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A Littlewood–Paley type inequality with applications to the elliptic Dirichlet problem

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Abstract. Let L be a strictly elliptic second order operator on a bounded domain $\Omega \subset \mathbb{R}^n$. Let u be a solution to $Lu = \operatorname{div} \vec{f}$ in Ω, $u = 0$ on $\partial\Omega$. Sufficient conditions on two measures, μ and ν defined on Ω, are established which imply that the $L^q(\Omega, d\mu)$ norm of $|\nabla u|$ is dominated by the $L^p(\Omega, dv)$ norms of $\operatorname{div} \vec{f}$ and $|\vec{f}|$. If we replace $|\nabla u|$ by a local Hölder norm of u, the conditions on μ and ν can be significantly weaker.

Introduction. The intent of this paper is to establish sufficient conditions on two measures, μ and ν, defined on a bounded domain Ω in $\mathbb{R}^n$, $n \geq 3$, so that

$$(1)\qquad \Big(\int_\Omega |\nabla u(x)|^q \, d\mu(x)\Big)^{1/q} \leq C\Big(\int_\Omega (|\operatorname{div} \vec{f}(x)|^p + |\vec{f}(x)|^p) \, d\nu(x)\Big)^{1/p}$$

if $2 < p \leq q < 2 + \varepsilon$, for any function $u(x)$ that is a weak solution to

$$\begin{cases} Lu(x) = \operatorname{div} \vec{f}(x), & x \in \Omega, \\ u(x')|_{\partial\Omega} = 0. \end{cases}$$

Here Ω is assumed to satisfy an exterior cone condition, $\vec{f} \in H^1(\Omega)$, and

$$L = \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x) \frac{\partial}{\partial x_j}\Big)$$

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is a strictly elliptic divergence form operator, in other words,

$$\sum_{1\le i,j\le n} \xi_i a_{i,j}(x)\xi_j \ge \lambda|\xi|^2 \quad \text{for some } \lambda > 0 \text{ and all } x \in \Omega.$$

L's coefficients are assumed to be symmetric, $a_{i,j}(x) = a_{j,i}(x)$, and bounded and measurable on Ω. Since Ω is bounded, for $\mu(\Omega) < \infty$ and $\nu(\Omega) < \infty$ (this is implied by the conditions that will be imposed on μ and ν), we can extend (1) to be valid for indices $0 < q < 2+\varepsilon$ and $2 < p < \infty$, simply by using Hölder's inequality. The constant C in (1) depends on Ω in any case, so this does not restrict the result too much for the wider range of indices. However, the presence of the gradient of u does introduce a restriction. Following a suggestion of Professor Wheeden, we replace $|\nabla u(x)|$ by a local Hölder norm of u at x, and prove sufficient conditions on μ and ν so that for $0 < q < \infty$, $1 < p < \infty$,

$$\Big(\int_\Omega \|u(x)\|_{H^\alpha}^q \, d\mu(x)\Big)^{1/q} \le C\Big(\int_\Omega (|\operatorname{div} \vec f(x)|^p)\, d\nu(x)\Big)^{1/p}, \tag{2}$$

with

$$\|u(x)\|_{H^\alpha} = \sup_{y\in P(x)} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

where

$$P(x) = \{z \in \Omega : |x_i - z_i| \le b\delta(x),\ i = 1,\dots,n\},$$
$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

The constant $b < 1$ is fixed and chosen so that both $P(x)$ and $\eta P(x) = \{z \in \Omega : |x_i - z_i| \le \eta b\delta(x),\ i = 1,\dots,n\}$ for $\eta_0 \ge \eta > 1$, $\eta_0 > 4$ fixed, are Whitney-type cubes in Ω centered at x. Corresponding results for solutions to the homogeneous Dirichlet problem

$$\begin{cases} Lu(x) = 0, & x \in \Omega, \\ u(x')|_{\partial\Omega} = f(x'), \end{cases}$$

are proved in [S].

Combining the results of this paper with those in [S] and [SW], and using superposition, we have established sufficient conditions on measures μ and ν on Ω, and on a boundary measure, $\varrho d\omega$, so that for $\|u(x)\|$, which denotes either $|\nabla u(x)|$ or $\|u(x)\|_{H^\alpha}$, it follows that

$$\Big(\int_\Omega \|u(x)\|^q \, d\mu(x)\Big)^{1/q} \tag{3}$$
$$\le C\Big(\Big(\int_\Omega (|\operatorname{div} \vec f(x)|^p + |\vec f(x)|^p\Big)\, d\nu(x)\Big)^{1/p} + \Big(\int_{\partial\Omega} |g(x')|^r \varrho(x')\, d\omega(x')\Big)^{1/r}\Big)$$

for solutions to

$$\begin{cases} Lu(x) = \operatorname{div} \vec{f}(x), & x \in \Omega, \\ u(x')|_{\partial\Omega} = g(x'), \end{cases}$$

if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. $\omega = \omega^{x_0}$ is the elliptic measure on $\partial\Omega$ generated by L, measured from a fixed point x_0 interior to Ω. So for $L = \Delta$, ω is harmonic measure.

