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Fixed points of meromorphic functions and of their differences and shifts

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Abstract. Let $f(z)$ be a finite order transcendental meromorphic function such that $\lambda(1/f(z)) < \sigma(f(z))$, and let $c \in \mathbb{C} \setminus \{0\}$ be a constant such that $f(z+c) \not\equiv f(z)+c$. We mainly prove that

$$\begin{aligned}\max\{\tau(f(z)), \tau(\Delta_c f(z))\} &= \max\{\tau(f(z)), \tau(f(z+c))\} \\ &= \max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \sigma(f(z)),\end{aligned}$$

where $\tau(g(z))$ denotes the exponent of convergence of fixed points of the meromorphic function $g(z)$, and $\sigma(g(z))$ denotes the order of growth of $g(z)$.

1. Introduction and results. We assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [10, 13, 15, 16]. In addition, we use $\sigma(f(z))$ to denote the order of growth of a meromorphic $f(z)$; and $\lambda(f(z))$ and $\lambda(1/f(z))$ to denote, respectively, the exponents of convergence of zeros and of poles of $f(z)$. We also use $\tau(f(z))$ to denote the exponent of convergence of fixed points of $f(z)$, which is defined as

$$\tau(f(z)) = \limsup_{r\to\infty} \frac{\log N\big(r, \frac{1}{f(z)-z}\big)}{\log r}.$$

Fixed points are an important topic in the theory of meromorphic functions. Bergweiler and Pang [3] proved the following theorem.

THEOREM A (see [3]). *Let f be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. Suppose that all but finitely many zeros and poles of f are multiple. Then $f' - R$ has infinitely many zeros.*

When $R = z$, Theorem A shows that $f'(z)$ has infinitely many fixed points under the assumption of the theorem.

Recently, a number of articles (e.g. [1, 2, 4–6, 8, 9, 11–14]) focus on complex difference equations and difference analogues of Nevanlinna's theory.

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The functions $f_1(z) = e^z + z$, $f_2(z) = e^z + z - 1$, and $f_3(z) = e^z + \frac{1}{2}z^2$ have the property that $f_1(z)$, $f_2(z+1) = ee^z + z$ and $\Delta_{2\pi i} f_3(z) = f_3(z + 2\pi i) - f_3(z) = 2\pi iz - 2\pi^2$ each have only finitely many fixed points. Even for meromorphic functions of small growth, Chen and Shon [4] showed that there exists a meromorphic function f_0 such that $\sigma(f_0) < 1$ and $\Delta_c f_0(z) = f_0(z+c) - f_0(z)$ has only finitely many fixed points. They also proved

Theorem B (see [4]). *Let $\phi(r)$ be a positive non-decreasing function on $[1,\infty)$ which satisfies $\lim_{r\to\infty}\phi(r) = \infty$. Then there exists a transcendental meromorphic function f with*

$$\limsup_{r\to\infty}\frac{T(r,f)}{r} < \infty \quad \textit{and} \quad \liminf_{r\to\infty}\frac{T(r,f)}{\phi(r)\log r} < \infty,$$

such that $g(z) = \Delta f(z) = f(z+1) - f(z)$ has only one fixed point and satisfies

$$\limsup_{r\to\infty}\frac{T(r,g)}{\phi(r)\log r} < \infty.$$

For a meromorphic function $f(z)$, its divided difference $\frac{f(z+c)-f(z)}{f(z)}$ may also have only finitely many fixed points: for example, if $f(z) = ze^z$ then $\frac{f(z+1)-f(z)}{f(z)} = \frac{(z+1)e-z}{z}$ has only finitely many fixed points. Chen and Shon [5] obtained the following results.

Theorem C (see [5]). *Let $c \in \mathbb{C}\setminus\{0\}$ be a constant and f be a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant, and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Suppose that $p(z)$ is a nonconstant polynomial. Then*

$$G(z) = \frac{f(z+c) - f(z)}{f(z)} - p(z)$$

has infinitely many zeros.

From Theorem C, we easily see that under the assumptions of Theorem C, the divided difference $G_1(z) = \frac{f(z+c)-f(z)}{f(z)}$ has infinitely many fixed points. The example $f(z) = ze^z$ shows that the result of Theorem C is sharp.

