ANNALES
POLONICI MATHEMATICI
109.2 (2013)

The zero distribution and uniqueness of difference-differential polynomials

by Kai Liu (Nanchang), Xin-Ling Liu (Nanchang) and Lian-Zhong Yang (Jinan)

Abstract. We consider the zero distribution of difference-differential polynomials of meromorphic functions and present some results which can be seen as the discrete analogues of the Hayman conjecture. In addition, we also investigate the uniqueness of difference-differential polynomials of entire functions sharing one common value. Our theorems improve some results of Luo and Lin [J. Math. Anal. Appl. 377 (2011), 441–449] and Liu, Liu and Cao [Appl. Math. J. Chinese Univ. 27 (2012), 94–104].

1. Introduction. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [9, 26]. In this paper, a meromorphic function f means meromorphic in the complex plane. If no poles occur, then f reduces to an entire function. Denote by $\rho(f)$ and $\rho_2(f)$ the order and the hyper-order of f respectively [11, 26]. If $f-a$ and $g-a$ have the same zeros, then we say that f and g *share the value* a *IM* (ignoring multiplicities). If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then f and g *share the value* a *CM* (counting multiplicities).

Given a meromorphic function $f(z)$, recall that $\alpha(z)\not\equiv 0,\infty$ is a *small function* with respect to $f(z)$, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure.

The following result is related to the Hayman conjecture [8, Theorem 10]. The conjecture was also considered later (see [1, 3, 2, 20]).

Theorem A ([3, Theorem 1]). *Let f be a transcendental meromorphic function. If $n\geq 1$ is a positive integer, then $f^nf'-1$ has infinitely many zeros.*

2010 *Mathematics Subject Classification*: Primary 30D35; Secondary 39A05.
Key words and phrases: zeros, difference-differential polynomials, value sharing, uniqueness.

DOI: 10.4064/ap109-2-3

Noting that $[f^{n+1}]' = (n+1)f^n f'$ in Theorem A, Chen [2, Theorem 1], Wang [22, Theorem 4], Wang and Fang [23, Corollary 1] extended Theorem A. The latter result can be stated as follows.

THEOREM B ([23, Corollary 1]). *Let f be a transcendental meromorphic function, and let n and k be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} - 1$ has infinitely many zeros.*

Extending Theorem A to difference polynomials, Laine and Yang [12, Theorem 2] investigated the zero distribution of $f(z)^n f(z+c) - a$ and proved the following result.

THEOREM C. *Let f be a transcendental entire function of finite order and c be a nonzero complex constant. If $n \geq 2$, then $f(z)^n f(z+c) - a$, where $a \in \mathbb{C}\backslash\{0\}$, has infinitely many zeros.*

Recently, Theorem C has been improved in different directions: the constant a was replaced by a nonzero polynomial in [16] or by a small function $a(z)$ in [14]. In addition, the papers [13, 14, 18, 27] are devoted to the cases of meromorphic functions f or more general difference products.

In the following, unless otherwise specified, we assume that c is a nonzero constant, n, m, k, s, t are positive integers, and $a(z)$ is a nonzero small function with respect to $f(z)$. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \ldots, a_n$ $(\neq 0)$ are complex constants and t is the number of distinct zeros of $P(z)$. Recently, Luo and Lin [18] obtained the following result.

THEOREM D ([18, Theorem 1]). *Let f be a transcendental entire function of finite order. If $n > t$, then $P(f)f(z+c) - a(z)$ has infinitely many zeros.*

Obviously, Theorem D is an improvement of Theorem C. Here, we complete this result of [18] by showing that the restriction $n > t$ in Theorem D is indispensable:

REMARK 1. The conclusion of Theorem D is not true if $n = t = 1$. This can be seen by taking $f(z) = e^z + 1$, $e^c = -1$. Then $f(z)f(z+c) - 1 = -e^{2z}$ has no zeros.

Moreover, the assertion of Theorem D may fail if $n = t = 2$. This can be seen by taking

$$f(z) = \frac{1}{e^z} + 1, \quad e^c = -1, \quad P(z) = \left(z + \frac{-1+\sqrt{3}i}{2}\right)\left(z + \frac{-1-\sqrt{3}i}{2}\right);$$

then $P(f)f(z+c) - 1 = -1/e^{3z}$ has no zeros.

In fact, the conclusion of Theorem D is not true for any natural positive integers n, t satisfying $n = t \geq 2$,. Taking

$$f(z) = \frac{1}{e^z} + 1, \quad e^c = -1, \quad P(z) = \left(z - 1 - \frac{1}{d_1}\right) \cdots \left(z - 1 - \frac{1}{d_n}\right),$$

where $d_i \neq 1$, $i = 1, \dots, n$, are the distinct zeros of $z^{n+1} - 1 = 0$, we get $P(f)f(z+c) - 1 = -1/e^{(n+1)z}$, which has no zeros.

It is interesting to investigate what we can get if f^n is replaced with $f^n f(z + c)$ in Theorem B, that is, to consider the zero distribution of difference-differential polynomials. Liu, Liu and Cao [15, Theorems 1.1 & 1.3] considered the zero distribution of $[f^n f(z+c)]^{(k)}$ and $[f^n \Delta_c f]^{(k)}$; their results are summarized in Theorem E below.

