Hodge type decomposition

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Abstract. In the space $\Lambda^p$ of polynomial $p$-forms in $\mathbb{R}^n$ we introduce some special inner product. Let $\mathbf{H}^p$ be the space of polynomial $p$-forms which are both closed and co-closed. We prove in a purely algebraic way that $\Lambda^p$ splits as the direct sum $d^*(\Lambda^{p+1}) \oplus \delta^*(\Lambda^{p-1}) \oplus \mathbf{H}^p$, where $d^*$ (resp. $\delta^*$) denotes the adjoint operator to $d$ (resp. $\delta$) with respect to that inner product.

1. Introduction and main result. To begin with, recall the classical Hodge decomposition theorem. Suppose $(M^n, g)$ is an oriented Riemannian manifold. Introduce the following notation: $\Lambda^p(M)$ is the space of all smooth differential $p$-forms on $M$ whereas $d$, $\delta$ and $\Delta = d\delta + \delta d$, are the differential, co-differential and the Laplace–Beltrami operator, respectively. Recall that for any $\omega \in \Lambda^p(M)$, $\delta \omega = (-1)^n(p+1)^*d^*\omega$, where $*$ is the Hodge operator. Moreover, let $\mathbf{H}^p(M)$ denote the space of $p$-forms which are both closed and co-closed: $\mathbf{H}^p(M) = \{ \omega \in \Lambda^p(M) : d\omega = \delta \omega = 0 \}$.

If $M$ is compact then the formula

$$(\omega|\eta)_M = \int_M \omega \wedge \ast \eta$$

defines an inner product in $\Lambda^p(M)$. Then $d^* = \delta$ and $\delta^* = d$, i.e., $d$ and $\delta$ are adjoint to each other. Take any $\omega \in \Lambda^p(M)$; then $(\omega|\Delta \omega)_M = (d\omega|d\omega)_M + (\delta \omega|\delta \omega)_M$. Therefore, $\omega$ is harmonic, i.e., $\Delta \omega = 0$, iff $\omega \in \mathbf{H}^p(M)$. The following is a classical result in analysis:

**Theorem 1.1** (Hodge decomposition theorem). On a compact oriented Riemannian manifold $M$, $\Lambda^p(M) = d\Lambda^{p-1}(M) \oplus \delta \Lambda^{p+1}(M) \oplus \mathbf{H}^p(M)$, or equivalently $\Lambda^p(M) = \delta^* \Lambda^{p-1}(M) \oplus d^* \Lambda^{p+1}(M) \oplus \mathbf{H}^p(M)$, where $\oplus$ denotes an orthogonal direct sum.

For the proof of the Hodge decomposition theorem we refer to the book of F. Warner ([4, Chapter 6]). Notice that a historical survey of the develop-
opment of the theory of elliptic operators is given in the beautiful article of L. Hörmander ([1]).

We treat \( \mathbb{R}^n \) as a Riemannian manifold equipped with the canonical inner product. If \( f \) is a polynomial of the form \( f = \sum_\alpha a_\alpha x^\alpha \) (here \( \alpha = (\alpha_1, \ldots, \alpha_n) \) denotes a multi-index) we put
\[
f(D) = \sum_\alpha a_\alpha D^\alpha, \quad D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}.
\]

Let \( \mathcal{P}_k \) (resp. \( \mathcal{H}_k \)) denote the space of all homogeneous (resp. harmonic homogeneous) polynomials on \( \mathbb{R}^n \) of degree \( k \). Define the inner product \( (\cdot, \cdot)_k \) in \( \mathcal{P}_k \) as follows: \( (f, g) = f(D)g \), for \( f, g \in \mathcal{P}_k \) (cf. [3, p. 139]). Clearly, for any \( f \in \mathcal{P}_k \), \( g \in \mathcal{P}_l \) and \( h \in \mathcal{P}_{k+l} \), \( (fg, h)_{k+l} = (f, g(D)h)_k \). In particular \((1), (\Delta f, h) = (f, -r^2 h) \) where \( r^2(x) \) is the square of the Euclidian norm of \( x \in \mathbb{R}^n \). As a result we get the well-known identity ([3, Thm. 2.12])
\[
\mathcal{P}_k = \mathcal{H}_k \oplus 1^{-2} \mathcal{P}_{k-2}.
\]

We may extend (using linearity) this inner product onto the space of all polynomials, putting \( (f, g) = 0 \) if \( f \in \mathcal{P}_k \), \( g \in \mathcal{P}_l \) and \( k \neq l \).