However, we find that the function $f(z) = e^z + z$ has no fixed point, but $f(z+1) = ee^z + z + 1$ and $\Delta_1 f(z) = f(z+1) - f(z) = (e-1)e^z + 1$ each have infinitely many fixed points. Thus, it is natural to ask about the relationships between fixed points of a meromorphic function $f(z)$ and its shift $f(z+c)$ and its difference $\Delta_c f(z) = f(z+c) - f(z)$.

In this article, we prove the following.

Theorem 1.1. *Let $f(z)$ be a finite order meromorphic function such that $\lambda(1/f(z)) < \sigma(f(z))$, and let $c \in \mathbb{C}\setminus\{0\}$ be a constant such that*

$f(z+c) \not\equiv f(z)+c$. *Set* $\Delta_c f(z) = f(z+c) - f(z)$. *Then*

$$\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \sigma(f(z)),$$
$$\max\{\tau(f(z)), \tau(f(z+c))\} = \sigma(f(z)),$$
$$\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \sigma(f(z)).$$

REMARK 1.1. (i) By Theorem 1.1, if $f(z)$ and c satisfy the assumptions of the theorem, then at least two of $\tau(f(z))$, $\tau(f(z+c))$ and $\tau(\Delta_c f(z))$ are equal to $\sigma(f(z))$.

(ii) Generally, one might think that $\tau(f(z)) = \tau(f(z+c))$ for a finite order meromorphic function $f(z)$. But in fact, generally,

$$\tau(f(z)) \neq \tau(f(z+c)).$$

For example, $f_1(z) = e^z + z$ satisfies $\tau(f_1(z)) = 0$ and $\tau(f_1(z+1)) = 1$ (where $f_1(z+1) = ee^z + z + 1$). This shows that

$$\tau(f_1(z)) \neq \tau(f_1(z+1)).$$

Similarly,

$$\tau(\Delta_1 f_1(z)) = 1 \neq \tau(f_1(z)) = 0;$$

and $f_2(z) = e^z + z - 2$ satisfies

$$\tau(\Delta_2 f(z)) = 1 \neq \tau(f_2(z+2)) = 0.$$

So an obvious question to ask is what conditions guarantee that

$$\tau(f(z)) = \tau(f(z+c)) = \tau(\Delta_c f(z)) = \sigma(f(z)).$$

The following two theorems answer this question.

THEOREM 1.2. *Let* $f(z)$ *be a finite order meromorphic function such that* $\lambda(1/f(z)) < \sigma(f(z))$, *and* $c \in \mathbb{C}\backslash\{0\}$ *be a constant such that* $f(z+c) \not\equiv f(z)$. *If* $f(z)$ *has a Borel exceptional value* $d \in \mathbb{C}$, *then*

$$\tau(f(z)) = \tau(f(z+c)) = \tau(\Delta_c f(z)) = \sigma(f(z)).$$

THEOREM 1.3. *Let* $f(z)$ *be a finite order meromorphic function such that* $\lambda(1/f(z)) < \sigma(f(z))$, *and let* $c \in \mathbb{C} \setminus \{0\}$ *be a constant such that* $f(z+c) \not\equiv f(z)$. *If all but finitely many zeros of* $f(z)$ *are multiple, then*

$$\tau(f(z)) = \tau(f(z+c)) = \sigma(f(z)).$$

REMARK 1.2. Theorem 1.1 fails if we replace "fixed points" with "zeros". For example, the function $f_2(z) = e^z$ has no zero, and $\Delta_1 f_2(z) = (e-1)e^z$ and $f_2(z+1) = e^{z+1}$ have no zero either.

REMARK 1.3. The condition "$f(z+c) \not\equiv f(z)+c$" cannot be omitted in Theorem 1.1. For example, for $f(z) = e^z + z$, both $f(z)$ and $\Delta_{2\pi i} f(z) = f(z+2i\pi) - f(z) = 2i\pi$ have only finitely many fixed points, and $c = 2\pi i$ satisfies $f(z+c) \equiv f(z) + c$.

But we do not know whether the condition "$\lambda(1/f(z)) < \sigma(f(z))$" may be omitted.