THEOREM E. *Let f be a transcendental entire function of finite order. If $n \geq k + 2$, then $[f(z)^n f(z + c)]^{(k)} - a(z)$ has infinitely many zeros. If $n \geq k+3$, then $[f(z)^n \Delta_c f]^{(k)} - a(z)$ has infinitely many zeros, unless f is a periodic function with period c.*

In this paper, we continue to investigate the zero distribution of difference-differential polynomials and obtain the following four theorems that improve Theorems D and E.

THEOREM 1.1. *Let f be a transcendental entire function with $\rho_2(f) < 1$. If $n \geq t(k+1)+1$, then $[P(f)f(z+c)]^{(k)} - a(z)$ has infinitely many zeros.*

REMARK 2. (1) Theorem 1.1 is an improvement of Theorem E in the case $t = 1$ and an improvement of Theorem D in the case $k = 0$.

(2) The conclusion of Theorem 1.1 does not remain valid if $\rho_2(f) = 1$. Indeed, take $f(z) = e^{e^z}$, $P(z) = z^n$, $k \geq 1$, $e^c = -n$, $a(z)$ a nonconstant polynomial. Then $[P(f)f(z+c)]^{(k)} - a(z) = -a(z)$ has finitely many zeros.

(3) The condition $a(z) \neq 0$ cannot be removed. Let $f(z) = e^z$, $P(z) = z^n$, $e^c = -1$. Then $[P(f)f(z+c)]^{(k)} = -(n+1)^k e^{(n+1)z}$ has no zeros.

THEOREM 1.2. *Let f be a transcendental entire function with $\rho_2(f) < 1$, which is not a periodic function with period c. If $n \geq (t+1)(k+1)+1$, then $[P(f)(\Delta_c f)^s]^{(k)} - a(z)$ has infinitely many zeros.*

REMARK 3. The condition $a(z) \neq 0$ cannot be removed in Theorem 1.2 either, as can be seen by taking $f(z) = e^z$, $P(z) = z^n$, $e^c = 2$ then $[P(f)\Delta_c f]^{(k)} = (n+1)^k e^{(n+1)z}$ has no zeros.

For the case that $f(z)$ is a transcendental meromorphic function we obtain the following counterparts of Theorems 1.1 and 1.2.

THEOREM 1.3. *Let f be a transcendental meromorphic function with $\rho_2(f) < 1$. If $n \geq t(k+1)+5$, then $[P(f)f(z+c)]^{(k)} - a(z)$ has infinitely many zeros.*

REMARK 4. Theorem 1.3 is a partial answer to a question raised by Luo and Lin [18, p. 448].

THEOREM 1.4. *Let f be a transcendental meromorphic function with $\rho_2(f) < 1$. If $n \geq (t+2)(k+1)+3+s$, then $[P(f)(\Delta_c f)^s]^{(k)} - a(z)$ has infinitely many zeros.*

COROLLARY 1.5. *Let $P(z), Q(z), H(z), A(z)$ be nonzero polynomials. If $H(z)$ is a nonconstant polynomial, then the nonlinear difference-differential equation*

$$[P(f)f(z+c)]^{(k)} - A(z) = Q(z)e^{H(z)} \tag{1.1}$$

has no transcendental entire (resp. meromorphic) solution f with $\rho_2(f) < 1$ provided that $n \geq t(k+1)+1$ (resp. $n \geq t(k+1)+5$). If $H(z)$ is a constant, then (1.1) has no transcendental entire solution f with $\rho_2(f) < 1$, and no transcendental meromorphic solution f with $\rho_2(f) < 1$, provided that $n \geq 2$.

COROLLARY 1.6. *Let $P(z), Q(z), H(z), A(z)$ be nonzero polynomials. If $H(z)$ is a nonconstant polynomial, then the nonlinear difference-differential equation*

$$[P(f)(\Delta_c f)^s]^{(k)} - A(z) = Q(z)e^{H(z)} \tag{1.2}$$

has no transcendental entire (resp. meromorphic) solution f with $\rho_2(f) < 1$ provided that $n \geq (t+1)(k+1)+s+1$ (resp. $n \geq (t+2)(k+1)+3+s$). If $H(z)$ is a constant, then (1.2) has no transcendental entire solution f with $\rho_2(f) < 1$, and no transcendental meromorphic solution f with $\rho_2(f) < 1$, provided that $n \geq 3$, unless f is a periodic function with period c.

Concerning the uniqueness of difference products of entire functions sharing one common value, some results can be found in [14, 15, 17, 18, 21, 27]. The main purpose is to obtain relationships between f and g when $P(f)f(z+c)$ and $P(g)g(z+c)$ share one common value. In fact, the special cases $P(z) = z^n$ and $P(z) = z^n(z^m - 1)$ have mostly been considered. Luo and Lin [18, Theorem 2] considered the case of general $P(z)$. In this paper, we also consider the uniqueness of difference-differential polynomials sharing one common value. Liu, Liu and Cao [15, Theorem 1.5] considered the uniqueness on $[f^n f(z+c)]^{(k)}$ and $[g^n g(z+c)]^{(k)}$ sharing one common value; their result can be stated as follows.

THEOREM F. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, and let $n \geq 2k+6$. If $[f(z)^n f(z+c)]^{(k)}$ and $[g(z)^n g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k (c_1 c_2)^n [(n+1)C]^{2k} = 1$, or $f = tg$, where $t^{n+1} = 1$.*

In this paper, we consider the uniqueness of entire functions of hyper-order less than 1 sharing one common value and get the following results.

