Asymptotics of solutions to the Dirichlet–Cauchy problem for parabolic equations in domains with edges

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Abstract. This paper is concerned with the Dirichlet–Cauchy problem for second order parabolic equations in domains with edges. The asymptotic behaviour of the solution near the edge is studied.

1. Introduction. We are concerned with initial boundary value problems (IBVP) for parabolic equations or systems in non-smooth domains. Such problems in domains with conical points have been studied in [3, 4, 5]; we investigated the solvability and asymptotics of solutions in a neighbourhood of the conical point. Solonnikov [10] dealt with the Neumann problem in domains with edges for the classical heat equation. By using the Fourier transform to reduce the problem to an elliptic boundary value problem with parameter, he proved the unique solvability and obtained coercive estimates of the solution in a weighted Hölder norm. Frolova [2] extended the solvability results of [10] to the case of boundary conditions involving derivatives with respect to both space variables and time.

In the present paper, we consider the first initial boundary value problem for second order parabolic equations in domains with edges. We modify the approach suggested in [2, 3] to demonstrate the asymptotic representation of the generalized solution of the problem in a neighbourhood of the edge.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial \Omega$ consisting of two surfaces $\Gamma_1$, $\Gamma_2$ which intersect along a manifold $l_0$. Assume that in a neighbourhood of each point of $l_0$ the set $\overline{\Omega}$ is diffeomorphic to a dihedral angle. For any $P \in l_0$, two half-spaces $T_1(P)$ and $T_2(P)$ tangent to $\Omega$, and a two-dimensional plane $\pi(P)$ normal to $l_0$, are defined. We denote by $\nu(P)$ the angle in the plane $\pi(P)$ (on the side of $\Omega$) bounded by the rays $R_1 = T_1(P) \cap \pi(P)$ and $R_2 = T_2(P) \cap \pi(P)$, and by $\beta(P)$ the aperture

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of this angle. Set \( Q_T = \Omega \times (0, T) \), \( S_T = \partial \Omega \times (0, T) \) for each \( T, 0 < T \leq \infty \).

For each multi-index \( p = (p_1, \ldots, p_n) \in \mathbb{N}^n \), \( |p| = p_1 + \cdots + p_n \), the symbol \( D^p u = \partial^{|p|} u / \partial x_1^{p_1} \cdots \partial x_n^{p_n} \) denotes the generalized derivative of order \( |p| \) with respect to \( x = (x_1, \ldots, x_n) \); \( u_{t_k} = \partial^k u / \partial t^k \) is the generalized derivative of order \( k \) with respect to \( t \).

We denote by \( H^l(\Omega), \tilde{H}^l(\Omega) \) the usual Sobolev spaces as in [1]. We denote by \( H^l_\alpha(\Omega) (\alpha \in \mathbb{R}) \) the weighted Sobolev space of all functions \( u \) defined on \( \Omega \) with the norm

\[
\|u\|_{H^l_\alpha(\Omega)}^2 = \sum_{0 \leq |p| \leq l} \int_\Omega (r^{2(\alpha+|p|-l)}|D^p u|^2 + |u|^2) \, dx,
\]

where \( r^2 = x_1^2 + x_2^2 \).

By \( H^{l,k}(Q_T, \gamma), H^{l,k}_\alpha(Q_T, \gamma) (\gamma \in \mathbb{R}) \) we denote the weighted Sobolev spaces of functions \( u \) defined on \( Q_T \) with the norms

\[
\|u\|_{H^{l,k}(Q_T, \gamma)}^2 = \int_{Q_T} \left( \sum_{0 \leq |p| \leq l} |D^p u|^2 + \sum_{j=1}^k |u_{\gamma j}|^2 \right) e^{-\gamma t} \, dx \, dt
\]

and

\[
\|u\|_{H^{l,k}_\alpha(Q_T, \gamma)}^2 = \int_{Q_T} \left( \sum_{0 \leq |p| \leq l} r^{2(\alpha+|p|-l)}|D^p u|^2 + \sum_{j=0}^k |u_{\gamma j}|^2 \right) e^{-\gamma t} \, dx \, dt.
\]

The space \( \tilde{H}^{l,k}(Q_T, \gamma) \) is the closure in \( H^{l,k}(Q_T, \gamma) \) of the set of all infinitely differentiable functions on \( Q_T \) which vanish near \( S_T \).

Denote by \( L_2(Q_T, \gamma), H^l_\alpha(Q_T, \gamma) \) the spaces of functions \( u(x,t) \) defined on \( Q_T \) with the norms

\[
\|u\|_{L_2(Q_T, \gamma)}^2 = \int_{Q_T} |u|^2 e^{-\gamma t} \, dx dt,
\]

\[
\|u\|_{H^l_\alpha(Q_T, \gamma)}^2 = \sum_{0 \leq |p| + k \leq l} \int_{Q_T} (r^{2(\alpha+|p|+k-\ell)}|D^p u_{\gamma k}|^2 + |u|^2) e^{-\gamma t} \, dx dt.
\]

Notice that if \( T < \infty \), then we can omit the weight \( \gamma \).