2. Proof of Theorem 1.1. We need the following lemmas.

LEMMA 2.1 (see [6, 9]). *Let $f(z)$ be a meromorphic function with $\sigma(f(z)) = \sigma < \infty$, and let c be a nonzero constant. Then for each ε $(0 < \varepsilon < 1)$,*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

LEMMA 2.2 (see [6, 9]). *Let $f(z)$ be a meromorphic function such that $\lambda(1/f(z)) = \lambda < \infty$ and let $\eta \neq 0$ be fixed. Then for each ε $(0 < \varepsilon < 1)$,*

$$N(r, f(z+\eta)) = N(r, f(z)) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Proof of Theorem 1.1. **First**, we prove that $\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \sigma(f(z))$. Suppose that $\tau(f(z)) < \sigma(f(z))$. Since z_0 is a pole of $f(z)-z$ if and only if z_0 is a pole of $f(z)$, we see that $\lambda(1/f(z)-z) = \lambda(1/f(z)) < \sigma(f(z))$. Thus, since $f(z)$ is of finite order, $f(z)-z$ can be written as

$$f(z) - z = z^s \frac{p_1(z)}{q_1(z)} e^{h(z)} = \frac{p(z)}{q(z)} e^{h(z)} = F(z)e^{h(z)}, \tag{2.1}$$

where: $h(z)$ is a nonconstant polynomial with $\deg h(z) = \sigma(f(z))$; s is an integer; if $s \geq 0$ then $p(z) = z^s p_1(z)$, $q(z) = q_1(z)$; if $s < 0$ then $p(z) = p_1(z)$, $q(z) = z^{-s} q_1(z)$, $p_1(z)$ and $q_1(z)$ are canonical products (or polynomials) formed by nonzero zeros and poles of $f(z)-z$ respectively; and $F(z) = p(z)/q(z)$, so that

$$\lambda(p(z)) = \sigma(p(z)) = \lambda(f(z)-z) = \tau(f(z)) < \sigma(f(z)), \tag{2.2}$$

$$\lambda(q(z)) = \sigma(q(z)) = \lambda\left(\frac{1}{f(z)-z}\right) = \lambda\left(\frac{1}{f(z)}\right) < \sigma(f(z)). \tag{2.3}$$

So,

$$\sigma(F(z)) = \max\{\sigma(p(z)), \sigma(q(z))\} < \sigma(f(z)). \tag{2.4}$$

Set

$$g(z) = \Delta_c f(z) - z = f(z+c) - f(z) - z. \tag{2.5}$$

Now we only need to prove $\lambda(g(z)) = \sigma(f(z))$. Substituting (2.1) into (2.5), we obtain

$$g(z) = F(z+c)e^{h(z+c)} - F(z)e^{h(z)} - (z-c) = E(z)e^{h(z)} - (z-c) \tag{2.6}$$

where

$$E(z) = F(z+c)e^{h(z+c)-h(z)} - F(z). \tag{2.7}$$

If $E(z) \equiv 0$, then by (2.1) and (2.7), we have $F(z+c)e^{h(z+c)} = F(z)e^{h(z)}$ and

$$f(z+c) = F(z+c)e^{h(z+c)} + z + c = F(z)e^{h(z)} + z + c = f(z) + c,$$

that is,

$$f(z+c) \equiv f(z) + c,$$

contrary to assumption. Thus, $E(z) \not\equiv 0$.

By Lemma 2.1, we see that

$$T(r, F(z+c)) = T(r, F(z)) + S(r, F(z)). \tag{2.8}$$

By (2.4), (2.7), (2.8) and $\deg[h(z+c) - h(z)] = \deg h(z) - 1$, we find that

$$\sigma(E(z)) < \deg h(z) = \sigma(g(z)) = \sigma(f(z)). \tag{2.9}$$

Now, by (2.6) and (2.9), $g(z)$ is of regular growth and $\sigma(g(z)) = \sigma(e^{h(z)})$, so that

$$N(r, g(z)) = N(r, E(z)) = o\{T(r, g(z))\}. \tag{2.10}$$

By (2.6), we obtain

$$g'(z) = E(z)e^{h(z)}\left[\frac{E'(z)}{E(z)} + h'(z)\right] - 1. \tag{2.11}$$

By (2.11), we see that $\frac{E'(z)}{E(z)} + h'(z) \not\equiv 0$. In fact, if $\frac{E'(z)}{E(z)} + h'(z) \equiv 0$, then $g'(z) \equiv -1$, a contradiction. So we obtain

$$\begin{aligned} N\left(r, \frac{1}{g'(z) - (-1)}\right) &= N\left(r, \frac{1}{E(z)\left(\frac{E'(z)}{E(z)} + h'(z)\right)}\right) \\ &\le T\left(r, E(z)\left(\frac{E'(z)}{E(z)} + h'(z)\right)\right). \end{aligned} \tag{2.12}$$

