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An alternative proof of Petty's theorem on equilateral sets

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Abstract. The main goal of this paper is to provide an alternative proof of the following theorem of Petty: in a normed space of dimension at least three, every 3-element equilateral set can be extended to a 4-element equilateral set. Our approach is based on the result of Kramer and Németh about inscribing a simplex into a convex body. To prove the theorem of Petty, we shall also establish that for any three points in a normed plane, forming an equilateral triangle of side p, there exists a fourth point, which is equidistant to the given points with distance not larger than p. We will also improve the example given by Petty and obtain the existence of a smooth and strictly convex norm in $\mathbb{R}^n$ for which there exists a maximal 4-element equilateral set. This shows that the theorem of Petty cannot be generalized to higher dimensions, even for smooth and strictly convex norms.

1. Introduction. Let X be a real n-dimensional vector space equipped with a norm $\|\cdot\|$. We say that a set $S \in X$ is *equilateral* if there is a $p > 0$ such that $\|x - y\| = p$ for all $x, y \in S$, $x \neq y$. We then say that S is a *p-equilateral set*. Let us denote the by $e(X)$ *equilateral dimension* of X, defined as the maximal cardinality of an equilateral set in X. For many classical spaces (like ℓ_p^n) determining the equilateral dimension is an open problem. It is not difficult to show that the equilateral dimension of the n-dimensional space equipped with the Euclidean norm is $n + 1$, and it is 2^n for the ℓ_∞ norm. It is known (see [11] and [13]) that 2^n is, in fact, an upper bound for the equilateral dimension of any normed space X of dimension n. Moreover, the bound is attained if and only if there exists a linear isometry between X and ℓ_∞^n.

It is believed that $n + 1$ is similarly a lower bound for the equilateral dimension of any n-dimensional normed space. For $n = 1$ and $n = 2$ this is an easy exercise. For $n = 3$ it has been proved by Petty and in the case $n = 4$ quite recently by Makeev [8]. For $n \geq 5$ only weaker estimates on the size of a maximal equilateral set are known (for the best bound to date see [15]).

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In the three-dimensional setting even more can be demonstrated. We have the following

THEOREM 1.1 (Petty [11]). *Let X be a real 3-dimensional vector space, equipped with a norm $\|\cdot\|$. Assume that $a, b, c \in X$ form a p-equilateral set in the norm $\|\cdot\|$. Then there exists $d \in X$ such that*

$$\|d-a\| = \|d-b\| = \|d-c\| = p.$$

In other words, every equilateral set of three elements can be extended to an equilateral set of four elements.

The main goal of this paper is to give an alternative proof of Petty's theorem (see Section 4). In his original reasoning Petty used a two-dimensional result called a monotonicity lemma (see Section 3.5 of [10]). Our approach will be based on a result of Kramer and Németh (Theorem 2.1).

We will need to investigate the properties of the circumcircle of an equilateral set in the plane. We shall prove

PROPOSITION 1.2. *Let $\|\cdot\|$ be a norm on the plane. Assume $a, b, c \in \mathbb{R}^2$ form a p-equilateral set in this norm. Then there exists $s \in \mathbb{R}^2$ such that*

$$\|a-s\| = \|b-s\| = \|c-s\| \leq p.$$

In other words, on the plane every equilateral set of size 3 has a circumcircle of radius not greater than the common distance.

This result has also appeared in [9, Lemma 2.4 and Theorem 3.1] in the case of a strictly convex norm, but we give a simpler proof.

In Section 5, we study an extension of the example given by Petty in [11]. For $n \geq 4$, he has constructed a norm in $\mathbb{R}^n$ and a 4-element equilateral set which cannot be extended to a 5-element equilateral set. In particular, Theorem 1.1 cannot be generalized to higher dimensions. However, the norm given by Petty is not smooth and not strictly convex. It is therefore natural to ask whether the smoothness or strict convexity of the norm would enable us to generalize Theorem 1.1 to higher dimensions. We answer this question negatively in

THEOREM 1.3. *For every $n \geq 4$ there exist a smooth and strictly convex norm $\|\cdot\|$ in $\mathbb{R}^n$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}^n$ forming a p-equilateral set such that there is no a_5 with*

$$\|a_5-a_1\| = \|a_5-a_2\| = \|a_5-a_3\| = \|a_5-a_4\| = p.$$

This improves the example given by Petty.