Theorem 1.7. *Let $f(z)$ and $g(z)$ be transcendental entire functions of hyper-order less than* 1*, and let $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value* 1 *CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

Theorem 1.8. *The conclusion of Theorem* 1.7 *is also valid if $n \geq 5k + 4m+12$ and $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value* 1 *IM.*

2. Some lemmas. For finite order transcendental meromorphic functions, the difference analogue of the logarithmic derivative lemma, given by Chiang and Feng [4, Corollary 2.5], Halburd and Korhonen [5, Theorem 2.1], [6, Theorem 5.6], plays an important part in considering the difference analogues of Nevanlinna theory. Afterwards, Halburd, Korhonen and Tohge improved the growth condition from $\rho < \infty$ to $\rho_2(f) < 1$ as follows.

Lemma 2.1 ([7, Theorem 5.1]). *Let f be a transcendental meromorphic function with $\rho_2(f) < 1$, and let ε be a sufficiently small number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}\right) = S(r,f) \tag{2.1}$$

for all r outside a set of finite logarithmic measure.

Lemma 2.2 ([7, Lemma 8.3]). *Let $T : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function and let $s \in (0,\infty)$. If the hyper-order of T is strictly less than* 1*, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1, \tag{2.2}$$

and $\delta \in (0, 1-\varsigma)$, then

$$T(r+s) = T(r) + o(T(r)/r^{\delta}) \tag{2.3}$$

as $r \to \infty$ outside a set of finite logarithmic measure.

From Lemma 2.2, we get the following lemma.

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f) \tag{2.4}$$

and

$$\begin{aligned} N(r, f(z+c)) &= N(r,f) + S(r,f), \\ N\left(r, \frac{1}{f(z+c)}\right) &= N(r, 1/f) + S(r,f). \end{aligned} \tag{2.5}$$

Combining the method of proof of [18, Lemma 5] with Lemma 2.1, we get the following result.

Lemma 2.4. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$T(r,F) = T(r,P(f)f(z)) + S(r,f) = (n+1)T(r,f) + S(r,f). \tag{2.6}$$

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$(n-1)T(r,f) + S(r,f) \le T(r,F) \le (n+1)T(r,f) + S(r,f). \tag{2.7}$$

Proof. Since $F(z) = P(f)f(z+c)$, we have

$$\frac{1}{P(f)f} = \frac{1}{F}\,\frac{f(z+c)}{f(z)}. \tag{2.8}$$

Using the first and second main theorem of Nevanlinna theory, Lemma 2.1 and the standard Valiron–Mohon'ko theorem [19], from (2.8) we get

$$\begin{aligned}(n+1)T(r,f) &\le T(r,F(z)) + T\left(r, \frac{f(z+c)}{f(z)}\right) + O(1) \\ &\le T(r,F(z)) + m\left(r, \frac{f(z+c)}{f(z)}\right) \\ &\quad + N\left(r, \frac{f(z+c)}{f(z)}\right) + O(1) \\ &\le T(r,F(z)) + N\left(r, \frac{f(z+c)}{f(z)}\right) + S(r,f) \\ &\le T(r,F(z)) + 2T(r,f) + S(r,f).\end{aligned} \tag{2.9}$$

Hence, $T(r,F) \ge (n-1)T(r,f) + S(r,f)$. It is easy to deduce that $T(r,F) \le (n+1)T(r,f) + S(r,f)$. Thus, (2.7) follows. ■

Remark. The following two examples show that (2.7) cannot be improved. If $f(z) = \tan z$, $P(z) = z^n$, $c_1 = \pi/2$, then

$$T(r, P(f)f(z+c_1)) = -\tan^{n-1} z = (n-1)T(r,f) + S(r,f).$$

If $f(z) = \tan z$, $P(z) = z^n$, $c_2 = \pi$, then

$$T(r, P(f)f(z+c_2)) = \tan^{n+1} z = (n+1)T(r,f) + S(r,f).$$

Using a similar method to the proof of Lemma 2.5, we can obtain the following two lemmas.

Lemma 2.6. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, and let s be a natural number. Then*

$$nT(r,f) + S(r,f) \le T(r, P(f)[f(z+c) - f(z)]^s) \le (n+s)T(r,f) + S(r,f). \tag{2.10}$$

Remark. The following two examples show that (2.10) also cannot be improved. If $f(z) = e^z$, $e^c = 2$, then

$$T(r, f(z)^n[f(z+c) - f(z)]^s) = T(r, e^{(n+s)z}) = (n+s)T(r,f) + S(r,f).$$

If $f(z) = e^z + z$, $c = 2\pi i$, then

$$T(r, f(z)^n[f(z+c) - f(z)]^s) = T(r, [e^z + z]^n(2\pi i)^s) = nT(r, f) + S(r, f).$$

LEMMA 2.7. *Let $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$. Then*

(2.11)
$$(n-s)T(r,f)+S(r,f) \le T(r, P(f)[f(z+c)-f(z)]^s) \le (n+2s)T(r,f)+S(r,f).$$

For the proof of Theorem 1.7, we need the following lemma. For the case of $k = 0$, $m = 1$, and f and g transcendental entire functions of finite order, the proof can be found in [27, proof of Theorem 6].