Let \[
L(x, t, \partial) u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t) u
\]
be a second order partial differential operator, where \( a_{ij}(x, t), b_i(x, t) \) and \( c(x, t) \) are real-valued functions on \( Q_T \) belonging to \( C^\infty(\overline{Q}_T) \). Moreover, suppose that \( a_{ij} = a_{ji}, i, j = 1, \ldots, n \), are continuous in \( x \in \overline{\Omega} \) uniformly
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with respect to \( t \in [0, T) \) and

\[
(1.1) \quad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0|\xi|^2
\]

for all \( \xi \in \mathbb{R}^n \setminus \{0\} \) and \((x,t) \in Q_T\), where \( \mu_0 = \text{const} > 0 \). We consider the problem

\[
(1.2) \quad u_t + L(x,t,\partial)u = f \quad \text{in} \; Q_T,
\]

with the initial condition

\[
(1.3) \quad u|_{t=0} = 0 \quad \text{on} \; \Omega,
\]

and the boundary condition

\[
(1.4) \quad u|_{S_T} = 0.
\]

Let us denote

\[
B(u,v;t) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx + \sum_{i=1}^{n} \int_{\Omega} b_i(x,t) \frac{\partial u}{\partial x_i} v \, dx + \int_{\Omega} c(x,t)uv \, dx,
\]

a time-dependent bilinear form. Applying condition (1.1) and similar arguments to the proof of Gårding’s inequality it follows that

\[
(1.5) \quad B(u,u;t) \geq \mu_0\|u\|_{H^1(\Omega)}^2 - \lambda_0\|u\|^2_{L_2(\Omega)}, \quad \text{a.e. } t \in [0, T),
\]

for all \( u \in \dot{H}^{1,1}(Q_T, \gamma) \), where \( \mu_0 = \text{const} > 0 \) and \( \lambda_0 = \text{const} \geq 0 \). Without loss of generality, we shall deal explicitly with the case when \( \lambda_0 = 0 \), since by the substitution \( v = e^{\lambda_0 t}u \), problem (1.2)–(1.4) can be transformed to a problem with \( \lambda_0 = 0 \).

We denote by \((\cdot, \cdot)\) the inner product in \( L_2(\Omega) \). A function \( u(x,t) \) is called a generalized solution in \( \dot{H}^{1,1}(Q_T, \gamma) \) of problem (1.2)–(1.4) if \( u(x,t) \in \dot{H}^{1,1}(Q_T, \gamma) \), \( u(x,0) = 0 \), and the equality

\[
(1.6) \quad (u_t,v) + B(u,v;t) = (f,v), \quad \text{a.e. } t \in [0, T),
\]

holds for all \( v \in \dot{H}^1(\Omega) \).

2. Preliminaries. In this section, we will present some results on the well-posedness of the problem in weighted Sobolev spaces and the regularity in the time variable.

THEOREM 2.1. Let \( f \in H^0_\alpha(Q_T, \gamma_0) \), \( \gamma_0 > 0 \), \( \alpha \in [0, 1] \), and suppose the coefficients of the operator \( L \) satisfy

\[
\sup\{|a_{ij}|, |a_{ij}x|, |b_i|, |c| : i, j = 1, \ldots, n; (x,t) \in Q_T\} \leq \mu.
\]

Then for each \( \gamma \geq \gamma_0 \), problem (1.2)–(1.4) has a unique generalized solution \( u \) in \( \dot{H}^{1,1}(Q_T, \gamma) \), and the following estimate holds:

\[
(2.1) \quad \|u\|^2_{\dot{H}^{1,1}(Q_T, \gamma)} \leq C\|f\|^2_{H^0_\alpha(Q_T, \gamma_0)}
\]
where \( C \) is a constant independent of \( u \) and \( f \). This solution depends continuously on \( f \).

Proof. Firstly, we will prove the existence by Galerkin’s approximation method. Let \( \{ \omega_k \}_{k=1}^{\infty} \) be an orthogonal basis of \( \tilde{H}^1(\Omega) \) which is orthonormal in \( L_2(\Omega) \). Put

\[
u^N(x, t) = \sum_{k=1}^{N} C^N_k(t) \omega_k(x)
\]

where \( C^N_k(t), t \in [0, T), k = 1, \ldots, N, \) is the solution of the following system of ordinary differential equations:

\[
(2.2) \quad (u^N_t, \omega_k) + B(u^N, \omega_k; t) = (f, \omega_k), \quad t \in [0, T), \quad k = 1, \ldots, N,
\]

with the initial conditions

\[
(2.3) \quad C^N_k(0) = 0, \quad k = 1, \ldots, N.
\]

Multiplying \( (2.2) \) by \( C^N_k(t) \), then summing over \( k \) from 1 to \( N \), we arrive at

\[
(u^N_t, u^N) + B(u^N, u^N; t) = (f, u^N), \quad t \in [0, T).
\]

This can be rewritten in the form

\[
(2.4) \quad \frac{d}{dt} \left( \|u^N\|_{L_2(\Omega)}^2 \right) + 2B(u^N, u^N; t) = 2(f, u^N).
\]