The method of obtaining the conditions on the measures follows that of Wheeden and Wilson [WW] in using dual operator norms. This argument utilizes a norm inequality that derives from Littlewood–Paley theory. The crucial Littlewood–Paley type inequality is proved in Theorem 1 for functions of the form

$$h(x) = \sum_{J \in \mathcal{F}} \lambda_J \varphi_{(J)}(x), \qquad \lambda_J \in \mathbb{R},$$

using methods closely allied to those in [W], [SW]. The $\varphi_{(J)}(x)$ are members of a family of functions that have certain decay and cancellation properties, and $\mathcal{F}$ is a finite family of dyadic cubes. The details about the $\varphi_{(J)}(x)$ are stated below in (a), (a′), (b) and (c). To prove the $\varphi_{(J)}(x)$ satisfy the necessary conditions, we utilize geometric properties of elliptic Green's functions on rough domains, proved by Grüter and Widman [GW] (see also [K]).

The results presented in this paper stem from a considerable body of work. References for situations in which an inequality of the form

$$\Big(\int_{\Omega} |\nabla u(x)|^q \, d\mu(x)\Big)^{1/q} \leq \Big(\int_{\partial\Omega} |g(x')|^p v(x') \, d\omega(x')\Big)^{1/p} \tag{4}$$

holds when u is harmonic are given in [SW]. The history for semi-discrete Littlewood–Paley results is mentioned in [W]. Wheeden and Wilson dealt with the case of the Dirichlet problem in the upper half space for harmonic $u(x)$. Sweezy and Wilson later found sufficient conditions for μ and $\varrho d\omega$ to ensure (4) for harmonic gradients on Lipschitz domains; they found that similar techniques allowed them to deal with elliptic functions on rough domains. A key part of their argument consisted in establishing a Littlewood–Paley type inequality for functions of the form $\sum_{J \in \mathcal{F}} \lambda_J \varphi_{(J)}(x)$ with minimal smoothness conditions assumed for the $\varphi_{(J)}$ and for the domain. They accomplished this by an argument in the spirit of Wilson's method of proving a semi-discrete Littlewood–Paley type inequality on $\mathbb{R}^n$ for smoother functions (see [W]). The fact that the operator L has an associated kernel function which has geometric decay was an essential ingredient to their proof. It remains an important ingredient in the case of a solution to the inhomogeneous equation.

In Section 1 of this paper the two main theorems are stated. To avoid becoming immersed in technical details too early we leave the proof of Theorem 1 until the last section of the paper. Section 2 contains the proof of Theorem 2, assuming that Theorem 1 is valid. Theorem 3 for a local Hölder norm of a solution u instead of the gradient of u is stated and proved after Theorem 2 in that section. Section 3 presents the proof of Theorem 1, the Littlewood–Paley type inequality.

1. The main results. To state the first main theorem, the Littlewood–Paley type inequality, we need to establish some definitions. First recall that a measure σ defined on a domain D is said to be A^∞ with respect to Lebesgue measure if for any cube $Q \subset D$ and any measurable subset E of Q, there are fixed constants C_0 and $\kappa > 0$ so that

$$\left(\frac{\sigma(E)}{\sigma(Q)}\right)^{\kappa} \le C_0 \frac{|E|}{|Q|} \quad \text{(see [CF]).}$$

We will utilize a family $\mathfrak{D}$ of dyadic cubes which includes all dyadic subcubes of a given (large) cube Q_0. When Q is a cube, $l(Q)$ will denote the side length of Q. If Q is a region that is comparable to a cube, say, Q is the image of an actual Euclidean cube under a Lipschitz map, then $l(Q)$ denotes a length comparable to the side length of the pre-image cube. One could also take $l(Q)$ to be the diameter of any such region. Q_0 is chosen so that $\Omega \subset Q_0$ and $l(Q_0) \sim \operatorname{diam}(\Omega)$. In order to have $\bigcup_{I\in\mathfrak{D}} I$ cover Ω completely, we take the dyadic subcubes I to be half closed, i.e. of the form $[a_1, b_1) \times \cdots \times [a_n, b_n)$. $\mathcal{W}$ is a collection of special dyadic cubes from $\mathfrak{D}$; these are Whitney-type cubes that lie inside Ω, are pairwise disjoint, and cover the interior of Ω. $I \in \mathcal{W}$ implies that $l(I) \simeq \operatorname{dist}(I, \partial\Omega)$, but the $I \in \mathcal{W}$ may be subcubes of a fixed proportion to the usual Whitney cube decomposition of Ω. We need to be sure that βI (the β-dilate of I, that is, the cube concentric with I and of side length $\beta l(I)$) is also a Whitney cube for any $1 \le \beta < \eta_0$, $\eta_0 > 4$. The point x_J will denote the geometric center of the dyadic cube J, and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