LEMMA 2.8. *Let f and g be transcendental entire functions with $\rho_2(f) < 1$, and c be a nonzero constant. If $n \ge m + 5$ and*

(2.12)
$$[f^n(f^m - 1)f(z+c)]^{(k)} = [g^n(g^m - 1)g(z+c)]^{(k)},$$

then $f = tg$, and $t^{n+1} = t^m = 1$.

Proof. From (2.12), we get $f^n(f^m-1)f(z+c) = g^n(g^m-1)g(z+c)+Q(z)$, where $Q(z)$ is a polynomial of degree at most $k - 1$. If $Q(z) \not\equiv 0$, then

$$\frac{f^n(f^m-1)f(z+c)}{Q(z)} = \frac{g^n(g^m-1)g(z+c)}{Q(z)} + 1.$$

From the second main theorem of Nevanlinna theory and Lemma 2.4, we obtain

$$\begin{aligned}
(n+m+1)T(r,f) &= T\left(r, \frac{f^n(f^m-1)f(z+c)}{Q(z)}\right) + S(r,f) \\
&\le \overline{N}\left(r, \frac{f^n(f^m-1)f(z+c)}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{f^n(f^m-1)f(z+c)}\right) \\
&\quad + \overline{N}\left(r, \frac{Q(z)}{g^n(g^m-1)g(z+c)}\right) + S(r,f) \\
&\le \overline{N}\left(r, \frac{1}{f^n(f^m-1)}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{g^n(g^m-1)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{g(z+c)}\right) + S(r,f) \\
&\le (m+2)T(r,f) + (m+2)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Similarly to the above, we have

$$(n+m+1)T(r,g) \le (m+2)T(r,f) + (m+2)T(r,g) + S(r,f) + S(r,g).$$

Thus, we get

$$\begin{aligned}
(n+m+1)[T(r,f) + T(r,g)] \le 2(m+2)&[T(r,f) + T(r,g)] \\
&+ S(r,f) + S(r,g),
\end{aligned}$$

which contradicts $n \ge m + 5$.

Hence, $Q(z) \equiv 0$. This implies that

$$f^n(f^m - 1)f(z+c) = g^n(g^m - 1)g(z+c). \tag{2.13}$$

Let $G(z) = f(z)/g(z)$. Assuming that $G(z)$ is not a constant, from (2.13) we get

$$g(z)^m = \frac{G(z)^n G(z+c) - 1}{G(z)^{n+m} G(z+c) - 1}. \tag{2.14}$$

If 1 is a Picard exceptional value of $G(z)^{n+m}G(z+c)$, applying the second main theorem of Nevanlinna theory, we get

$$\begin{aligned} T(r, G^{n+m}G(z+c)) &\le \overline{N}(r, G^{n+m}G(z+c)) \\ &\quad + \overline{N}\left(r, \frac{1}{G^{n+m}G(z+c)}\right) + \overline{N}\left(r, \frac{1}{G^{n+m}G(z+c) - 1}\right) + S(r,G) \\ &\le 2T(r, G(z)) + 2T(r, G(z+c)) + S(r,G) \\ &\le 4T(r, G(z)) + S(r,G). \end{aligned} \tag{2.15}$$

Combining (2.15) with Lemma 2.5, we infer that

$$(n + m - 1)T(r, G) \le 4T(r, G(z)) + S(r, G),$$

which contradicts $n \ge m + 5$.

Therefore, 1 is not a Picard exceptional value of $G(z)^{n+m}G(z+c)$. Thus, there exists z_0 such that $G(z_0)^{n+m}G(z_0 + c) = 1$. We now distinguish two cases.

Case 1: $G(z)^{n+m}G(z + c) \not\equiv 1$. From (2.14) and since $g(z)$ is an entire function, we get $G(z_0)^n G(z_0 + c) = 1$, thus $G(z_0)^m = 1$. Therefore,

$$\overline{N}\left(r, \frac{1}{G^{n+m}G(z+c) - 1}\right) \le \overline{N}\left(r, \frac{1}{G^m - 1}\right) \le mT(r,G) + S(r,G). \tag{2.16}$$

By (2.16), Lemma 2.3, and the second main theorem,

$$\begin{aligned} T(r, G^{n+m}G(z+c)) &\le \overline{N}(r, G^{n+m}G(z+c)) + \overline{N}\left(r, \frac{1}{G^{n+m}G(z+c)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G^{n+m}G(z+c) - 1}\right) + S(r,G) \\ &\le (m+2)T(r, G(z)) + 2T(r, G(z+c)) + S(r,G) \\ &\le (m+4)T(r, G(z)) + S(r,G). \end{aligned}$$

On the other hand,

$$\begin{aligned} (n+m)T(r,G) = T(r, G^{n+m}) &\le T(r, G^{n+m}G(z+c)) + T(r, G(z+c)) + O(1) \\ &\le (m+5)T(r, G(z)) + S(r,G), \end{aligned}$$

which contradicts $n \ge m + 5 \ge 6$.