For a survey on equilateral sets in finite-dimensional normed spaces, see [14].

2. Preliminaries. We shall recall some standard definitions of Banach space theory and convex geometry. A convex and compact set $C \subset \mathbb{R}^n$ is called a *convex body* if its interior is non-empty. A convex body C is *symmetric* if it is symmetric with respect to the origin. It is *smooth* if every point on the boundary of C lies on exactly one supporting hyperplane of C. A convex body C is *strictly convex* if it does not contain a segment on the boundary. The unit ball of any norm in $\mathbb{R}^n$ is a symmetric convex body and vice versa—every symmetric convex body is the unit ball of exactly one norm. We say that a norm $\|\cdot\|$ is smooth or strictly convex if so is the corresponding unit ball. A sphere in the given norm on the plane is called a *circle*.

A set $P \subset \mathbb{R}^n$ is called a *convex polytope* if it is the convex hull of a finite number of points, i.e. $P = \operatorname{conv}\{p_1, \ldots, p_m\}$. If there does not exist a proper subset S of $\{p_1, \ldots, p_m\}$ such that $P = \operatorname{conv} S$, then the points p_i are the *vertices* of P. A convex polytope $P \subset \mathbb{R}^n$ with exactly $n+1$ vertices is called a *simplex*. A simplex is *non-degenerate* if it is not contained in an affine subspace of dimension $n-1$. A (positive) *homothet* of a set $A \subset \mathbb{R}^n$ is $\lambda A + v = \{\lambda a + v : a \in A\}$, where $\lambda > 0$ and $v \in \mathbb{R}^n$ are arbitrary.

We shall use the following theorem of Kramer and Németh, which gives sufficient conditions on a convex body $C \subset \mathbb{R}^n$ for every non-degenerate simplex of $\mathbb{R}^n$ to have a homothet inscribed in C.

THEOREM 2.1 (Kramer & Németh [6]). *Let C be a smooth and strictly convex body in $\mathbb{R}^n$ and let $p_0, p_1, \ldots, p_n$ be the vertices of a non-degenerate simplex. Then there exist $z \in \mathbb{R}^n$ and $r > 0$ such that the points $z + rp_0$, $z + rp_1, \ldots, z + rp_n$ lie on the boundary of C.*

It is worth pointing out that while smoothness is necessary, strict convexity can in fact be dropped, as shown independently by Gromov [4] and Makeev [7]. For our purposes, however, the weaker version of the theorem is sufficient.

It is much harder to guarantee the uniqueness of an inscribed homothet. However, in the two-dimensional case we have the following

PROPOSITION 2.2. *Let C be a strictly convex body in the plane and let p_0, p_1, p_2 be the vertices of a non-degenerate triangle. Then there exists at most one pair of $z \in \mathbb{R}^2$ and $r > 0$ such that the points $z+rp_0, z+rp_1, z+rp_2$ lie on the boundary of C.*

Proof. An elementary proof can be found in [10, Section 3.2]. ∎

Combining these two results, we obtain

COROLLARY 2.3. *Let C be a smooth and strictly convex body in the plane. Then, for every non-degenerate triangle, there exists exactly one homothet of C passing through its vertices.*

REMARK 2.4. A generalization of Proposition 2.2 to higher dimensions does not hold in the following strong sense: if every simplex in $\mathbb{R}^n$ (where $n \geq 3$) has at most one homothet inscribed in a fixed convex body C, then C must be an ellipsoid. This characterization of finite-dimensional Hilbert space was established by Goodey [3].

To take advantage of the preceding results we have to assume smoothness and strict convexity of a norm. To reduce the general case to this setting, we will use a special kind of smooth and strictly convex approximation. The proof of Theorem 1.3 will also rely on this technique. We have the intuitive

PROPOSITION 2.5. *Assume that* $\|\cdot\|$ *is a norm in* $\mathbb{R}^n$ *and unit vectors* $\pm p_1, \ldots, \pm p_m$ *are the vertices of a symmetric convex polytope. Then for every* $\varepsilon > 0$ *there exists a smooth and strictly convex norm* $\|\cdot\|_0$ *in* $\mathbb{R}^n$ *such that*

$$(1-\varepsilon)\|x\|_0 \leq \|x\| \leq (1+\varepsilon)\|x\|_0$$

for every $x \in \mathbb{R}^n$, *and* $\|p_i\|_0 = 1$ *for* $i = 1, \ldots, m$.