When $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, the function $g^*(f)(x) = g^*(x)$ is defined by

$$g^*(x) = \left(\sum_{J\in\mathcal{F}} \frac{\lambda_J^2}{|J|}\left(1 + \frac{|x - x_J|}{l(J)}\right)^{-n}\right)^{1/2}.$$

It is a discrete version of the g_λ^* function of classical Littlewood–Paley theory. Notice that the order of decay in g^*, as defined here, is slightly less than that said to be optimal in [W]. The reason for this is the order of decay for the Green function, $G(x, y)$, which appears in $u(x) = \int_\Omega G(x, y) \operatorname{div} \vec{f}(y)\, dy$, the integral representation for solutions to Poisson's equation.

The four conditions that will be assumed to hold for the family $\{\varphi_{(J)}(x)\}$ are:

$$\text{(a)}\quad |\varphi_{(J)}(x)| \le C l(J)^{2-n/2}\left(1+\frac{|x-x_J|}{l(J)}\right)^{2-n} \qquad \text{for all } x\in\Omega,$$

$$\text{(a}'\text{)}\quad |\varphi_{(J)}(x)| \le C\delta(x)^{\alpha} l(J)^{2-n/2-\alpha}\left(1+\frac{|x-x_J|}{l(J)}\right)^{2-n-\alpha} \qquad \text{for all } x\in\Omega,$$

$$\text{(b)}\quad |\varphi_{(J)}(x)-\varphi_J(y)| \le C|x-y|^{\alpha} l(J)^{2-n/2-\alpha} \times\left(1+\frac{|x-x_J|}{l(J)}+\frac{|y-x_J|}{l(J)}\right)^{2-n-\alpha} \qquad \text{for all } x,y\in\Omega,$$

$$\text{(c)}\quad \int\Big|\sum_{J\in\mathcal{F}}\lambda_J\varphi_{(J)}(x)\Big|^2\,dx \le C\sum_{J\in\mathcal{F}}\lambda_J^2.$$

THEOREM 1. *Suppose that* $f(x)=\sum_{J\in\mathcal{F}}\lambda_J\varphi_{(J)}(x)$ *is a function defined on* Ω, *where* $\mathcal{F}$ *is a finite set of dyadic cubes from* $\mathcal{W}$, *and the* $\{\varphi_{(J)}\}_{J\in\mathcal{F}}$ *are a family of functions that satisfy conditions* (a), (a′), (b), *and* (c), *and* $\varphi_{(J)}(x)=0$ *if* $x\in Q_0\setminus\Omega$. *Then, if* $\sigma\in A^{\infty}(Q_0,dx)$, *there is a constant* $C=C(n,\alpha,p,\Omega,\kappa,C_0)$ *such that, for any* $0<p<\infty$,

$$\|f\|_{L^p(Q_0,d\sigma)} \le C\|g^*\|_{L^p(Q_0,d\sigma)}.$$

The major application of Theorem 1 of concern here is to demonstrate sufficient conditions on measures μ and ν so that (1) is valid. To state these conditions we need to recall what it means for a measure to satisfy a reverse Hölder condition on a domain D with respect to Lebesgue measure. This is written as $\mu\in B^r(D,dx)$, with $r>1$, if for every cube $Q\subset D$,

$$\left(\frac{1}{|Q|}\int_Q\left(\frac{d\mu}{dx}\right)^r dx\right)^{1/r} \le C\left(\frac{1}{|Q|}\int_Q\frac{d\mu}{dx}\,dx\right).$$

It is true that any measure satisfying a reverse Hölder condition with respect to Lebesgue measure is also an A^p measure on D, for some exponent p. (A non-negative $L^1_{\mathrm{loc}}(D)$ function w is in $A^p(D,dx)$ if

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{w}\right)^{p'-1}dx\right)^{1/(p'-1)} \le A_0$$

for all cubes $Q\subset D$.) A measure μ is in $A^p(D,dx)$ if $d\mu/dx\in A^p(D,dx)$.) We let p' denote the Hölder conjugate index for p, that is, $1/p+1/p'=1$. Any measure that is either an A^p measure on D, or satisfies a reverse Hölder condition on D, is also an A^{∞} measure on D (see [CF]).