By (2.6) and (2.9), we see that $T(r, g(z)) = T(r, e^{h(z)}) + S(r, g(z))$. Since $h(z)$ is a polynomial, (2.9) yields

$$T\left(r, E(z)\left(\frac{E'(z)}{E(z)} + h'(z)\right)\right) = S(r, g(z)). \tag{2.13}$$

Hence, by (2.12) and (2.13),

$$N\left(r, \frac{1}{g'(z) - (-1)}\right) = S(r, g(z)). \tag{2.14}$$

By the Milloux inequality, (2.10) and (2.14),

$$\begin{aligned} T(r, g(z)) &\le N(r, g(z)) + N\left(r, \frac{1}{g(z)}\right) \\ &\quad + N\left(r, \frac{1}{g'(z) - (-1)}\right) + S(r, g(z)) \\ &= N\left(r, \frac{1}{g(z)}\right) + S(r, g(z)). \end{aligned} \tag{2.15}$$

By (2.15), we obtain $\lambda(g(z)) = \sigma(g(z)) = \sigma(f(z))$. Hence, $\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \sigma(f(z))$.

Secondly, we prove $\max\{\tau(f(z)),\tau(f(z+c))\}=\sigma(f(z))$.

Suppose that $\tau(f(z))<\sigma(f(z))$. Then (2.1)–(2.4) hold. Set

$$g_1(z)=f(z+c)-z=F(z+c)e^{h(z+c)}+c. \tag{2.16}$$

Then $\sigma(g_1(z))=\sigma(f(z+c))=\sigma(f(z))$. By Lemma 2.1 and (2.4),

$$\begin{aligned} T(r,F(z+c)) &= T(r,F(z))+S(r,F(z)) \\ &= S(r,g_1(z))=o\{T(r,g_1(z)\}. \end{aligned} \tag{2.17}$$

By (2.4), (2.16) and (2.17), $g_1(z)$ is of regular growth. Thus, we deduce that

$$N(r,g_1(z))=N(r,F(z+c))\le T(r,F(z))=o\{T(r,g_1(z))\} \tag{2.18}$$

and

$$\begin{aligned} N\left(r,\frac{1}{g_1(z)-c}\right) &= N\left(r,\frac{1}{F(z+c)}\right)\le T(r,F(z+c)) \\ &= T(r,F(z))+S(r,F(z))=o\{T(r,g_1(z)\}. \end{aligned} \tag{2.19}$$

By the second fundamental theorem, (2.16), (2.18) and (2.19),

$$\begin{aligned} T(r,g_1(z)) &\le N(r,g_1(z))+N\left(r,\frac{1}{g_1(z)}\right) \\ &\quad +N\left(r,\frac{1}{g_1(z)-c}\right)+S(r,g(z)) \\ &= N\left(r,\frac{1}{g_1(z)}\right)+S(r,g_1(z)). \end{aligned} \tag{2.20}$$

By (2.20), we obtain $\lambda(g_1(z))=\sigma(g_1(z))=\sigma(f(z))$. Hence, $\max\{\tau(f(z)),\tau(f(z+c))\}=\sigma(f(z))$.

Thirdly, we prove $\max\{\tau(\Delta_c f(z)),\tau(f(z+c))\}=\sigma(f(z))$. Because $\sigma(f(z+c))=\sigma(f(z))=\sigma$ and $\lambda(1/f(z))<\sigma(f(z))$, we see that $N(r,f(z))$ is of order σ_1 $(<\sigma)$. By Lemma 2.2, we obtain

$$N(r,f(z+c))=N(r,f(z))+O(r^{\sigma_1-1+\varepsilon})+O(\log r). \tag{2.21}$$

This gives

$$\lambda\left(\frac{1}{f(z+c)-z}\right)=\lambda\left(\frac{1}{f(z+c)}\right)=\lambda(1/f(z))=\sigma_1<\sigma(f(z)).$$

