)$ and $u_2 \in H^{k+2,0}_\alpha(Q_{z,T},\gamma)$. By the assumption (i), this implies that u is differentiable with respect to t. Then, we can see that the functions $c(z,\cdot)$ and $u_1(\cdot,t)$ are differentiable with respect to t. Combining (4.2) and (4.6), we conclude that

$$c_{t^h} = d \in L_2(Q_T,\gamma), \quad (u_1)_{t^h} = u_2 \in H^{k+2,0}_\alpha(Q_{z,T},\gamma).$$

The proof is complete. ∎

Next, we have the following theorem.

THEOREM 4.2. *Suppose that the hypotheses of Lemma* 4.1 *are satisfied. Then the following representation holds:*

$$u(x,t) = c(x,t) r^{\pi/\omega} \Phi(z,\varphi,t) + u_1(x,t)$$

where $c_{t^s} \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_T,\gamma)$ *and* $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q_T,\gamma)$ *for all* $s \le h$.

Proof. From Lemma 4.1, we have the representation:

$$u(x,t) = c(z,t) r^{\pi/\omega} \Phi(z,t,\varphi) + u_1(x,t) \tag{4.7}$$

where $c_{t^s} \in L_2(Q_T,\gamma)$ and $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q_{z,T},\gamma)$ for all $s \le h$. Consider the differential operator

$$D_1 = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi};$$

in the coordinates x_1, x_2, it is

$$D_1 = \varPhi_1 \frac{\partial}{\partial x_1} + \varPhi_2 \frac{\partial}{\partial x_2}$$

where $\varPhi_1, \varPhi_2$ are infinitely differentiable. From representation (4.7), we find

$$D_1 u = \frac{\pi}{\omega} c(z,t) r^{\pi/\omega - 1} \varPhi_3(z,t,\varphi) + D_1 u_1(x,t). \tag{4.8}$$

Moreover,

$$u_1 \in H_\alpha^{k+2,0}(Q_{z,T}, \gamma), \qquad \int\limits_{Q_{z,T}} \left(r^{2\alpha} \frac{\partial^{k+2} u_1}{\partial x_1^{k_1} \partial x_2^{k_2}} \right) e^{-\gamma t}\, dx_1\, dx_2\, dt < \infty. \tag{4.9}$$

By arguments analogous to the proof of Lemma 4.1, we obtain

$$u_z, u \in H_\alpha^{k+1,0}(Q_T, \gamma).$$

Therefore,

$$\begin{aligned} &(D_1 u)_z \in H_\alpha^{k,0}(Q_T, \gamma), \\ &\int\limits_{Q_T} (r^{2(\alpha-k)} |(D_1 u)_z|) e^{-\gamma t}\, dx\, dt \le \int\limits_{Q_T} (r^{2\alpha} |f|^2) e^{-\gamma_0 t}\, dx\, dt < \infty. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$r^{-\pi/\omega+1} D_1 u_1 \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_{z,T}, \gamma), \qquad r^{-\pi/\omega+1} (D_1 u)_z \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_T, \gamma).$$

On the other hand, equality (4.8) yields

$$(r^{-\pi/\omega+1} D_1 u)_y = (r^{-\pi/\omega+1} D_1 u_1)_y.$$

Consequently,

$$r^{-\pi/\omega+1} D_1 u \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_T, \gamma). \tag{4.11}$$

Now write

$$c_1(x,t) = \frac{\omega}{\pi} r^{-\pi/\omega+1} D_1 u \varPhi_3.$$

Then (4.11) implies $c_1 \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_T, \gamma)$. From Lemma 2 in [9], we conclude that there is $\widetilde{c}_1 \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_T, \gamma)$ with $(\widetilde{c}_1)_{t^s} \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_T, \gamma)$ for all $s \le h$ such that

$$\int\limits_{Q_T} (|c_1 - \widetilde{c}_1|^2 r^{2(\alpha+\pi/\omega-k-2)}) e^{-\gamma t}\, dx\, dt < \infty. \tag{4.12}$$

Ultilizing (4.8) and the fact that $u_1 \in H_\alpha^{k+2,0}(Q_{z,T}, \gamma)$, we get

$$\int\limits_{Q_T} (|c - c_1|^2 r^{2(\alpha+\pi/\omega-k-2)}) e^{-\gamma t}\, dx\, dt < \infty. \tag{4.13}$$