By the Cauchy inequality and the Hardy inequality, for all \( \alpha \in [0, 1] \) we have

\[
(2.5) \quad |(f, u^N)| \leq \|r^\alpha f\|_{L_2(\Omega)} \|r^{-\alpha} u^N\|_{L_2(\Omega)} \leq C \|f\|_{H^0_0(\Omega)} \|r^{-1} u^N\|_{L_2(\Omega)} \\
\leq C \|f\|_{H^0_0(\Omega)} \|u^N\|_{H^1(\Omega)} \leq C(\varepsilon) \|f\|_{H^0_0(\Omega)}^2 + \varepsilon \|u^N\|_{H^1(\Omega)}^2
\]

for any small \( \varepsilon \), where \( C = C(\varepsilon) \) is a constant independent of \( N, f, t \). Combining the estimate above and \( (1.5) \), we deduce from \( (2.4) \) that

\[
(2.6) \quad \frac{d}{dt} \left( \|u^N(\cdot, t)\|_{L_2(\Omega)}^2 \right) + 2(\mu_0 - \varepsilon) \|u^N(\cdot, t)\|_{H^1(\Omega)}^2 \leq C \|f(\cdot, t)\|_{H^0_0(\Omega)}^2
\]

for a.e. \( t \in [0, T) \). Multiplying \( (2.6) \) by \( e^{-\gamma t} \), then integrating with respect to \( t \) from 0 to \( \tau \), \( \tau \in (0, T) \), we obtain

\[
\int_0^\tau e^{-\gamma t} \left( \frac{d}{dt} \|u^N\|_{L_2(\Omega)}^2 \right) dt + 2(\mu_0 - \varepsilon) \int_0^\tau e^{-\gamma t} \|u^N\|_{H^1(\Omega)}^2 dt \leq C \|f\|_{H^0_0(Q_T; \gamma_0)}^2
\]

for a.e. \( t \in [0, T) \).
The inequalities above yield

\begin{equation}
\int_0^\tau e^{-\gamma t} \|u^N\|_{H^1(\Omega)}^2 \, dt \leq C \|f\|_{H^0_0(Q,T;\gamma_0)}^2, \quad \forall \tau \in (0,T).
\end{equation}

Since the right-hand side of (2.7) is independent of \(\tau\), we get

\begin{equation}
\|u^N\|_{H^{1,0}(Q,T;\gamma)}^2 \leq C \|f\|_{H^0_0(Q,T;\gamma_0)}^2
\end{equation}

where \(C\) is a constant independent of \(u, f\) and \(N\).

Multiplying (2.2) by \(e^{-\gamma t} dC_k^N / dt\), then summing over \(k\) from 1 to \(N\), we obtain

\begin{equation}
e^{-\gamma t}(u^N_t, u^N_t) + e^{-\gamma t} B(u^N, u^N_t; t) = e^{-\gamma t}(f, u^N_t)
\end{equation}

for a.e. \(0 \leq t < T\). To simplify notation, write \(u_x = \partial u / \partial x_i\); then

\begin{equation}
e^{-\gamma t} B(u^N, u^N_t; t) = \int \sum_{i,j=1}^n e^{-\gamma t} a_{ij} u^N_{x_j} u^N_{x_i} \, dx
+ \left( \int \sum_{i=1}^n b_i u^N_{x_i} u^N_t \, dx + \int_{\Omega} cu^N u^N_t \, dx \right) e^{-\gamma t}
=: I + II e^{-\gamma t}.
\end{equation}

It is easily seen that

\begin{equation}
I = \frac{1}{2} \int \sum_{i,j=1}^n \frac{\partial}{\partial t} \left[ e^{-\gamma t} a_{ij} u^N_{x_j} u^N_{x_i} \right] \, dx - \frac{1}{2} \int \sum_{i,j=1}^n \frac{\partial}{\partial t} e^{-\gamma t} a_{ij} u^N_{x_j} u^N_{x_i} \, dx.
\end{equation}

Furthermore,

\begin{align*}
|II| & \leq C(\epsilon) \|u^N\|_{H^1(\Omega)}^2 + \epsilon \|u^N_t\|_{L^2(\Omega)}^2,

|\langle f, u^N_t \rangle| & \leq C(\epsilon) \|f\|_{L^2(\Omega)}^2 + \epsilon \|u^N_t\|_{L^2(\Omega)}^2.
\end{align*}

Combining the above inequalities and (2.9)–(2.11), we deduce

\begin{align*}
e^{-\gamma t} \|u^N_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \int \sum_{i,j=1}^n \frac{\partial}{\partial t} [e^{-\gamma t} a_{ij} u^N_{x_j} u^N_{x_i}] \, dx & 
\leq C(\epsilon) e^{-\gamma t} \|u^N\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + 2\epsilon e^{-\gamma t} \|u^N_t\|_{L^2(\Omega)}^2

+ \frac{1}{2} \int \sum_{i,j=1}^n \frac{\partial}{\partial t} e^{-\gamma t} a_{ij} u^N_{x_j} u^N_{x_i} \, dx.
\end{align*}
Since $a_{ij}, \partial a_{ij}/\partial t, e^{-\gamma t}$ are bounded, using Cauchy’s inequality, we get

$$e^{-\gamma t} \|u_t^N\|^2_{L_2(\Omega)} + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial t} [e^{-\gamma t} a_{ij} u_{x_j}^N u_{x_i}^N] dx \leq C_1(\epsilon) e^{-\gamma t} \|u^N\|^2_{H^1(\Omega)} + \|f\|_{L_2(\Omega)}^2 + 2\epsilon e^{-\gamma t} \|u_t^N\|^2_{L_2(\Omega)}.$$ 

Choosing $\epsilon = 1/4$, then integrating (2.12) with respect to $t$ from 0 to $\tau$ ($0 < \tau < T$), we find

$$\|u_t^N\|^2_{L_2(\Omega,\tau)} + \int_\Omega \sum_{i,j=1}^n \int_0^\tau \frac{\partial}{\partial t} [e^{-\gamma t} a_{ij} u_{x_j}^N u_{x_i}^N] d\tau dx \leq C[\|u^N\|^2_{H^1,0(\Omega,\tau)} + \|f\|_{L_2(\Omega,\tau)}^2].$$

By a simple calculation using (1.1), we obtain

$$\|u_t^N\|^2_{L_2(\Omega,\tau)} \leq C[\|u^N\|^2_{H^1,0(\Omega,\tau)} + \|f\|_{L_2(\Omega,\tau)}^2].$$

Letting $\tau \rightarrow T$ and using (2.8) we find that

$$\|u_t^N\|^2_{L_2(\Omega,T)} \leq C\|f\|^2_{H_0^0(\Omega,T)}. $$

It follows readily from (2.8) and (2.14) that

$$\|u^N\|^2_{H^{1.1}(\Omega,T)} \leq C\|f\|^2_{H_0^0(\Omega,T)},$$

where $C$ is a constant independent of $u, f$ and $N$.