Proof. See [2] for a much more general result. ∎

3. Circumcircle of an equilateral set in the plane. In this section we prove Proposition 1.2. Results from the preceding section are not required here. We begin with a simple

LEMMA 3.1. *Let* $\|\cdot\|$ *be a norm in the plane and let* $v \in \mathbb{R}^2$ *be a non-zero vector. Consider the mapping* f, *defined on the unit disc* B *of the norm* $\|\cdot\|$ *by*

$$f(x) = \max\{t \geq 0 : \|x + tv\| = 1\}.$$

Then f *is continuous and bounded by* $2/\|v\|$.

Proof. For any $x \in B$ and any $t > 2/\|v\|$, we have

$$\|\|x + tv\| \geq \|tv\| - \|x\| > 2 - 1 = 1.$$

This proves the boundedness statement.

To prove the continuity of f, let D be the diameter of B orthogonal to v and denote by $P : B \to D$ the orthogonal projection. It is clear that f factors as $f = f \circ P$ and it is therefore enough to check the continuity of the restriction $\tilde{f} = f|_D$.

Let $(x_n)_{n\in\mathbb{N}} \subset D$ be a sequence converging to $x \in D$. As $\tilde{f}$ is bounded, it is enough to check that every convergent subsequence of $(\tilde{f}(x_n))_{n\in\mathbb{N}}$ converges to $\tilde{f}(x)$. So, assume that $\lim_{k\to\infty} \tilde{f}(x_{n_k}) = y$ for some subsequence $(\tilde{f}(x_{n_k}))_{k\in\mathbb{N}}$. Since

$$\|x + yv\| = \lim_{k\to\infty} \|x_{n_k} + \tilde{f}(x_{n_k})v\| = 1,$$

we have $y \leq \tilde{f}(x)$, by the definition of f.

On the other hand, it is immediate to check that $\tilde{f}$ is a concave function. As concave functions defined on closed intervals are lower semicontinuous, we must have $y \geq \tilde{f}(x)$. Therefore $y = \tilde{f}(x)$ and the lemma is proved. ■

Proof of Proposition 1.2. We can suppose that $p = 1$ and denote the unit circle in the norm $\|\cdot\|$ by S. Define $f, g : S \to \mathbb{R}$ by

$$f(x) = \max\{t \geq 0 : \|x + t(c - a)\| = 1\},$$
$$g(x) = \max\{t \geq 0 : \|x + t(b - a)\| = 1\}.$$

By the preceding lemma, the mappings f and g are continuous and bounded by 2. At the same time, $f(a - c) = g(a - b) = 2$. Therefore $f(x) \geq g(x)$ for $x = a - c$ and $f(x) \leq g(x)$ for $x = a - b$.

The intermediate value theorem implies that on the closed arc of the unit circle between the points $a - b$ and $a - c$, there exists a point z such that $f(z) = g(z) = r$ for some $r \geq 0$ (see Figure 1). For that z we have

$$\|z + r(c - a)\| = \|z + r(b - a)\| = 1.$$

If $r > 0$, then the points $z, z + r(c - a), z + r(b - a)$ are the vertices of a non-degenerate triangle inscribed in the unit circle, which is a homothet of the triangle with vertices a, b, c. Therefore, we can circumscribe a circle of radius $1/r$ around the triangle with vertices a, b, c. It suffices to show that $r \geq 1$ (and in particular $r > 0$).