CASE 2: $G(z)^{n+m}G(z+c) \equiv 1$. Thus,

$$(n+m)T(r,G) = T(r,G(z+c)) + S(r,G) = T(r,G(z)) + S(r,G),$$

which also contradicts $n \geq m+5$. Thus, G must be a constant, hence $f(z) = tg(z)$, where t is a nonzero constant. From $f^n(f^m-1)f(z+c) \equiv g^n(g^m-1)g(z+c)$, we deduce that $t^m = t^{n+1} = 1$, where n, m are positive integers. ∎

LEMMA 2.9. *If $n \geq k+1$, then there are no transcendental entire functions f and g with hyper-order less than 1 satisfying*

$$[f^n(f^m-1)f(z+c)]^{(k)} \cdot [g^n(g^m-1)g(z+c)]^{(k)} = 1. \tag{2.17}$$

Proof. Assume that f and g are transcendental entire functions of hyper-order less than 1 satisfying (2.17). From (2.17) and $n \geq k+1$, neither f nor g has zeros. Thus, $f(z) = e^{b(z)}$ and $g(z) = e^{d(z)}$, where $b(z), d(z)$ are entire functions of order less than 1. Substituting these into (2.17), we get

$$[e^{nb(z)}(e^{mb(z)}-1)e^{b(z+c)}]^{(k)}[e^{nd(z)}(e^{md(z)}-1)e^{d(z+c)}]^{(k)} = 1. \tag{2.18}$$

Let

$$(n+m)b(z) + b(z+c) = B_1(z), \quad nb(z) + b(z+c) = B_2(z),$$
$$(n+m)d(z) + d(z+c) = D_1(z), \quad nd(z) + d(z+c) = D_2(z).$$

It is easy to see that $B_1(z)$ and $B_2(z)$ are not constants at the same time: otherwise, $b(z)$ is a constant, thus $f(z)$ must be a constant.

We next proceed to show that one of $B_1(z)$ and $B_2(z)$ must be a constant for any positive integer k. The equation (2.18) can be written as

$$(e^{B_1(z)} - e^{B_2(z)})^{(k)}(e^{D_1(z)} - e^{D_2(z)})^{(k)} = 1.$$

Thus, we obtain

$$\begin{aligned}(e^{B_1} - e^{B_2})^{(k)} &= (B_1'^k + M_k)e^{B_1} - (B_2'^k + N_k)e^{B_2}\\ &= [(B_1'^k + M_k)e^{B_1-B_2} - (B_2'^k + N_k)]e^{B_2},\end{aligned}$$

where $M_k = M_k(B_1', B_1'', \ldots, B_1^{(k)})$ is a differential polynomial of B_1' of degree $k-1$, and $N_k = N_k(B_2', B_2'', \ldots, B_2^{(k)})$ is a differential polynomial of B_2' of degree $k-1$.

Remarking that 0 is the only Picard exceptional value of $e^{B_1(z)-B_2(z)}$, we get $B_1'^k + M_k(B_1', B_1'', \ldots, B_1^{(k)}) \equiv 0$ or $B_2'^k + N_k(B_2', B_2'', \ldots, B_2^{(k)}) \equiv 0$. In the former case, from the Clunie lemma [11, Theorem 2.4.2] we get $m(r, B_1') = S(r, B_1')$. This implies that the entire function $B_1(z)$ must be a constant. In the latter case we similarly deduce that B_2 is a constant.

If $B_1(z) \equiv B_1$ is a constant, then $f(z)^{n+m}f(z+c) = e^{B_1}$. From Lemma 2.4, we get $T(r,f) = S(r,f)$, a contradiction. If $B_2(z) \equiv B_2$ is a constant, then $f(z)^n f(z+c) = e^{B_2}$, and from Lemma 2.4, we also get $T(r,f) = S(r,f)$, a contradiction. ∎

LEMMA 2.10 ([26]). *Let f be a nonconstant meromorphic function, and k be a positive integer. Then*

$$T(r, f^{(k)}) \le T(r,f) + k\overline{N}(r,f) + S(r,f). \tag{2.19}$$

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, 1/(f-a))$ the counting function of the zeros of $f-a$ where an m-fold zero is counted m times if $m \le p$ and p times if $m > p$. Similarly, $N_p(r,f)$ denotes the counting function of the poles of f where an m-fold pole is counted m times if $m \le p$ and p times if $m > p$.

LEMMA 2.11 ([10, Lemma 2.3]). *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, 1/f^{(k)}) \le T(r, f^{(k)}) - T(r,f) + N_{p+k}(r, 1/f) + S(r,f), \tag{2.20}$$

$$N_p(r, 1/f^{(k)}) \le k\overline{N}(r,f) + N_{p+k}(1/f) + S(r,f). \tag{2.21}$$

LEMMA 2.12 ([25, Lemma 3]). *Let F and G be nonconstant meromorphic functions. If F and G share the value 1 CM, then one of the following three cases holds:*

- (i) $\max\{T(r,F), T(r,G)\} \le N_2(r,1/F) + N_2(r,F) + N_2(r,1/G) + N_2(r,G) + S(r,F) + S(r,G)$,
- (ii) $F = G$,
- (iii) $F \cdot G = 1$.

For the proof of Theorem 1.8, we need the following lemma.