According to (2.15), by standard weak convergence arguments, the sequence $\{u^N\}_{N=1}^\infty$ has a subsequence convergent to a function $u \in H^{1,1}(Q,T,\gamma)$, which is a generalized solution of problem (1.2)–(1.4). Moreover, it follows from (2.15) that inequality (2.1) holds.

Finally, we will prove the uniqueness of the generalized solution. It suffices to check that the only generalized solution of problem (1.2)–(1.4) with $f \equiv 0$ is $u \equiv 0$. By setting $v = u(\cdot, t)$ in (1.6) (for $f \equiv 0$), we get

$$\frac{d}{dt}(\|u(\cdot, t)\|^2) + 2B(u, u; t) = 0.$$ 

By (1.5), we have

$$\frac{d}{dt}(\|u\|^2_{L_2(\Omega)}) + 2\mu_0 \|u\|^2_{H^1(\Omega)} \leq 0 \text{ for a.e. } t \in [0, T).$$

Since $u|_{t=0} = 0$, it follows that $u = 0$ on $Q_T$. By (2.15), we also see that the solution $u$ depends continuously on $f$.

By the same arguments used in the proof of Theorem 2.1 together with inductive arguments (cf. [3]), we obtain the following theorem:

**Theorem 2.2.** Let $h \in \mathbb{N}^*$, and assume that

(i) $\sup\{a_{ijk+1}, |b_{ik}|, |c_{jk}| : i, j = 1, \ldots, n; (x, t) \in \overline{Q}_T, k \leq h\} \leq \mu$,

(ii) $f_{ik} \in H_0^0(Q_T, \gamma_0)$ for all $k \leq h$; $f_{ik}(x, 0) = 0$ for all $0 \leq k \leq h - 1$. 


Then for each \( \gamma \geq \gamma_0 \), the generalized solution \( u \in \dot{H}^{1,1}(Q_T, \gamma) \) of problem (1.2)–(1.4) has derivatives with respect to \( t \) up to order \( h \), and

\[
(2.16) \quad \| u_{t^h} \|_{\dot{H}^{1,1}(Q_T, \gamma)}^2 \leq C \sum_{j=0}^{h} \| f_j \|_{\dot{H}^{0,0}(Q_T, \gamma_0)}^2
\]

where \( C \) is a constant independent of \( u \) and \( f \).

3. Regularity of the generalized solution. We reduce the operator with coefficients at \( P \in l_0 \), \( t \in [0, T) \),

\[
L^{(2)}_0 := -\sum_{i,j=1}^{2} a_{ij}(P, t) \frac{\partial^2}{\partial x_i \partial x_j},
\]

to its canonical form. After a linear transformation of coordinates that realizes this reduction, \( T_1 \) and \( T_2 \) go over into hyperplanes \( T_1' \) and \( T_2' \), respectively, the angle between which is denoted by \( \omega(P, t) \). It is easy to see that \( \omega(P, t) \) does not depend on the method by which \( L^{(2)}_0 \) is reduced to its canonical form. The function \( \omega(P, t) \) is infinitely differentiable, and \( \omega(P, t) > 0 \).

**Theorem 3.1.** Let the assumptions of Theorem 2.2 be satisfied for a given positive integer \( h \). Furthermore, let \( \alpha \in [0, 1] \), \( 1 - \alpha < \pi/\omega \). Then the generalized solution \( u \in \dot{H}^{1,1}(Q_T, \gamma) \) of problem (1.2)–(1.4) has derivatives with respect to \( t \) up to order \( h \), \( u_{t^h} \in H^{2,0}_{\alpha}(Q_T, \gamma) \) and

\[
\| u_{t^h} \|_{H^{2,0}_{\alpha}(Q_T, \gamma)}^2 \leq C \sum_{k=0}^{h} \| f_k \|_{H^{0,0}_{\alpha}(Q_T, \gamma_0)}^2,
\]

where \( C \) is a constant independent of \( u, f \).

**Proof.** We use induction on \( h \). Firstly, we consider the case \( h = 0 \). It is easy to see that \( u(\cdot, t_0) \), \( t_0 \in (0, T) \), is the generalized solution of the problem

\[
L(x, t_0, \partial)u = F(x, t_0) \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0,
\]

where \( F(x, t_0) = f(x, t_0) - u_t(x, t_0) \in H^0_{\alpha}(\Omega) \). From [8, Thm. 2], we get \( u(\cdot, t_0) \in H^2_{\alpha}(\Omega) \) and

\[
(3.1) \quad \| u(\cdot, t_0) \|_{H^2_{\alpha}(\Omega)} \leq C [\| F(\cdot, t_0) \|_{H^0_{\alpha}(\Omega)} + \| u \|_{L^2(\Omega)}] \leq C [\| f \|_{H^0_{\alpha}(\Omega)} + \| u_t \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}].
\]