Recall that any \( p \)-form \( \omega \) in \( \mathbb{R}^n \) has a unique expression
\[
\omega = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^n \omega_{i_1, \ldots, i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p},
\]
where the functions \( \omega_{i_1, \ldots, i_p} \), called coefficients, are skew-symmetric with respect to the indices. Denote by \( \Lambda^p \) the space of all \( p \)-forms in \( \mathbb{R}^n \) that have polynomial coefficients. We put \( \Lambda^p = \{0\} \) if \( p < 0 \). Let \( \mathfrak{S}^p \) and \( \mathbf{H}^p \) denote, respectively, the space of all harmonic polynomial \( p \)-forms and its subspace of all polynomial \( p \)-forms which are both closed and co-closed:
\[
\mathfrak{S}^p = \{ \omega \in \Lambda^p : \Delta \omega = (d \delta + \delta d) \omega = 0 \}, \quad \mathbf{H}^p = \{ \omega \in \Lambda^p : d \omega = \delta \omega = 0 \}.
\]

Consider the vector field \( \nu \) and the 1-form \( \nu^* \) defined by
\[
\nu_x = x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n}, \quad \nu^*_x = x^1 dx^1 + \cdots + x^n dx^n,
\]
where \( x = (x^1, \ldots, x^n) \in \mathbb{R}^n \). One sees that \( \nu^* \nu = r^2 \). Let \( \varepsilon_{\nu} \) (resp. \( \iota_{\nu} \)) denote the exterior (resp. interior) product, i.e., \( \varepsilon_{\nu} \omega = \nu^* \omega \wedge \omega \) and \( \iota_{\nu} \omega = \omega(\nu, \cdot, \ldots, \cdot) \). Manifestly, \( \varepsilon_{\nu}^2 = 0 \) and \( \iota_{\nu}^2 = 0 \). Since \( \iota_{\nu} \) is an anti-derivation, \( \iota_{\nu} \varepsilon_{\nu} \omega = \nu^* \nu \omega - \varepsilon_{\nu} \iota_{\nu} \omega \). Therefore, we get
\[
(\iota_{\nu} \varepsilon_{\nu} + \varepsilon_{\nu} \iota_{\nu}) \omega = r^2 \omega.
\]

One can easily check that
\[
d \varepsilon_{\nu} = -\varepsilon_{\nu} d, \quad \delta \iota_{\nu} = -\iota_{\nu} \delta.
\]

(1) Since we define the Laplace operator as \( \Delta = d \delta + \delta d \), our \( \Delta \) on smooth real-valued functions has the sign such that \( \Delta u = -\left( \partial / (\partial x^1)^2 + \cdots + \partial / (\partial x^n)^2 \right) u \) on \( \mathbb{R}^n \).
Equip the space $\Lambda^p$ with the inner product $(\cdot|\cdot)$ as follows: for any $\omega, \eta \in \Lambda^p$ we put

$$\omega|\eta = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^n (\omega_{i_1, \ldots, i_p}, \eta_{i_1, \ldots, i_p}),$$

where $\omega_{i_1, \ldots, i_p}$'s and $\eta_{i_1, \ldots, i_p}$'s denote the coefficients of $\omega$ and $\eta$, respectively.