LEMMA 2.13 ([24, Lemma 2.3]). *Let F and G be nonconstant meromorphic functions sharing the value 1 IM. Let*

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}.$$

If $H \not\equiv 0$, then

$$\begin{aligned} T(r,F) + T(r,G) &\le 2\big(N_2(r,1/F) + N_2(r,F) + N_2(r,1/G) + N_2(r,G)\big) \\ &\quad + 3\big(\overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}(r,G) + \overline{N}(r,1/G)\big) \\ &\quad + S(r,F) + S(r,G). \end{aligned} \tag{2.22}$$

3. Proofs of Theorems 1.1 and 1.2. Let $F(z) = P(f)f(z+c)$. From Lemma 2.4, we know that $F(z)$ is not a constant, and $S(r,F) = S(r,F^{(k)}) = S(r,f)$ follows. Assume that $F(z)^{(k)} - \alpha(z)$ has only finitely many zeros. Combining the second main theorem for three small functions [9, Theorem

2.5] and (2.20) with f a transcendental entire function, we get

$$(3.1)\quad T(r,F^{(k)}) \le \overline{N}(r,F^{(k)}) + \overline{N}\left(r,\frac{1}{F^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-\alpha(z)}\right) + S(r,F^{(k)})$$
$$\le N_1\left(r,\frac{1}{F^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-\alpha(z)}\right) + S(r,F^{(k)})$$
$$\le T(r,F^{(k)}) - T(r,F) + N_{k+1}(r,1/F) + S(r,F^{(k)}).$$

Combining (2.7) with (3.1), we get

$$(n+1)T(r,f) + S(r,f) = T(r,F) \le N_{k+1}(r,1/F) + S(r,f)$$
$$\le t(k+1)\overline{N}(r,1/f) + N(r,1/f(z+c)) + S(r,f)$$
$$\le [t(k+1)+1]T(r,f) + S(r,f),$$

contrary to $n \ge t(k+1)+1$. Thus, Theorem 1.1 is proved.

Set $G(z) = P(f)[\Delta_c f]^s$. Suppose that $G(z)^{(k)} - \alpha(z)$ has only finitely many zeros. Using a similar method to the above and Lemma 2.6, we get

$$nT(r,f) + S(r,f) \le T(r,G) \le N_{k+1}(r,1/G) + S(r,f)$$
$$\le t(k+1)\overline{N}(r,1/f) + (k+1)\overline{N}\left(r,\frac{1}{f(z+c)-f(z)}\right) + S(r,f)$$
$$\le (t+1)(k+1)T(r,f) + S(r,f),$$

contradicting $n \ge (t+1)(k+1)+1$. Thus, we get the proof of Theorem 1.2.

4. Proofs of Theorems 1.3 and 1.4. Let $F(z) = P(f)f(z+c)$. From Lemma 2.5, we know that $F(z)$ is not a constant, and $S(r,F) = S(r,F^{(k)}) = S(r,f)$ follows. Assume that $F(z)^{(k)} - \alpha(z)$ has only finitely many zeros. Combining the second main theorem for three small functions [9, Theorem 2.5] and (2.20) with f a transcendental meromorphic function, we have

(4.1)

$$T(r,F^{(k)}) \le \overline{N}(r,F^{(k)}) + \overline{N}\left(r,\frac{1}{F^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-\alpha(z)}\right) + S(r,F^{(k)})$$
$$\le \overline{N}(r,f) + \overline{N}(r,f(z+c)) + N_1\left(r,\frac{1}{F^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-\alpha(z)}\right) + S(r,F^{(k)})$$
$$\le 2T(r,f) + T(r,F^{(k)}) - T(r,F) + N_{k+1}(r,1/F) + S(r,F^{(k)}).$$

Combining (2.7) with (4.1), we obtain

$$\begin{aligned}(n-1)T(r,f)+S(r,f) \le T(r,F) &\le 2T(r,f)+N_{k+1}(r,1/F)+S(r,f)\\ &\le t(k+1)\overline{N}(r,1/f)+N\left(r,\frac{1}{f(z+c)}\right)+2T(r,f)+S(r,f)\\ &\le [t(k+1)+3]T(r,f)+S(r,f),\end{aligned}$$

which contradicts $n \ge t(k+1)+5$. Thus, Theorem 1.3 is proved.

Set $G(z) = P(f)[\Delta_c f]^s$. Suppose that $G(z)^{(k)} - \alpha(z)$ has only finitely many zeros. Similarly to the above, using Lemma 2.6 we get

$$\begin{aligned}&(n-s)T(r,f)+S(r,f) \le T(r,G) \le 2T(r,f)+N_{k+1}(r,1/G)+S(r,f)\\ &\quad\le 2T(r,f)+t(k+1)\overline{N}(r,1/f)+(k+1)\overline{N}\left(r,\frac{1}{f(z+c)-f(z)}\right)+S(r,f)\\ &\quad\le [(t+2)(k+1)+2]T(r,f)+S(r,f),\end{aligned}$$

which contradicts $n \ge (t+2)(k+1)+3+s$. Thus, we get the proof of Theorem 1.4.

5. Proof of Theorem 1.7. Let $F = [f^n(f^m-1)f(z+c)]^{(k)}$, $G = [g^n(g^m-1)g(z+c)]^{(k)}$. By assumption, F and G share the value 1 CM. From (2.19) and since f is a transcendental entire function,

$$T(r,F) \le T(r,f^n(f^m-1)f(z+c))+S(r,f). \tag{5.1}$$

Combining (5.1) with Lemma 2.4, we get $S(r,F)=S(r,f)$. We also have $S(r,G)=S(r,g)$. From (2.20), we obtain

$$\begin{aligned}N_2(r,1/F) &= N_2\left(r,\frac{1}{[f^n(f^m-1)f(z+c)]^{(k)}}\right)\\ &\le T(r,F)-T(r,f^n(f^m-1)f(z+c))\\ &\quad+N_{k+2}\left(r,\frac{1}{f^n(f^m-1)f(z+c)}\right)+S(r,f).\end{aligned} \tag{5.2}$$