Now suppose that $\tau(f(z+c))<\sigma(f(z))$. Since $\lambda\left(\frac{1}{f(z+c)-z}\right)<\sigma(f(z))$, we see that $f(z+c)-z$ can be written as

$$f(z+c)-z=\frac{p^*(z)}{q^*(z)}e^{h^*(z)}=F^*(z)e^{h^*(z)}, \tag{2.22}$$

where $h^*(z)$ is a nonconstant polynomial with $\deg h^*(z)=\sigma(f(z+c))$,

$F^*(z) = p^*(z)/q^*(z)$, and $p^*(z)$ and $q^*(z)$ are entire functions such that

$$\begin{aligned}\lambda(p^*(z)) = \sigma(p^*(z)) &= \lambda(f(z+c) - z)\\ &= \tau(f(z+c)) < \sigma(f(z+c)),\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\lambda(q^*(z)) = \sigma(q^*(z)) &= \lambda\left(\frac{1}{f(z+c)-z}\right)\\ &= \lambda\left(\frac{1}{f(z+c)}\right) < \sigma(f(z+c)).\end{aligned} \tag{2.24}$$

So,

$$\sigma(F^*(z)) = \max\{\sigma(p^*(z)), \sigma(q^*(z))\} < \sigma(f(z+c)). \tag{2.25}$$

Thus, by (2.22) we obtain

$$\Delta_c f(z) = f(z+c) - f(z) = F^*(z)e^{h^*(z)} - F^*(z-c)e^{h^*(z-c)} + c. \tag{2.26}$$

Set $g_2(z) = \Delta_c f(z) - z$. Then

$$g_2(z) = E^*(z)e^{h^*(z)} + c - z, \tag{2.27}$$

where $E^*(z) = F^*(z) - F^*(z-c)e^{h^*(z-c)-h^*(z)}$. As $\deg(h^*(z-c) - h^*(z)) = \deg h^*(z) - 1$ and (2.27), we see that $\sigma(E^*(z)) < \sigma(g_2(z)) = \sigma(e^{h^*(z)}) = \sigma(f(z))$. Using the same method as in the proof of the first step, we deduce that $\lambda(g_2(z)) = \sigma(g_2(z)) = \sigma(f(z))$.

Hence, $\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \sigma(f(z))$. ∎

3. Proof of Theorem 1.2.

We need the following lemma.

LEMMA 3.1 (see [7, pp. 69–70] or [16, pp. 79–80]). *Suppose that $n \geq 2$ and let $f_j(z)$, $j=1,\dots,n$, be meromorphic functions and $g_j(z)$, $j=1,\dots,n$, be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z)\exp\{g_j(z)\} \equiv 0$;
(ii) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not constant;*
(iii) *when* $1 \leq j \leq n$ *and* $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$, $j = 1, \dots, n$.

Proof of Theorem 1.2. Since $f(z)$ has a Borel exceptional value $d \in \mathbb{C}$ and $\lambda(1/f(z)) < \sigma(f(z))$, we see that $f(z)$ can be written as

$$f(z) = d + \frac{p(z)}{q(z)}e^{h_1(z)} = d + F_1(z)e^{h_1(z)}, \tag{3.1}$$

where $h_1(z)$ is a nonconstant polynomial with $\deg h_1(z) = \sigma(f(z))$, $F_1(z) = p(z)/q(z)$, and $p(z), q(z)$ are nonzero entire functions such that

$$\lambda(p(z)) = \sigma(p(z)) = \lambda(f(z) - d) < \sigma(f(z)), \tag{3.2}$$

$$\lambda(q(z)) = \sigma(q(z)) = \lambda\left(\frac{1}{f(z) - d}\right) = \lambda(1/f(z)) < \sigma(f(z)). \tag{3.3}$$

So,

$$\sigma(F_1(z)) = \max\{\sigma(p(z)), \sigma(q(z))\} < \sigma(f(z)). \tag{3.4}$$

First, we prove that $\tau(f(z)) = \sigma(f(z))$. Suppose that $\tau(f(z)) < \sigma(f(z))$. Since z_0 is a pole of $f(z) - z$ if and only if it is a pole of $f(z)$, we see that $\lambda(1/f(z) - z) = \lambda(1/f(z)) < \sigma(f(z))$. Thus, $f(z) - z$ can be written as

$$f(z) - z = \frac{a(z)}{b(z)} e^{h_2(z)} = F_2(z) e^{h_2(z)} \tag{3.5}$$

where $h_2(z)$ is a nonconstant polynomial with $\deg h_2(z) = \sigma(f(z))$, $a(z), b(z)$ are nonzero entire functions, and $F_2(z) = a(z)/b(z)$, so that

$$\lambda(a(z)) = \sigma(a(z)) = \lambda(f(z) - z) = \tau(f(z)) < \sigma(f(z)), \tag{3.6}$$

$$\lambda(b(z)) = \sigma(b(z)) = \lambda\left(\frac{1}{f(z) - z}\right) = \lambda(1/f(z)) < \sigma(f(z)). \tag{3.7}$$