It turns out ([2, Thm. 2.2.1]) that the operators $\delta$ and $-\varepsilon_\nu$, and $d$ and $\iota_\nu$, are adjoint to each other:

$$\delta^* = -\varepsilon_\nu, \quad d^* = \iota_\nu.$$

If $\omega$ is a polynomial form such that $\iota_\nu \omega = \varepsilon_\nu \omega = 0$ then, by (1.2), $\omega = 0$. Since $i^2_\nu = \varepsilon^2_\nu = 0$, we see that $\varepsilon_\nu(A^{p-1}) \cap \iota_\nu(A^{p+1}) = \{0\}$. Moreover, by (1.4) the spaces $\mathbf{H}^p$ and $\varepsilon_\nu(A^{p-1}) \oplus \iota_\nu(A^{p+1})$ are mutually orthogonal.

The purpose of this paper is to prove, in a purely algebraic way, the following

**Theorem 1.2 (Hodge type decomposition).** For any $0 \leq p \leq n$, the space $\Lambda^p$ splits as the direct sum $\Lambda^p = \iota_\nu \Lambda^{p+1} \oplus \varepsilon_\nu \Lambda^{p-1} \oplus \mathbf{H}^p$, or equivalently $\Lambda^p = d^* \Lambda^{p+1} \oplus \delta^* \Lambda^{p-1} \oplus \mathbf{H}^p$.

**2. Some preparations.** Denote by $\Lambda^p_k$ and $\mathcal{F}^p_k$ the space of all forms in $\mathbb{R}^n$ that have coefficients from $\mathcal{P}_k$ and $\mathcal{H}_k$, respectively. It is convenient to put $\Lambda^p_k = \{0\}$ if either $p < 0$ or $k < 0$. Let $\mathbf{H}^p_k = \mathbf{H}^p \cap \Lambda^p_k = \{\omega \in \Lambda^p_k : d\omega = \delta \omega = 0\}$. Since for any differential $p$-form $\omega$ on $\mathbb{R}^n$,

$$(\Delta \omega)_{i_1, \ldots, i_p} = \Delta \omega_{i_1, \ldots, i_p},$$

we see that $\omega \in \mathcal{F}^p$ iff all coefficients of $\omega$ are harmonic polynomials. In particular, $\mathcal{F}^p_k = \mathcal{F}^p \cap \Lambda^p_k$ and $\mathbf{H}^p_k = \mathcal{F}^p \cap \mathbf{H}^p$. Moreover, the decomposition (1.1) translates immediately to $\Lambda^p_k$ where we have

$$(2.1) \quad \Lambda^p_k = \mathcal{F}^p_k \oplus \frac{1}{r^2} \Lambda^p_{k-2}.$$

In the proof of the main theorem we will need to use some formulæ from [2]. We list them below.

For any $\omega \in \Lambda^p_k$ we have ([2, Prop. 2.2.1])

$$\delta \varepsilon_\nu \omega = -\varepsilon_\nu \delta \omega - (n - p + k)\omega, \quad (2.2)$$

$$d \iota_\nu \omega = -\iota_\nu d \omega + (p + k)\omega, \quad (2.3)$$

Applying the second identity in (1.3) and (2.3) we see that for any polynomial form $\omega$,

$$\Delta \iota_\nu \omega = \iota_\nu \Delta \omega + 2\delta \omega. \quad (2.4)$$
Define the spaces $\chi_{p,k}, \chi^0_{p,k}$ and the operator $I_{p,k}$ as follows:

$$
\chi_{p,k} = \mathcal{F}_k^p \cap \ker \delta, \quad \chi^0_{p,k} = \chi_{p,k} \cap \ker \iota_\nu,
$$

$$
I_{p,k} = \varepsilon_\nu - c_k r^2 d, \quad \text{where} \quad c_k = \begin{cases} 
1/(n + 2k - 4) & \text{if } k \geq 2, \ 0 \ & \text{otherwise},
\end{cases}
$$

We have the following decompositions ([2, (3.3.2), (3.3.3)]):

$$
(2.5) \quad \chi_{p,k} = \chi^0_{p,k} \oplus I d\chi^0_{p-1,k+1},
$$

$$
(2.6) \quad \mathcal{F}_k^p = \chi^0_{p,k} \oplus I d\chi^0_{p-1,k+1} \oplus \varepsilon_\nu d\chi^0_{p-2,k} \oplus I_{p,k}(\chi^0_{p-1,k-1}).
$$