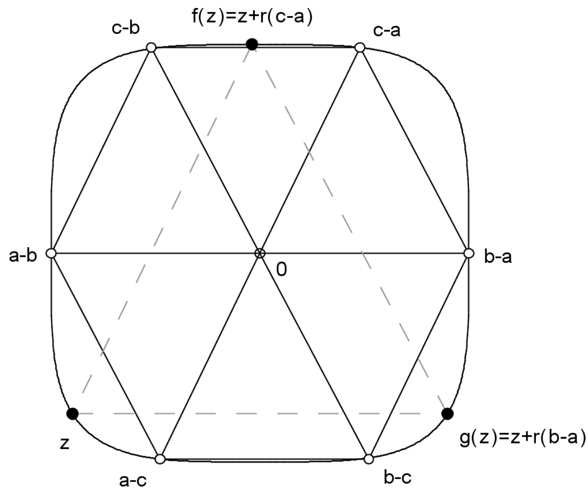

Fig. 1. Proof of Proposition 1.2

To do so, consider the parallelogram with vertices $0, c - b, a - b, a - c$. The points z and $z + r(c - a)$ lie outside this parallelogram and the segment connecting them is parallel to one of its sides (as $(c - b) - (a - b) = c - a$). Hence $r \geq 1$. ■

4. Proof of Petty's theorem. In this section we prove Theorem 1.1. We will only need the two- and three-dimensional cases of the following lemma, but we prove it in general form.

LEMMA 4.1. *If $p_0, p_1, \ldots, p_n$ are the vertices of a non-degenerate simplex in $\mathbb{R}^n$, then the points $p_i - p_j$, where $0 \le i \ne j \le n$, are the vertices of the convex polytope* $\operatorname{conv}\{p_i - p_j : 0 \le i \ne j \le n\}$.

Proof. For a contradiction suppose that some of these points can be written as a convex combination of the others. Without loss of generality, we can assume that it is $p_n - p_0$.

Set $q_i = p_i - p_0$ for $i = 0, 1, \ldots, n$. Because the simplex with vertices $p_0, p_1, \ldots, p_n$ is non-degenerate, the vectors q_i are linearly independent for $i = 1, \ldots, n$. We will show that if

$$q_n = \sum_{0 \le k \ne l \le n} t_{k,l}(p_k - p_l),$$

where $t_{k,l} \in [0,1]$ and $\sum_{0 \le k \ne l \le n} t_{k,l} = 1$, then $t_{n,0} = 1$ and $t_{k,l} = 0$ for $(k,l) \ne (n,0)$. We have

$$\begin{aligned} q_n &= \sum_{0 \le k \ne l \le n} t_{k,l}(p_k - p_l) = \sum_{0 \le k \ne l \le n} t_{k,l}(q_k - q_l) \\ &= \sum_{k=1}^{n} \Big(\sum_{\substack{0 \le l \le n \\ l \ne k}} t_{k,l} - \sum_{\substack{0 \le l \le n \\ l \ne k}} t_{l,k} \Big) q_k. \end{aligned}$$

Since the vectors $q_1, \ldots, q_n$ are linearly independent, comparing the coefficients of q_n gives

$$\sum_{l=0}^{n-1} t_{n,l} - \sum_{l=0}^{n-1} t_{l,n} = 1.$$

Because the numbers $t_{k,l}$ are non-negative and their sum is 1, we obtain

$$\sum_{l=0}^{n-1} t_{n,l} = 1 \quad \text{and} \quad t_{k,l} = 0 \text{ for } k \ne n.$$

If $t_{n,0} = 1$ and $t_{n,l} = 0$ for $1 \le l \le n-1$ the claim follows. So, assume that $t_{n,l} \ne 0$ for some $1 \le l \le n-1$. But then it is easy to see that the coefficient of q_l in the above combination is $-t_{n,l} \ne 0$, contrary to the linear independence of $q_1, \ldots, q_n$. This proves the lemma. ∎

With the results of the preceding sections at hand, we are ready to give an alternative proof of Petty's theorem.

Proof of Theorem 1.1. Suppose that $p = 1$. First we assume that the norm $\|\cdot\|$ is smooth and strictly convex. Let π be the affine plane containing a, b, c. In a similar fashion to Makeev [8], we will consider sections of the unit ball B with planes parallel to π and use the continuity of these cuts. All such sections which are not empty and not single-point are smooth and strictly convex bodies in the plane. From Corollary 2.3 we know that in all such sections we can inscribe exactly one homothet of the triangle with vertices a, b, c. To be more precise, let v be the unit vector perpendicular to π and denote $B_t = B \cap (\pi + tv)$. Let $t_1 < t_2$ be such that