Multiplying the above inequality with \( e^{-t_0} \), then integrating with respect to \( t_0 \) from 0 to \( T \) and using the estimates from Theorem 2.2, we obtain

\[
\| u \|_{H^2_{\alpha}(Q_T, \gamma)}^2 \leq C \| f \|_{H^0_{\alpha}(Q_T, \gamma_0)}^2.
\]

Thus, the assertion is valid for \( h = 0 \).
Next, suppose that the assertion is true for \( h - 1 \); we will prove it for \( k = h \). Differentiating (1.2) \( h \) times with respect to \( t \), we find

\[
L u_{th} = f_{th} - u_{th+1} - \sum_{k=0}^{h-1} \binom{k}{h} L_{th-k} u_{tk} =: F.
\]

By the assumptions of the theorem and the induction assumption, this implies that \( f_{th} \in H^0_\alpha(\Omega) \), \( u_{th+1} \in L_2(\Omega) \subset H^0_\alpha(\Omega) \), and \( u_{tk} \in H^0_\alpha(\Omega) \), \( k \leq h - 1 \). Therefore, \( F(\cdot, t_0) \in H^0_\alpha(\Omega) \) for a.e. \( t_0 \in (0, T) \). By using again \[8\] Thm. 2, we get \( u_{th} \in H^2_\alpha(\Omega) \) for a.e. \( t_0 \in (0, T) \) and

\[
\begin{align*}
\|u_{th}\|^2_{H^2_\alpha(\Omega)} &\leq C \|F\|^2_{H^0_\alpha(\Omega)} \\
&\leq C \left[ \|f_{th}\|^2_{H^0_\alpha(\Omega)} + \|u_{th+1}\|^2_{L_2(\Omega)} + \sum_{k=0}^{h-1} \|u_{tk}\|^2_{L_2(\Omega)} \right].
\end{align*}
\]

Multiplying (3.3) with \( e^{-t_0 \gamma} \), then integrating with respect to \( t_0 \) from 0 to \( T \) and using again the estimates from Theorem 2.2 we obtain

\[
\|u_{th}\|^2_{H^{2,0}_{\alpha}(Q_T, \gamma)} \leq C \sum_{k=0}^{h} \|f_{tk}\|^2_{H^0_\alpha(Q_T, \gamma_0)}.
\]

This means that the assertion of the theorem is valid for \( k = h \). ■

**Theorem 3.2.** Assume that \( f, f_t \in H^h_\alpha(Q_T, \gamma_0), f_{tk}(x, 0) = 0 \) for all \( k \leq h - 1 \), and

\[
h + 1 - \alpha < \pi/\omega, \quad \alpha \in [0, 1].
\]

Then the generalized solution \( u \) of problem (1.2)–(1.4) is in \( H^{2+h}_\alpha(Q_T, \gamma) \). Moreover,

\[
\|u_{th}\|^2_{H^{2+h}_\alpha(Q_T, \gamma)} \leq C (\|f\|^2_{H^h_\alpha(Q_T, \gamma_0)} + \|f_t\|^2_{H^h_\alpha(Q_T, \gamma_0)}),
\]

where \( C \) is a constant independent of \( u, f \).

**Proof.** We have

\[
\begin{align*}
\|u\|^2_{H^2_\alpha(Q_T, \gamma)} &= \sum_{|p|+k\leq 2} \int_{Q_T} (r^{2(\alpha+|p|+k-2)} |D^p u_{tk}|^2 + |u|^2) e^{-\gamma t} \, dx \, dt \\
&= \sum_{|p|\leq 2} \int_{Q_T} (r^{2(\alpha+|p|-2)} |D^p u|^2 + |u|^2) e^{-\gamma t} \, dx \, dt \\
&\quad + \sum_{|p|\leq 1} \int_{Q_T} (r^{2(\alpha+|p|-1)} |D^p u_t|^2) e^{-\gamma t} \, dx \, dt + \int_{Q_T} r^{2\alpha} |u_{tt}|^2 e^{-\gamma t} \, dx \, dt \\
&= \|u\|^2_{H^{2,0}_{\alpha}(Q_T, \gamma)} + \|u_t\|^2_{H^1_{\alpha,0}(Q_T, \gamma)} + \|u_{tt}\|^2_{H^0_\alpha(Q_T, \gamma)} \\
&= \sum_{k=0}^{2} \|u_{tk}\|^2_{H^{2-k,0}_{\alpha}(Q_T, \gamma)}.
\end{align*}
\]
Therefore, $u \in H^{2}_{\alpha}(Q_{T}, \gamma)$ by Theorem 3.1. Moreover, we have

$$\|u\|_{H^{2}_{\alpha}(Q_{T}, \gamma)}^{2} = \sum_{k=0}^{2} \|u_{tk}\|_{H^{2-k,0}_{\alpha}(Q_{T}, \gamma)}^{2} \leq C(\|f\|_{H^{0}_{\alpha}(Q_{T}, \gamma)}^{2} + \|f_{t}\|_{H^{0}_{\alpha}(Q_{T}, \gamma)}^{2}).$$

Thus, the assertion is valid for $h = 0$. Suppose it is true for $h - 1$. It is easy to see that

$$(3.5) \quad \|u\|_{H^{2+h}_{\alpha}(Q_{T}, \gamma)}^{2} = \sum_{k=0}^{h+2} \|u_{tk}\|_{H^{h+2-k,0}_{\alpha}(Q_{T}, \gamma)}^{2}.$$  