Notice that in (2.6) some subspaces may degenerate sometimes. In particular, by (2.3) it follows easily ([2, (3.2.1)]) that

$$
(2.7) \quad \chi^0_{p,0} = \{0\} \quad \text{if } p > 0.
$$

**Remark.** The orthogonal decomposition (2.6) above is the key step in the proof of Hodge type decomposition. In fact, (2.6) is the very special case of the more general decomposition [2, Thm. 3.3.1] of the kernel of the operator $L = ad\delta + b d\delta, a, b > 0$.

The last part of this section has a technical character. From (1.3) and (2.4) it follows that $\iota_\nu \mathbf{H}^{p+1}_{k-1} \subset \chi^0_{p,k}$. On the other hand, if $p + k > 0$ then taking $\omega \in \chi^0_{p,k}$ we see that $\omega = \iota_\nu \eta$, where $\eta = (p + k)^{-1} d\omega$. Clearly, $\eta \in \mathbf{H}^{p+1}_{k-1}$. So we have

$$
(2.8) \quad \iota_\nu \mathbf{H}^{p+1}_{k-1} = \chi^0_{p,k} \quad \text{if } p + k > 0.
$$

If $k = p = 0$ then clearly $\mathbf{H}^0_0 = \mathbb{R}$ (the space of constant functions). Suppose that either $p > 0$ or $k > 0$. Let $\omega \in \mathbf{H}^p_k$. In the light of (2.5) and the relation $\mathbf{H}^p_k \subset \chi_{p,k}$ we may write $\omega = \omega' + \omega''$, where $\omega' \in \chi^0_{p,k}$ and $\omega'' \in d\chi^0_{p-1,k+1}$. Since $\omega$ and $\omega''$ are closed, $d\omega' = 0$. Therefore, $0 = \iota_\nu d\omega' = -dt_\nu d\omega' + (p + k) \omega' = (p + k) \omega'$. Since $p + k > 0$, $\omega' = 0$. Thus $\mathbf{H}^p_k \subset d\chi^0_{p-1,k+1}$.

On the other hand, one easily checks that $d\chi^0_{p-1,k+1} \subset \mathbf{H}^p_k$. We have proved that

$$
(2.9) \quad \mathbf{H}^p_k = \begin{cases} 
\mathbb{R} & \text{if } p = k = 0, \\
d\chi^0_{p-1,k+1} & \text{otherwise.}
\end{cases}
$$

Let us complete the section with the following observation: if either $p \neq 1$ or $k \neq 1$ then

$$
(2.10) \quad I_{p,k}(\chi^0_{p-1,k-1}) \oplus r^2 A^p_{k-2} \subset \iota_\nu d\varepsilon_\nu(A^p_{k-2}) \oplus \iota_\nu \varepsilon_\nu(t_\nu A^p_{k-2}).
$$

Indeed, the inclusion $r^2 A^p_{k-2} \subset \iota_\nu d\varepsilon_\nu(A^p_{k-2}) \oplus \varepsilon_\nu(t_\nu A^p_{k-2})$ follows from (1.2). Now we prove that $I_{p,k}(\chi^0_{p-1,k-1}) \subset \iota_\nu d\varepsilon_\nu(A^p_{k-2}) \oplus \varepsilon_\nu(t_\nu A^p_{k-2})$. If $p = 0$ or $k = 0$ then the inclusion is trivial. Suppose that $p, k > 0$. Take $\omega \in$
$I_{p,k}(x_p^{0,k-1})$, $\omega = I_{p,k}\eta$, $\eta \in x_p^{0,k-1}$. Using (1.2) and (2.3) one easily checks that $\omega = \iota_\nu \epsilon_\nu \psi' + \epsilon_\nu \iota_\nu \psi''$, where

$$\psi' = -c_k d\eta, \quad \psi'' = \frac{1 - (p + k - 2)c_k}{p + k - 2} d\eta.$$