So, by (3.6) and (3.7), we have

$$\sigma(F_2(z)) = \max\{\sigma(a(z)), \sigma(b(z))\} < \sigma(f(z)). \tag{3.8}$$

By (3.1) and (3.5), we obtain

$$F_1(z) e^{h_1(z)} - F_2(z) e^{h_2(z)} + F_0(z) e^{h_0(z)} = 0, \tag{3.9}$$

where $F_0(z) = d - z, h_0(z) = 0$.

If $\deg(h_1(z) - h_2(z)) = \deg h_1(z)$, then since $e^{h_1(z)}, e^{h_2(z)}, e^{h_1(z) - h_2(z)}$ are of regular growth, by (3.4) and (3.8) we obtain

$$T(r, F_j(z)) = o\{T(r, e^{h_k(z) - h_s(z)}\}, \qquad j = 0, 1, 2;\ 0 \le s < k \le 2. \tag{3.10}$$

Now Lemma 3.1, (3.9) and (3.10) yield

$$F_1(z) \equiv F_2(z) \equiv F_0(z) \equiv d - z \equiv 0,$$

a contradiction.

If $\deg(h_1(z) - h_2(z)) < \deg h_1(z)$, then by (3.9) we obtain

$$e^{h_1(z)} (F_1(z) - F_2(z) e^{h_2(z) - h_1(z)}) + d - z = 0. \tag{3.11}$$

If $F_1(z) - F_2(z) e^{h_2(z) - h_1(z)} \equiv 0$, then $d - z \equiv 0$, a contradiction; if $F_1(z) - F_2(z) e^{h_2(z) - h_1(z)} \not\equiv 0$, then the order of growth of the left side of (3.11) is $\deg h_1(z) = \sigma(f(z))$, also a contradiction. Hence $\tau(f(z)) = \sigma(f(z))$.

Secondly, we prove that $\tau(f(z + c)) = \sigma(f(z))$. By Lemma 2.1,

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f(z)).$$

Using a similar method to the proof of the first step, we conclude that $\tau(f(z+c)) = \sigma(f(z))$.

Thirdly, we prove that $\tau(\Delta_c f(z)) = \sigma(f(z))$. Suppose that $\tau(\Delta_c f(z)) < \sigma(f(z))$. As in the proof of the first step, $\Delta_c f(z) - z$ can be written as

$$\Delta_c f(z) - z = f(z+c) - f(z) - z = F_3(z)e^{h_3(z)} \tag{3.12}$$

where $h_3(z)$ is a polynomial with $\deg h_3(z) \le \sigma(f(z))$ and $F_3(z)$ ($\not\equiv 0$) is a meromorphic function such that

$$\sigma(F_3(z)) < \sigma(f(z)). \tag{3.13}$$

Substituting (3.1) into (3.12), we obtain

$$F_1(z+c)e^{h_1(z+c)} - F_1(z)e^{h_1(z)} - F_3(z)e^{h_3(z)} - z = 0,$$

that is,

$$e^{h_1(z)}(F_1(z+c)e^{h_1(z+c)-h_1(z)} - F_1(z)) - F_3(z)e^{h_3(z)} - z = 0. \tag{3.14}$$

We claim that

$$F_1(z+c)e^{h_1(z+c)-h_1(z)} - F_1(z) \not\equiv 0. \tag{3.15}$$

In fact, if $F_1(z+c)e^{h_1(z+c)-h_1(z)} - F_1(z) \equiv 0$, then by (3.1), we obtain $f(z+c) \equiv f(z)$, contrary to assumption.

From (3.4), (3.13)–(3.15) and $\deg(h_1(z+c) - h_1(z)) = \deg h_1(z) - 1$, we obtain

$$\deg h_3(z) = \deg h_1(z) = \sigma(f(z)) = k. \tag{3.16}$$

If $\deg(h_3(z) - h_1(z)) = k$, then by Lemma 3.1 and (3.14),

$$F_1(z+c)e^{h_3(z+c)-h_1(z)} - F_1(z) \equiv F_3(z) \equiv z \equiv 0,$$

a contradiction.