Combining Lemma 2.4 with (5.2), we get

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,f^n(f^m-1)f(z+c))+S(r,f)\\ &\le T(r,F)-N_2(r,1/F)+N_{k+2}\left(r,\frac{1}{f^n(f^m-1)f(z+c)}\right)+S(r,f).\end{aligned} \tag{5.3}$$

From (2.21), we obtain

$$\begin{aligned}N_2(r,1/F) &\le N_{k+2}\left(r,\frac{1}{f^n(f^m-1)f(z+c)}\right)+S(r,f)\\ &\le (k+2)N(r,1/f)+N\left(r,\frac{1}{f^m-1}\right)+N\left(r,\frac{1}{f(z+c)}\right)+S(r,f)\\ &\le (k+m+3)T(r,f)+S(r,f).\end{aligned} \tag{5.4}$$

Similarly, we obtain

$$(5.5)\quad (n+m+1)T(r,g) \le T(r,G) - N_2(r,1/G) + N_{k+2}\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) + S(r,g)$$

and

$$(5.6)\qquad N_2(r,1/G) \le (k+m+3)T(r,g) + S(r,g).$$

If (i) of Lemma 2.12 is satisfied, then we get

$$\max\{T(r,F),T(r,G)\} \le N_2(r,1/F) + N_2(r,1/G) + S(r,F) + S(r,G).$$

Thus, combining the above with (5.3)–(5.6), we obtain

$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] &\le 2N_{k+2}\left(r, \frac{1}{f^n(f^m-1)f(z+c)}\right)\\ &\quad + 2N_{k+2}\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) + S(r,f) + S(r,g)\\ &\le 2(k+m+3)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),\end{aligned}$$

contradicting $n \ge 2k+m+6$. Hence, $F = G$ or $F \cdot G = 1$. From Lemmas 2.8 and 2.9, we get $f = tg$ for $t^m = t^{n+1} = 1$. Thus, we get the proof of Theorem 1.7.

6. Proof of Theorem 1.8. Let $F = [f^n(f^m-1)f(z+c)]^{(k)}$, $G = [g^n(g^m-1)g(z+c)]^{(k)}$. We will show that $F = G$ or $F \cdot G = 1$ under the assumptions of Theorem 1.8.

Assume that $H \not\equiv 0$, where H is defined in Lemma 2.13. Then from (2.22), we get

$$\begin{aligned}T(r,F)+T(r,G) &\le 2\big(N_2(r,1/F)+N_2(r,1/G)\big)+3\big(\overline{N}(r,1/F)+\overline{N}(r,1/G)\big)\\ &\quad + S(r,F) + S(r,G).\end{aligned}$$

Combining the above with (5.3)–(5.6) and (2.21), we obtain

$$\begin{aligned}(n+m+1)(T(r,f)+T(r,g)) &\le T(r,F)+T(r,G)\\ &\quad + N_{k+2}\left(r, \frac{1}{f^n(f^m-1)f(z+c)}\right) + N_{k+2}\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right)\\ &\quad - N_2(r,1/F) - N_2(r,1/G) + S(r,f) + S(r,g)\\ &\le 2N_{k+2}\left(r, \frac{1}{f^n(f^m-1)f(z+c)}\right) + 2N_{k+2}\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right)\\ &\quad + 3\big(\overline{N}(r,1/F)+\overline{N}(r,1/G)\big) + S(r,f) + S(r,g)\\ &\le (5k+5m+12)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),\end{aligned}$$

which contradicts $n \ge 5k+4m+12$.

Thus, $H \equiv 0$. The idea of the following proof is due to Yang and Yi [26]. Integrating H twice, we obtain

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \qquad G = \frac{(a-b-1) - (a-b)F}{Fb - (b+1)}, \tag{6.1}$$

which implies that $T(r, F) = T(r, G) + O(1)$. We will consider three cases:

Case 1: $b \neq 0, -1$. If $a - b - 1 \neq 0$, then by (6.1), we get

$$\overline{N}(r, 1/F) = \overline{N}\left(r, \frac{1}{G - \frac{a-b-1}{b+1}}\right).$$

By the second main theorem, (2.20) and (2.21),

$$\begin{aligned}(n+m+1)T(r,g) &\leq T(r,G) + N_k\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) \\ &\qquad - N(r, 1/G) + S(r,g) \\ &\leq N_k\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) + \overline{N}\left(r, \frac{1}{G - \frac{a-b-1}{b+1}}\right) + S(r,g) \\ &\leq (k+m+1)T(r,g) + (k+m+2)T(r,f) + S(r,f) + S(r,g).\end{aligned} \tag{6.2}$$

Similarly,

$$\begin{aligned}(n+m+1)T(r,f) \leq (k+m+1)T(r,f) + (k+m+2)T(r,g) \\ + S(r,f) + S(r,g).\end{aligned}$$

From (6.2) and the above,

$$\begin{aligned}&(n+m+1)[T(r,f) + T(r,g)] \\ &\qquad \leq (2k+2m+3)[T(r,f) + T(r,g)] + S(r,f) + S(r,g),\end{aligned}$$

which contradicts $n \geq 5k + 4m + 12$.