3. Proof of the Hodge type decomposition. To prove Theorem 1.2 it suffices to show that for any $p, k \geq 0$,

$$A_k^p = \iota_\nu A_{k-1}^{p+1} \oplus \epsilon_\nu A_{k-1}^{p-1} \oplus H_k^p. \tag{3.1}$$

To prove (3.1) we apply induction with respect to $k$. Before doing this, we check some very special case of (3.1) separately. Namely, we have

$$A_1^1 = \iota_\nu A_0^2 \oplus \epsilon_\nu A_0^0 \oplus H_1^1. \tag{3.2}$$

Proof of (3.2). Clearly, $A_1^1 = S_1^1$, $A_0^0 = x_0^{0,0} = \mathbb{R}$ and $A_0^2 = S_0^2$. Formula (2.6) implies that it suffices to show that $x_1^{0,1} = \iota_\nu S_0^2$. By (1.3), $\iota_\nu S_0 \subset x_1^{0,1}$. Let $\omega \in x_1^{0,1}$. Put $\eta = (1/2)d\omega$; then one can easily check that $\iota_\nu \eta = \omega$. Thus $x_1^{0,1} = \iota_\nu S_0^2$.

Proof of (3.1) by induction with respect to $k$. Suppose that $k = 0$. If $p = 0$ then (3.1) is a direct consequence of the equalities $A_0^0 = S_0^0 = H_0^0 = \mathbb{R}$. If $p > 0$ then (3.1) follows from (2.9), (2.6), (2.7) and the equality $A_0^p = S_0^p$.

Suppose that (3.1) holds for $k-1$, $k \geq 1$. Take any $p \geq 0$. We may assume that either $p \neq 1$ or $k \neq 1$ (see (3.2)). Using (1.4), (2.2) and (2.3) we find that $\iota_\nu \iota_\nu (A_k^{p-1})$ and $\iota_\nu (H_k^{p+1})$, and $\iota_\nu \iota_\nu (A_k^p)$ and $\iota_\nu (H_k^{p-1})$ are mutually orthogonal. Thus, by induction hypothesis we have

$$\begin{align*}
\epsilon_\nu (A_{k-1}^{p-1}) &= \epsilon_\nu (\iota_\nu (A_k^{p-2}) \oplus \epsilon_\nu (H_k^{p-1})), \\
\iota_\nu (A_{k-1}^{p+1}) &= \iota_\nu \epsilon_\nu (A_k^{p-2}) \oplus \iota_\nu (H_k^{p+1}).
\end{align*} \tag{3.3}$$

From (2.1), (2.6), (2.8), (2.9), (2.10) and (3.3) we get

$$\begin{align*}
A_k^p &\supset \epsilon_\nu (A_{k-1}^{p-1}) \oplus \iota_\nu (A_{k-1}^{p+1}) \oplus H_k \\
&= (3.3) \oplus \epsilon_\nu \iota_\nu (A_k^{p-2}) \oplus \iota_\nu \epsilon_\nu (A_k^{p-2}) \oplus \epsilon_\nu (H_k^{p-1}) \oplus \iota_\nu (H_k^{p+1}) \oplus H_k \oplus H_k \oplus H_k \\
&= (2.9)/(2.8) \oplus \epsilon_\nu \iota_\nu (A_k^{p-2}) \oplus \iota_\nu \epsilon_\nu (A_k^{p-2}) \oplus \epsilon_\nu d^p_{p-2,k} \oplus \chi_{p,k}^0 \oplus H_k \\
&\supset r^2 A_k^{p-2} \oplus I_{p,k}(x_{p-1,k-1}) \oplus \chi_{p,k}^0 \oplus \epsilon_\nu d^p_{p-2,k} \oplus H_k \\
&= (2.6) \oplus r^2 A_k^{p-2} \oplus S_k^p \\
&\supset A_k^p,
\end{align*}$$

which proves (3.1).

Induction completes the proof.
References


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Received 10.10.2005
and in final form 15.12.2006 (1641)