$$\# B_t > 1 \ \Leftrightarrow \ t \in (t_1, t_2).$$

Corollary 2.3 implies that for every $t \in (t_1, t_2)$ there exists exactly one pair of $z \in \pi$ and $r > 0$ such that

$$\|z + ra + tv\| = \|z + rb + tv\| = \|z + rc + tv\| = 1.$$

Denote by $z : (t_1, t_2) \to \pi$ and $r : (t_1, t_2) \to (0, \infty)$ the resulting mappings. We shall verify that the mapping $(z, r) : (t_1, t_2) \to \pi \times (0, \infty)$ is continuous. Indeed, suppose that $t_n \to t$. It is easy to see that z and r are bounded, and therefore we can pick a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that $(z(t_{n_k}))_{k \in \mathbb{N}}$ and $(r(t_{n_k}))_{k \in \mathbb{N}}$ converge to some z' and r' respectively. It follows from the continuity of the norm that

$$\|z' + r'a + tv\| = \|z' + r'b + tv\| = \|z' + r'c + tv\| = 1,$$

and taking the uniqueness into account we conclude that $z' = z(t)$ and $r' = r(t)$. This proves our claim.

In particular, the mapping r is continuous. Let $s \in (t_1, t_2)$ be such that B_s is a section containing 0. Then B_s is a smooth, strictly convex and symmetric convex body obtained by restricting the norm $\|\cdot\|$ to the two-dimensional vector space parallel to π. By Theorem 1.2, it follows that $r(s) \geq 1$. Moreover, if $t \to t_1$, then $\operatorname{diam} B_t \to 0$. Therefore continuity implies that $r(t) = 1$ for some $t \in (t_1, t_2)$. Then

$$\|z(t) + tv + a\| = \|z(t) + tv + b\| = \|z(t) + tv + c\| = 1,$$

and hence $d = -(z(t) + tv)$ is the desired point. This completes the proof in the case of a smooth and strictly convex norm.

Now let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^3$. We shall reduce this case to the previous one by application of Lemma 4.1 and Proposition 2.5. As the points $a - b, b - c, c - a, b - a, c - b, a - c$ are the vertices of a symmetric convex hexagon, Proposition 2.5 implies that for every $n \in \mathbb{N}$ we can find a smooth and strictly convex norm $\|\cdot\|_n$ such that

$$\left(1 - \frac{1}{n}\right)\|x\|_n \leq \|x\| \leq \left(1 + \frac{1}{n}\right)\|x\|_n$$

for every $x \in \mathbb{R}^3$ and

$$\|a - b\|_n = \|b - c\|_n = \|c - a\|_n = 1.$$

We have already proved that for every $n \in \mathbb{N}$ we can find $d_n \in \mathbb{R}^3$ such that

$$\|d_n - a\|_n = \|d_n - b\|_n = \|d_n - c\|_n = 1.$$

It is clear that the sequence $(d_n)_{n\in\mathbb{N}}$ is bounded, so it contains some subsequence $(d_{n_k})_{k\in\mathbb{N}}$ convergent to a point $d \in \mathbb{R}^3$. From the inequalities

$$1 - \frac{1}{n_k} \le \|d_{n_k} - a\| \le 1 + \frac{1}{n_k}$$

it follows that $\|d - a\| = 1$ and analogously $\|d - b\| = \|d - c\| = 1$. ■

5. Existence of a smooth and strictly convex norm with a 4-element maximal equilateral set. In $\mathbb{R}^n$ we can introduce a norm $\|\cdot\|$ by $\|(x_1, x_2, \dots, x_n)\| = |x_1| + \sqrt{x_2^2 + x_3^2 + \dots + x_n^2}$. Petty has proved that for $n \ge 4$ it is possible to find a 4-element equilateral set in this normed space, which is maximal with respect to inclusion. In [16] Swanepoel and Villa have generalized this example to every space of the form $X \oplus_1 \mathbb{R}$, where X has at least one smooth point on the unit sphere. Spaces arising in this way are never smooth or strictly convex, however. In the same paper the authors have also proved that some of the ℓ_p^n spaces (with $n \ge 4$ and $1 < p < 2$) contain 5-element equilateral sets which are maximal with respect to inclusion. It remains to answer if a smooth and strictly convex space of dimension $n \ge 4$ can possess such a 4-element equilateral set. Using Proposition 2.5 we will obtain, in a non-constructive way, the existence of a space with this property.