If $\deg(h_3(z) - h_1(z)) < k$, then (3.14) can be rewritten as

$$e^{h_1(z)}\big(F_1(z+c)e^{h_1(z+c)-h_1(z)} - F_1(z) - F_3(z)e^{h_3(z)-h_1(z)}\big) - z = 0, \tag{3.17}$$

and using a similar method to the proof of the first step, we obtain a contradiction.

Hence, $\tau(\Delta_c f(z)) = \sigma(f(z))$. ∎

4. Proof of Theorem 1.3.

We need the following lemma.

LEMMA 4.1. *Let $f(z)$ be a meromorphic function with $\lambda(f(z)) = \lambda < \infty$ and let $\eta \neq 0$ be fixed. Then for each ε ($0 < \varepsilon < 1$),*

$$N\left(r, \frac{1}{f(z+\eta)}\right) = N\left(r, \frac{1}{f(z)}\right) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Proof. Use the same method as in the proof of Lemma 2.1. ∎

Proof of Theorem 1.3. We only prove that $\tau(f(z+c)) = \sigma(f(z))$ since the method of the proof of $\tau(f(z)) = \sigma(f(z))$ is the same.

If $\lambda(f(z)) < \sigma(f(z))$, then $f(z)$ has Borel exceptional value 0, so that by Theorem 1.2, we see that $\tau(f(z+c)) = \sigma(f(z))$.

Now we suppose that $\lambda(1/f(z)) < \lambda(f(z)) = \sigma(f(z))$. By Lemmas 2.2 and 4.1,

$$\lambda\bigg(\frac{1}{f(z+c)}\bigg) < \lambda(f(z+c)) = \sigma(f(z)). \tag{4.1}$$

Suppose that $\tau(f(z+c)) < \sigma(f(z))$. Since $\lambda\big(\frac{1}{f(z+c)-z}\big) = \lambda\big(\frac{1}{f(z+c)}\big) < \sigma(f(z))$, we see that, as in the proof of Theorem 1.1, $f(z+c)-z$ can be written as

$$f(z+c) - z = F(z)e^{h(z)}, \tag{4.2}$$

where $h(z)$ is a nonconstant polynomial and $F(z)$ is a meromorphic function such that $\sigma(F(z)) < \sigma(f(z)) = \deg h(z)$. Thus, by (4.2),

$$\begin{aligned} f'(z+c) &= F(z)e^{h(z)}\bigg(\frac{F'(z)}{F(z)} + h'(z)\bigg) + 1 \\ &= f(z+c)\bigg(\frac{F'(z)}{F(z)} + h'(z)\bigg) - z\bigg(\frac{F'(z)}{F(z)} + h'(z)\bigg) + 1, \end{aligned}$$

so that

$$\frac{f'(z+c)}{f(z+c)} = \bigg(\frac{F'(z)}{F(z)} + h'(z)\bigg) - \bigg(z\frac{F'(z)}{F(z)} + zh'(z) - 1\bigg)\frac{1}{f(z+c)}. \tag{4.3}$$

We claim that

$$z\frac{F'(z)}{F(z)} + zh'(z) - 1 \not\equiv 0.$$

In fact, otherwise $\frac{F'(z)}{F(z)} + h'(z) = \frac{1}{z}$. By integrating, we obtain $F(z)e^{h(z)} = \alpha z$, where α ($\neq 0$) is a constant. This contradicts (4.2).

Since $\sigma(F(z)) < \sigma(f(z)) = \lambda(f(z+c))$ and $h'(z)$ is a polynomial, we see that

$$\sigma\bigg(\frac{F'(z)}{F(z)} + h'(z)\bigg) < \sigma(f(z)), \quad \sigma\bigg(z\frac{F'(z)}{F(z)} + zh'(z) - 1\bigg) < \sigma(f(z))$$

and there exists a point z_0 which is a multiple zero of $f(z+c)$, and is neither a zero of $z\frac{F'(z)}{F(z)} + zh'(z) - 1$, nor a pole of $\frac{F'(z)}{F(z)} + h'(z)$. Thus, the right side of (4.3) has a multiple pole at $z = z_0$, but the left side of (4.3) has only a simple pole at $z = z_0$, is a contradiction.

Hence $\tau(f(z+c)) = \sigma(f(z))$. ∎

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