We will prove that

$$(3.6) \quad u_{tk} \in H^{h+2-k,0}_{\alpha}(Q_{T}, \gamma), \quad k = 0, \ldots, h,$$

and

$$(3.7) \quad \|u_{tk}\|_{H^{h+2-k,0}_{\alpha}(Q_{T}, \gamma)}^{2} \leq C \sum_{s=0}^{k} \|f_{ts}\|_{H^{h-k,0}_{\alpha}(Q_{T}, \gamma)}^{2}, \quad k \leq h.$$  

By using Theorem 3.1, this holds for $k = h$. Suppose that it holds for $k = h, h-1, \ldots, j+1$; we will prove it for $k = j$. Returning once more to (3.2) ($h = j$), we get

$$Lu_{tj} = f_{tj} - u_{tj+1} - \sum_{k=0}^{j-1} \binom{j}{k} Lu_{tj-k} u_{tk} =: F_{1}.$$  

Notice that $f_{tj} \in H^{h}_{\alpha}(\Omega) \subset H^{h-j}_{\alpha}(\Omega)$ for a.e. $t \in (0, T)$ (by the assumptions of the theorem), $u_{tj+1} \in H^{h-j+1}_{\alpha}(\Omega) \subset H^{h-j}_{\alpha}(\Omega)$ for a.e. $t \in (0, T)$ (by (3.6) which holds for $k = j + 1$), $u_{tk} \in H^{h+1-k}_{\alpha}(\Omega) \subset H^{h-j}_{\alpha}(\Omega)$, $k = 0, \ldots, j - 1$ (by the induction assumption for $k = h - 1$).

This implies that $F_{1}(\cdot, t) \in H^{h-j}_{\alpha}(\Omega)$ for a.e. $t \in (0, T)$. From [8, Thm. 2], we obtain

$$u_{tj} \in H^{h+2-j}_{\alpha}(\Omega) \quad \text{for a.e.} \ t \in (0, T)$$

and

$$(3.8) \quad \|u_{tj}\|_{H^{h+2-j}_{\alpha}(\Omega)}^{2} \leq C \|F_{1}\|_{H^{h-j}_{\alpha}(\Omega)}^{2}$$

$$\quad \leq C \left[ \|f_{tj}\|_{H^{h-j}_{\alpha}(\Omega)}^{2} + \|u_{tj+1}\|_{H^{h-j}_{\alpha}(\Omega)}^{2} + \sum_{k=0}^{j-1} \|u_{tk}\|_{H^{h-j}_{\alpha}(\Omega)}^{2} \right].$$  

Multiplying (3.8) with $e^{-\gamma t}$, then integrating with respect to $t$ from 0 to $T$, we arrive at

$$\|u_{tj}\|_{H^{h+2-j,0}_{\alpha}(Q_{T}, \gamma)}^{2} \leq C \sum_{k=0}^{j} \|f_{tk}\|_{H^{h-j,0}_{\alpha}(Q_{T}, \gamma)}^{2}.$$
This means that (3.6) and (3.7) are true for \( k = j \), so they hold for all \( k = 0, 1, \ldots, h \). From (3.5), we get

\[
\| u \|_{H^{h+2}_\alpha(Q_T, \gamma)}^2 \leq C \sum_{k=0}^{h+1} \| f_{tk} \|_{H^{h-k,0}_\alpha(Q_T, \gamma)}^2 = C(\| f \|_{H^h_\alpha(Q_T, \gamma)}^2 + \| f_t \|_{H^h_\alpha(Q_T, \gamma)}^2).
\]

The proof is complete. \( \blacksquare \)

4. Asymptotics of the solution in a neighbourhood of the edge.

In the previous section, we have seen that if \( k + 1 - \alpha < \pi/\omega \), \( \alpha \in [0, 1] \) and \( f, f_t \in H^{k}_\alpha(Q_T, \gamma_0) \), then the solution \( u \) is in \( H^{2+k}_\alpha(Q_T, \gamma_0) \). Now we study the solution in the case \( \pi/\omega < k + 1 - \alpha \). In this case we can obtain for \( u \) an asymptotic representation in a neighbourhood of \( l_0 : x_1 = x_2 = 0 \). To start, we denote \( y_1 = x_1, y_2 = x_2, y = (y_1, y_2), z_i = x_{i+2}, z = (z_1, \ldots, z_{n-2}), r = x_1^2 + x_2^2 \), \( (r, \varphi) \) are the polar coordinates of \( y = (y_1, y_2) \in \Omega_z = \Omega \cap \{ z = \text{const} \} \).

Let \( Q_{z,T} = \Omega_z \times (0, T) \).

**Lemma 4.1.** Suppose that the following hypotheses are satisfied:

(i) \( f_{ts} \in H^{k,0}_\alpha(Q_T, \gamma_0) \) for all \( s \leq h \); \( f_{ts}(x, 0) = 0 \) for all \( s \leq h - 1 \).

(ii) \( k - \alpha < \pi/\omega < k + 1 - \alpha < 2\pi/\omega \), \( \alpha \in [0, 1] \).

Let \( u \) be the solution of (1.2)–(1.4) with \( u \equiv 0 \) outside some neighbourhood of \( l_0 \). Then

\[
u(y, z, t) = c(z, t) r^{\pi/\omega} \Phi(z, \varphi, t) + u_1(y, z, t)
\]

where \( c_{ts} \in L_2(Q_T, \gamma), \Phi \in C^\infty \) and \( (u_1)_{ts} \in H^{k+2,0}_\alpha(Q_{z,T}, \gamma) \) for all \( s \leq h \).