Proof of Theorem 1.3. In $\mathbb{R}^n$ consider the ℓ_1 norm (which of course is neither smooth nor strictly convex), i.e. $\|(x_1, \dots, x_n)\| = |x_1| + \dots + |x_n|$. Let

$$\begin{aligned}
a_1 &= (1, 0, \dots, 0), \qquad a_2 = (-1, 0, \dots, 0),\\
a_3 &= \left(0, \frac{1}{n-1}, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}\right),\\
a_4 &= \left(0, -\frac{1}{2(n-1)}, -\frac{3}{2(n-1)}, -\frac{1}{n-1}, -\frac{1}{n-1}, \dots, -\frac{1}{n-1}\right).
\end{aligned}$$

It is immediate to check that the a_i's form a 2-equilateral set in the ℓ_1 norm. We will prove that there does not exist $x \in \mathbb{R}^n$ which extends this set to a 5-point equilateral set. Indeed, suppose that $x = (x_1, \dots, x_n)$ is such a point. From the equalities

$$\|x - a_1\| = \|x + a_1\| = 2$$

we conclude that $x_1 = 0$ and $|x_2| + |x_3| + \dots + |x_n| = 1$. Combining this with

$$\|x - a_2\| = \|x - a_3\| = 2,$$

we get

$$
\begin{aligned}
2 &= |x_2| + |x_3| + \cdots + |x_n| + 1 \\
&= \left|x_2 - \frac{1}{n-1}\right| + \left|x_3 - \frac{1}{n-1}\right| + \cdots + \left|x_n - \frac{1}{n-1}\right| \\
&= \left|x_2 + \frac{1}{2(n-1)}\right| + \left|x_3 + \frac{3}{2(n-1)}\right| + \left|x_4 + \frac{1}{n-1}\right| + \cdots + \left|x_n + \frac{1}{n-1}\right|.
\end{aligned}
$$

We shall show that this system of equations does not have a solution. Indeed, the usual triangle inequality easily implies that

$$
|x_2| + |x_3| + \cdots + |x_n| + 1 \geq \left|x_2 - \frac{1}{n-1}\right| + \left|x_3 - \frac{1}{n-1}\right| + \cdots + \left|x_n - \frac{1}{n-1}\right|
$$

and

$$
\begin{aligned}
|x_2| + |x_3| + \cdots + |x_n| + 1 \geq {}& \left|x_2 + \frac{1}{2(n-1)}\right| + \left|x_3 + \frac{3}{2(n-1)}\right| \\
&+ \left|x_4 + \frac{1}{n-1}\right| + \cdots + \left|x_n + \frac{1}{n-1}\right|.
\end{aligned}
$$

Moreover, in the inequality $|a| + |b| \geq |a+b|$ equality holds exactly when a and b are of the same sign. Therefore, the vector $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is the only possible solution of the system under consideration, but it is not a unit vector. This proves our claim.

It is not hard to check by hand the linear independence of the vectors $a_2 - a_1$, $a_3 - a_1$ and $a_4 - a_1$, which implies that a_1, a_2, a_3, a_4 are the vertices of a non-degenerate tetrahedron. By Lemma 4.1 we know that set $\{a_i - a_j : 1 \leq i \neq j \leq 4\}$ is the set of vertices of its convex hull. Applying Proposition 2.5, we can pick a sequence $\|\cdot\|_k$ of smooth and strictly convex norms such that

$$
\left(1 - \frac{1}{k}\right)\|x\|_k \leq \|x\| \leq \left(1 + \frac{1}{k}\right)\|x\|_k
$$

for every $x \in \mathbb{R}^n$ and $\|a_i - a_j\|_k = 2$ for $1 \leq i \neq j \leq 4$ and $k \in \mathbb{N}$.