Thus, $a - b - 1 = 0$, so

$$F = \frac{(b+1)G}{bG + 1}. \tag{6.3}$$

Since F is an entire function, (6.3) yields $\overline{N}(r, \frac{1}{G+1/b}) = 0$. Using the same method as above, we get

$$\begin{aligned}(n+m+1)T(r,g) &\leq T(r,G) + N_k\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) \\ &\qquad - N(r, 1/G) + S(r,g) \\ &\leq N_k\left(r, \frac{1}{g^n(g^m-1)g(z+c)}\right) + \overline{N}\left(r, \frac{1}{G+1/b}\right) + S(r,g) \\ &\leq (k+m+1)T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction.

CASE 2: $b = 0$, $a \neq 1$. From (6.1), we have

$$F = \frac{G + a - 1}{a},$$

and we get a contradiction as above. Thus, $a = 1$ follows, which implies that $F = G$.

CASE 3: $b = -1$, $a \neq -1$. From (6.1), we obtain

$$F = \frac{a}{a + 1 - G}.$$

and again we get a contradiction. Hence $a = -1$. Thus, $F \cdot G = 1$. From Lemmas 2.8 and 2.9, we get $f = tg$ for $t^m = t^{n+1} = 1$. Thus, we get the proof of Theorem 1.8.

7. Discussion. In this paper, we investigated the uniqueness of difference-differential polynomial of entire functions sharing one common value. It remains an open question under what conditions Theorem 1.7 holds for meromorphic functions f, g with $\rho_2(f) < 1$ and $\rho_2(g) < 1$. In addition, if $[f^n(f^m - 1)\Delta_c f]^{(k)}$ and $[g^n(g^m - 1)\Delta_c g]^{(k)}$ share one common value, we believe that $f = tg$ for $t^m = t^{n+1} = 1$. Unfortunately, we have not succeeded in proving that.

Acknowledgements. The authors would like to thank the referees for valuable suggestions. This work was partially supported by the NSFC (No. 11171013, 11101201), the NSF of Shandong (No. ZR2010AM030), the NSF of Jiangxi (No. 20132BAB211003) and the YFED of Jiangxi (No. GJJ13078) of China.

References

[1] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamer. 11 (1995), 355–373.

[2] H. H. Chen, *Yoshida functions and Picard values of integral functions and their derivatives*, Bull. Austral. Math. Soc. 54 (1996), 373–381.

[3] H. H. Chen and M. L. Fang, *On the value distribution of $f^n f'$*, Sci. China Ser. A 38 (1995), 789–798.

[4] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. 16 (2008), 105–129.

[5] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. 314 (2006), 477–487.

[6] R. G. Halburd and R. J. Korhonen, *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations*, J. Phys. A 40 (2007), 1–38.

[7] R. G. Halburd, R. J. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, arXiv:0903.3236.

[8] W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. of Math. 70 (1959), 9–42.
[9] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[10] I. Lahiri and A. Sarkar, *Uniqueness of a meromorphic function and its derivative*, J. Inequal. Pure Appl. Math. 5 (2004) Art.20.
[11] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Stud. Math. 15, de Gruyter, Berlin, 1993.
[12] I. Laine and C. C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A 83 (2007), 148–151.
[13] K. Liu, *Value distribution of differences of meromorphic functions*, Rocky Mountain J. Math. 5 (2011), 1567–1584.
[14] K. Liu, X. L. Liu and T. B. Cao, *Value distributions and uniqueness of difference polynomials*, Adv. Difference Equations 2011, art. ID 234215, 12 pp.
[15] K. Liu, X. L. Liu and T. B. Cao, *Some results on zeros and uniqueness of difference-differential polynomials*, Appl. Math. J. Chinese Univ. 27 (2012), 94–104.
[16] K. Liu and L. Z. Yang, *Value distribution of the difference operator*, Arch. Math. (Basel) 92 (2009), 270–278.
[17] K. Liu, C. H. Zhang and L. Z. Yang, *Uniqueness of entire functions and difference polynomials*, submitted.
[18] X. D. Luo and W. C. Lin, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl. 377 (2011), 441–449.
[19] A. Z. Mohon'ko, *The Nevanlinna characteristics of certain meromorphic functions*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 14 (1971), 83–87 (in Russian).
[20] E. Mues, *Über ein Problem von Hayman*, Math. Z. 164 (1979), 239–259.
[21] X. G. Qi, L. Z. Yang and K. Liu, *Uniqueness and periodicity of meromorphic functions concerning the difference operator*, Comput. Math. Appl. 60 (2010), 1739–1746.
[22] Y. F. Wang, *On Mues' conjecture and Picard values*, Sci. China 36 (1993), 28–35.
[23] Y. F. Wang and M. L. Fang, *Picard values and normal families of meromorphic functions with multiple zeros*, Acta Math. Sinica 14 (1998), 17–26.
[24] J. F. Xu and H. X. Yi, *Uniqueness of entire functions and differential polynomials*, Bull. Korean Math. Soc. 44 (2007), 623–629.
[25] C. C. Yang and X. H. Hua, *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. 22 (1997), 395–406.
[26] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer, 2003.
[27] J. L. Zhang, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. 367 (2010), 401–408.

Kai Liu, Xin-Ling Liu
Department of Mathematics
Nanchang University
Nanchang, Jiangxi
330031, P.R. China
E-mail: liukai418@126.com
sdliuxinling@hotmail.com

Lian-Zhong Yang
Shandong University
School of Mathematics
Jinan, Shandong
250100, P.R. China
E-mail: lzyang@sdu.edu.cn

Received 22.2.2012
and in final form 31.8.2012 (2706)