We can rewrite representation (4.7) in the form

$$u(x,t) = \widetilde{c}_1(x,t)r^{\pi/\omega}\Phi(z,\varphi,t) + [c - \widetilde{c}_1]r^{\pi/\omega}\Phi(z,\varphi,t) \\ = \widetilde{c}_1(x,t)r^{\pi/\omega}\Phi(z,\varphi,t) + u_2(x,t) \tag{4.14}$$

where $u_2 \in H_\alpha^{k+2,0}(Q_{z,T},\gamma)$ for all $z \in l_0$. Since u is differentiable with respect to z and $u_2 = u - \widetilde{c}_1 r^{\pi/\omega}\Phi$, we see that $Lu_2 \in H_\alpha^k(\Omega)$ and

$$\int_\Omega r^{2(\alpha-k-2)}|u_2|\,dx < \infty.$$

By Lemma 2 in [7], we obtain $u_2 \in H_\alpha^{k+2}(\Omega)$. Hence, $u_2 \in H_\alpha^{k+2,0}(Q_T,\gamma)$.

To prove $(u_2)_{t^s} \in H_\alpha^{k+2,0}(Q_T,\gamma)$ for all $s \le h$, we can use arguments analogous to the proof of Lemma 4.1. ■

Acknowledgements. This work was supported by Vietnam's National Foundation for Science and Technology Development (NAFOSTED:101.01-2011.30).

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[2] E. V. Frolova, *An initial boundary value problem with a noncoercive boundary condition in domains with edges*, Zap. Nauchn. Sem. POMI 213 (1994), 206–223 (in Russian); English transl.: J. Math. Sci. 84 (1997), 948–959.

[3] N. M. Hung, *Boundary problems for nonstationary systems in domains with a nonsmooth boundary*, Doctoral dissertation, Mech. Math. Department MSU, Moscow, 1999.

[4] N. M. Hung and N. T. Anh, *Regularity of solutions of initial-boundary value problems for parabolic equations in domains with conical points*, J. Differential Equations 245 (2008), 1801–1818.

[5] N. M. Hung and P. T. Duong, *On the smoothness of generalized solution for parabolic systems in domains with conical points on the boundary*, Ukrainian Math. J. 56 (2004), 854–864.

[6] A. Yu. Kokotov and B. A. Plamenevskiĭ, *On the asymptotic of solutions to the Neumann problem for hyperbolic systems in domains with conical points*, St. Petersburg Math. J. 16 (2005), 477–506.

[7] V. A. Kondrat'ev, *Boundary-value problems for elliptic equations in domains with conic or corner points*, Trudy Moskov. Mat. Obshch. 16 (1967), 209–292 (in Russian); English transl.: Trans. Moscow Math. Soc. 1967, 227–313.

[8] V. A. Kondrat'ev, *On the smoothness of the solution of the Dirichlet problem for second order elliptic equations in a piecewise smooth domain*, Differentsial'nye Uravneniya 6 (1970), 1831–1843 (in Russian); English transl.: Differential Equations 6 (1970), 1392–1401.

[9] V. A. Kondrat'ev, *Singularities of the solution of the Dirichlet problem for a second order elliptic equation in a neighborhood of an edge*, Differentsial'nye Uravneniya 13 (1977), 2026–2032 (in Russian); English transl.: Differential Equations 13 (1977), 1411–1415.

[10] V. A. Solonnikov, *On the solvability of the classical initial-boundary value problem for the heat equation in a dihedral angle*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklova 138 (1984), 146–180 (in Russian); English transl.: J. Soviet Math. 32 (1986), 526–546.

Vu Trong Luong (corresponding author)
Department of Mathematics
Taybac University
Quyettam, Sonla City
Vietnam
E-mail: vutrongluong@gmail.com

Nguyen Manh Hung
Department of Mathematics
Hanoi National University of Education
136 Xuanthuy, Caugiay, Hanoi
Vietnam
E-mail: hungnmmath@hnue.edu.vn

Do Van Loi
Department of Mathematics
Hongduc University
Thanhhoa City, Thanhhoa
Vietnam
E-mail: 37loilinh@gmail.com

Received 4.10.2011
and in final form 5.9.2012 (2573)