Suppose that our assertion is not true. This means that for any smooth and strictly convex norm in $\mathbb{R}^n$ (where $n \geq 4$), every 4-point equilateral set can be extended to a 5-point equilateral set. In particular, for every $k \in \mathbb{N}$ there exists $x_k \in \mathbb{R}^n$ such that

$$
\|x_k - a_1\|_k = \|x_k - a_2\|_k = \|x_k - a_3\|_k = \|x_k - a_4\|_k = 2.
$$

As $(x_k)_{k \in \mathbb{N}}$ is bounded, it has a subsequence convergent to $x \in \mathbb{R}^n$. Now it is easy to see that x extends a_1, a_2, a_3, a_4 to a 5-point equilateral set in the ℓ_1 norm. This contradicts the previous part of the reasoning, and the conclusion follows. ∎

6. Concluding remarks. As we already mentioned in the introduction, probably the most natural question in the field of equilateral sets which remains open is

QUESTION 6.1 (see [5], [11], [12]). *Does the inequality* $e(X) \geq n+1$ *hold for every normed space* X *of dimension* n*?*

It is reasonable to ask if the approach that led us to the alternative proof of Petty's theorem can be helpful in answering Question 6.1 also in higher dimensions. Using a similar approach Makeev has proved the case $n = 4$ in [8]. It is also the largest dimension for which the answer to Question 6.1 is known to be affirmative. Reduction of the general case to the situation in which the norm is smooth and strictly convex can be done in exactly the same way as in the presented proof. We can therefore try to use Theorem 2.1. On the other hand, Proposition 2.2 cannot be generalized to higher dimensions and in consequence it seems that establishing a continuous behaviour of the spheres, circumscribed about the variable simplex, is a serious technical issue. However, the main difficulty lies probably in obtaining the higher-dimensional analogue of Theorem 1.2. We already know from the previous section that such an analogue could hold only for certain equilateral sets, i.e. we would have to find an equilateral set with the sphere of small radius passing through its vertices. This requires a different idea than for the planar case.

References

[1] M. Fabian, P. Habala, P. Hájek, V. M. Santalucía, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books Math. 8, Springer, 2001.

[2] M. Ghomi, *Optimal smoothing for convex polytopes*, Bull. London Math. Soc. 36 (2004), 483–492.

[3] P. R. Goodey, *Homothetic ellipsoids*, Math. Proc. Cambridge Philos. Soc. 93 (1983), 25–34.

[4] M. L. Gromov, *Simplexes inscribed in a hypersurface*, Mat. Zametki 5 (1969), 81–89 (in Russian).

[5] B. Grünbaum, *On a conjecture of H. Hadwiger*, Pacific J. Math. 11 (1961), 215–219.

[6] H. Kramer and A. B. Németh, *The application of Brouwer's fixed point theorem to the geometry of convex bodies*, An. Univ. Timişoara Ser. Şti. Mat. 13 (1975), 33–39.

[7] V. V. Makeev, *The degree of a mapping in some problems of combinatorial geometry*, J. Soviet Math. 51 (1987), 2544–2546.

[8] V. V. Makeev, *Equilateral simplices in normed 4-space*, J. Math. Sci. (N.Y.) 140 (2007), 548–550.

[9] H. Martini and M. Spirova, *Covering discs in Minkowski planes*, Canad. Math. Bull. 52 (2009), 424–434.

[10] H. Martini, K. J. Swanepoel and G. Weiss, *The geometry of Minkowski spaces—a survey. Part I*, Expo. Math. 19 (2001), 97–142; Erratum, ibid., 364.

[11] C. M. Petty, *Equilateral sets in Minkowski spaces*, Proc. Amer. Math. Soc. 29 (1971), 369–374.

[12] K. Schütte, *Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand*, Math. Ann. 150 (1963), 91–98.

[13] P. S. Soltan, *Analogues of regular simplexes in normed spaces*, Dokl. Akad. Nauk SSSR 222 (1975), 1303–1305 (in Russian); English transl.: Soviet Math. Dokl. 16 (1975), 787–789.

[14] K. J. Swanepoel, *Equilateral sets in finite-dimensional normed spaces*, in: Seminar of Mathematical Analysis, D. G. Álvarez et al. (eds.), Secretariado de Publicationes, Universidad de Sevilla, Sevilla, 2004, 195–237.

[15] K. J. Swanepoel and R. Villa, *A lower bound for the equilateral number of normed spaces*, Proc. Amer. Math. Soc. 136 (2008), 127–131.

[16] K. J. Swanepoel and R. Villa, *Maximal equilateral sets*, arXiv:1109.5063 [math.MG].

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