**Proof.** Using (i), we deduce from Theorem 3.2 that \( u_{ts} \in H^{k+1,0}_\alpha(Q_T, \gamma) \) for all \( s \leq h \), in particular, \( u_s \in H^{k}_\alpha(\Omega) \) and \( u_{tz} \in H^{k}_\alpha(\Omega) \) for a.e. \( t \in (0, T) \). On the other hand, we have

\[
Lu_z = f_z - u_{tz} - L_z u =: f_1
\]

where

\[
L_z = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ijz} \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^n b_{iz} \frac{\partial}{\partial x_i} + c_z
\]

and \( f_1 \in H^{k-1}_\alpha(\Omega) \) for a.e. \( t \in (0, T) \). Using Theorem 3.1 we obtain \( u_z \in H^{k+1}_\alpha(\Omega) \) for a.e. \( t \in (0, T) \). Therefore, equality (1.2) can be rewritten in the form

(4.1) \[
L^{(2)}_0 u = F
\]

where \( F \in H^{k}_\alpha(\Omega) \) for a.e. \( t \in (0, T) \). Now we can apply Theorem 1' of \( \text{[9]} \) to get

(4.2) \[
u(y, z, t) = c(z, t) r^{\pi/\omega} \Phi(z, \varphi, t) + u_1(y, z, t)
\]
where $\Phi \in C^\infty$, $u_1 \in H^{k+2}_\alpha(\Omega_z)$ and
\[
|c(z, t)|^2 \leq C(\|F\|_{H^k_\alpha(\Omega_z)}^2 + \|u\|_{L^2_2(\Omega_z)}^2),
\]
\[
\|u_1\|_{H^{k+2}_\alpha(\Omega_z)}^2 \leq C(\|F\|_{H^k_\alpha(\Omega_z)}^2 + \|u\|_{L^2_2(\Omega_z)}^2), \quad z \in l_0, t \in (0, T).
\]

Hence, $c \in L_2(QT, \gamma)$, $u_1 \in H^{k+2, 0}_\alpha(Qz_T, \gamma)$. This implies that the conclusion holds for $h = 0$. Suppose it is true for $h - 1$; we will prove it for $k = h$.

Denoting $v = u_{th}$, and differentiating (4.1) $h$ times with respect to $t$, we find

(4.3) \[
L_0^{(2)} v = F_{th} - \sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} u_{th-j}.
\]

Setting $S_0 = r^{\pi/\omega} \Phi$, we have

(4.4) \[
\sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} u_{th-j} = \sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} (cS_0)_{th-j} + \sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} (u_1)_{th-j}.
\]

The first term of the right-hand side of (4.4) can be rewritten in the following form:

\[
\sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} (cS_0)_{th-j} = \sum_{j=1}^{h} \binom{h}{j} L_{0t}^{(2)} \left( \sum_{i=0}^{h-j} \binom{h-j}{i} c_{th-j-i} S_{0t^i} \right)
\]
\[
= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=0}^{h-j} \binom{h-j}{i} c_{th-j-i} L_{0t}^{(2)} S_{0t^i}
\]
\[
= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{th-j-i} L_{0t}^{(2)} S_{0t^i}
\]
\[
+ \sum_{j=1}^{h} \binom{h}{j} c_{th-j} L_{0t}^{(2)} S_{0}
\]
\[
= \sum_{j=0}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{th-j-i} L_{0t}^{(2)} S_{0t^i}
\]
\[
+ \sum_{j=1}^{h} \binom{h}{j} c_{th-j} L_{0t}^{(2)} S_{0} - \sum_{i=1}^{h} \binom{h}{i} c_{th-i} L_{0}^{(2)} S_{0t_i}
\]
\[
= F_1 - \sum_{i=1}^{h} \binom{h}{i} c_{th-i} L_{0}^{(2)} S_{0t_i}.
\]
From the assumptions of the lemma and the inductive assumptions, this implies that \( F_1 \in H^k_\alpha(\Omega_z) \). Hence, from (4.4) we obtain
\[
\sum_{j=1}^{h} \binom{h}{j} L^{(2)}_{0t^j} u_{t^j} = F_2 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{i-1}} L^{(2)}_{0t^i}
\]
where \( F_2 \in H^k_\alpha(\Omega_z) \). Employing the equality above, we infer from (4.3) that
\[
L^{(2)}_{0t} v = F_3 + \sum_{i=1}^{h} \binom{h}{i} c_{t^{i-1}} L^{(2)}_{0t^i}
\]
where \( F_3 \in H^k_\alpha(\Omega_z) \). Analogously to the case \( h = 0 \), we get
\[
v - \sum_{i=1}^{h} \binom{h}{i} c_{t^{i-1}} S_{0t^i} = d(z,t)S_0 + u_2(y,z,t).
\]
Therefore,
\[
u = \sum_{i=1}^{h} \binom{h}{i} c_{t^{i-1}} S_{0t^i} + d(z,t)S_0 + u_2(y,z,t)
\]
where \( d \in L_2(Q_T,\gamma_0) \) and \( u_2 \in H^{k+2,0}_\alpha(Q_{z,T},\gamma) \). By the assumption (i), this implies that \( u \) is differentiable with respect to \( t \). Then, we can see that the functions \( c(z,\cdot) \) and \( u_1(\cdot,t) \) are differentiable with respect to \( t \). Combining (4.2) and (4.6), we conclude that
\[
c_{t^h} = d \in L_2(Q_T,\gamma), \quad (u_1)_{t^h} = u_2 \in H^{k+2,0}_\alpha(Q_{z,T},\gamma).
\]
The proof is complete. ■

Next, we have the following theorem.

**Theorem 4.2.** Suppose that the hypotheses of Lemma 4.1 are satisfied. Then the following representation holds:
\[
u(x,t) = c(x,t)r^{\pi/\omega}\Phi(z,\varphi,t) + u_1(x,t)
\]
where \( c_{ts} \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_T,\gamma) \) and \( u_{1s} \in H^{k+2,0}_\alpha(Q_{z,T},\gamma) \) for all \( s \leq h \).

**Proof.** From Lemma 4.1, we have the representation:
\[
u(x,t) = c(z,t)r^{\pi/\omega}\Phi(z,t,\varphi) + u_1(x,t)
\]
where \( c_{ts} \in L_2(Q_T,\gamma) \) and \( u_{1s} \in H^{k+2,0}_\alpha(Q_{z,T},\gamma) \) for all \( s \leq h \). Consider the differential operator
\[
D_1 = \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \varphi};
\]
in the coordinates $x_1, x_2$, it is
\[ D_1 = \Phi_1 \frac{\partial}{\partial x_1} + \Phi_2 \frac{\partial}{\partial x_2} \]
where $\Phi_1, \Phi_2$ are infinitely differentiable. From representation (4.7), we find
\[ D_1 u = \frac{\pi}{\omega} c(z, t) r^{\pi/\omega - 1} \Phi_3(z, t, \varphi) + D_1 u_1(x, t). \]
Moreover,
\[ u_1 \in H^{k+2,0}_\alpha(Q_{z,T}, \gamma), \quad \int_{Q_{z,T}} \left( r^{2\alpha} \frac{\partial^{k+2} u_1}{\partial x_1^{k_1} \partial x_2^{k_2}} \right) e^{-\gamma t} \, dx_1 \, dx_2 \, dt < \infty. \]
By arguments analogous to the proof of Lemma 4.1, we obtain
\[ u_{z}, u \in H^{k+1,0}_{\alpha}(Q_{T}, \gamma). \]
Therefore,
\[ (D_1 u)_z \in H^{k,0}_{\alpha}(Q_{T}, \gamma), \]
\[ \int_{Q_{T}} (r^{2(\alpha-k)} |(D_1 u)_z|) e^{-\gamma t} \, dx \, dt \leq \int_{Q_{T}} (r^{2\alpha} |f|^2) e^{-\gamma_0 t} \, dx \, dt < \infty. \]
Combining (4.9) and (4.10), we get
\[ r^{-\pi/\omega + 1} D_1 u_1 \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_{z,T}, \gamma), \quad r^{-\pi/\omega + 1} (D_1 u)_z \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_{T}, \gamma). \]
On the other hand, equality (4.8) yields
\[ (r^{-\pi/\omega + 1} D_1 u)_y = (r^{-\pi/\omega + 1} D_1 u_1)_y. \]
Consequently,
\[ r^{-\pi/\omega + 1} D_1 u \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_{T}, \gamma). \]
Now write
\[ c_1(x, t) = \frac{\omega}{\pi} r^{-\pi/\omega + 1} D_1 u \Phi_3. \]
Then (4.11) implies $c_1 \in H^{k+1,0}_{\alpha+\pi/\omega-1}(Q_{T}, \gamma)$. From Lemma 2 in [9], we conclude that there is $\tilde{c}_1 \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_{T}, \gamma)$ with $(\tilde{c}_1)_t \in H^{k+2,0}_{\alpha+\pi/\omega}(Q_{T}, \gamma)$ for all $s \leq h$ such that
\[ \int_{Q_{T}} (|c_1 - \tilde{c}_1|^2 r^{2(\alpha+\pi/\omega-k-2)}) e^{-\gamma t} \, dx \, dt < \infty. \]
Using (4.11) and the fact that $u_1 \in H^{k+2,0}_{\alpha}(Q_{z,T}, \gamma)$, we get
\[ \int_{Q_{T}} (|c - c_1|^2 r^{2(\alpha+\pi/\omega-k-2)}) e^{-\gamma t} \, dx \, dt < \infty. \]
We can rewrite representation (4.7) in the form
\begin{equation}
(4.14) \quad u(x, t) = \tilde{c}_1(x, t)r^{\pi/\omega}\Phi(z, \varphi, t) + [c - \tilde{c}_1]r^{\pi/\omega}\Phi(z, \varphi, t) + u_2(x, t)
\end{equation}
where \(u_2 \in H^{k+2,0}_\alpha(Q_{z,T}, \gamma)\) for all \(z \in l_0\). Since \(u\) is differentiable with respect to \(z\) and \(u_2 = u - \tilde{c}_1r^{\pi/\omega}\Phi\), we see that \(Lu_2 \in H^k_{\alpha}(\Omega)\) and
\[
\int_{\Omega} r^{2(\alpha-k-2)}|u_2| \, dx < \infty.
\]
By Lemma 2 in [7], we obtain \(u_2 \in H^{k+2}_{\alpha}(\Omega)\). Hence, \(u_2 \in H^{k+2,0}_{\alpha}(Q_{T,T}, \gamma)\).

To prove \((u_2)_t^{s} \in H^{k+2,0}_{\alpha}(Q_{T,T}, \gamma)\) for all \(s \leq h\), we can use arguments analogous to the proof of Lemma 4.1.

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