To avoid cluttering up the statement of Theorem 2 we define, for any dyadic cube Q_j in $\mathfrak{D}$,

$$M(Q_j) = \max\left\{ l(Q_j)^{n/2+1}\left(\int_{2Q_j}\left(\frac{d\nu}{dx}\right)^{-2/(p-2)} dx\right)^{(p-2)/2p}, \left(\int_{Q_0}\left(1+\frac{|y-x_{Q_j}|}{l(Q_j)}\right)^{-np'/2} d\sigma(y)\right)^{1/p'}\right\}.$$

Suppose that $L=\sum_{1\le i,j\le n}(\partial/\partial x_i)(a_{i,j}(x)\partial/\partial x_j)$ is a strictly elliptic divergence form operator on the domain Ω, as described above. Suppose also that μ, a Borel measure defined on Ω, satisfies a reverse Hölder condition of order $((q+\varepsilon)/q)'$ with respect to Lebesgue measure on Ω, i.e. $\mu \in B^{(q+\varepsilon)/\varepsilon}(\Omega, dx)$.

THEOREM 2. *Let μ and ν be Borel measures on Ω with ν finite and absolutely continuous with respect to Lebesgue measure, and suppose that $d\sigma(x) = (d\nu/dx)^{1-p'}\,dx$ satisfies the condition $A^\infty(\Omega, dx)$. Let u be a solution to $Lu = \operatorname{div}\vec f$ on Ω, $u|_{\partial\Omega}=0$, $\vec f \in H^1(\Omega)$. If there is a constant $C_0>0$ so that for every dyadic cube Q_j in $\mathcal{W}$,*

$$\mu(Q_j)^{1/q}M(Q_j) \le C_0 l(Q_j)^{n+1},$$

then there is a constant $C=C(n,p,q,\alpha,b,\kappa,\Omega,\lambda,\eta_0,\varepsilon)$ such that

$$\left(\int_\Omega |\nabla u(x)|^q\,d\mu(x)\right)^{1/q} \le C\left(\int_\Omega (|\operatorname{div}\vec f(x)|^p + |\vec f(x)|^p)\,d\nu(x)\right)^{1/p} \tag{1}$$

for $2<p\le q<2+\varepsilon$. For $0<q\le q_0$ and $2<p_0\le p<\infty$, the same inequality is valid upon replacing C by $C\mu(\Omega)^{1/q-1/q_0}\nu(\Omega)^{1/p_0-1/p}$, $C=C(p_0,q_0,n,\alpha,b,\kappa,\Omega,\lambda,\eta_0,\varepsilon)$, for some fixed pair of indices $2<p_0\le q_0<2+\varepsilon$.

We will start with a brief discussion of the condition on the measures μ and ν given in Theorem 2. Then we will prove Theorem 2 assuming that Theorem 1 is valid. Subsequently we prove Theorem 3, the version of Theorem 2 with $\|u(x)\|_{H^\alpha}$ replacing $|\nabla u(x)|$ in (1). The companion result, sufficient conditions on a measure μ on Ω and a boundary measure $\varrho d\omega$ on $\partial\Omega$ so that $\|\,\|u(x)\|_{H^\alpha}\|_{L^q(\Omega,d\mu)} \le C\|g\|_{L^p(\partial\Omega,\varrho d\omega)}$ when $Lu=0$ in Ω, $u|_{\partial\Omega}=g$, is proved in [S].

2. Proof of Theorem 2. The condition on the measures μ and ν given in Theorem 2 may look complicated, but in fact it is closely related to well known properties of measures such as A^p conditions, geometric decay and the concept of Carleson measures. To gain an idea of what the condition in Theorem 2 can mean for the relation between μ and ν, we look at some simple examples. The domain Ω is bounded, so we need only consider the case when $l(Q_j) \le 1$.

If ν is taken to be Lebesgue measure, then Theorem 2's condition becomes that μ, on the Whitney-type cubes Q_j in Ω, must satisfy $\mu(Q_j)^{1/q} \le Cl(Q_j)^{n/2}$ with $C = C(p', n, \mathrm{diam}(\Omega))$. This follows from taking

$$M(Q_j) = \Big(\int_{Q_0} \Big(1 + \frac{|y - x_{Q_j}|}{l(Q_j)} \Big)^{-np'/2} d\sigma(y) \Big)^{1/p'}$$

and performing a standard estimate of the integral over Q_0 by dividing Q_0 into dyadic annular regions centered at Q_j. An elementary calculation shows that this condition also implies that $\mu(Q_j)^{1/q}$ multiplied by the other term,

$$l(Q_j)^{n/2+1} \Big(\int_{2Q_j} (d\nu/dx)^{-2/(p-2)}\, dx \Big)^{(p-2)/2p},$$

in the definition of $M(Q_j)$, is also less than or equal to $Cl(Q_j)^{n+1}$ since $d\nu/dx = 1$.

As a second example, consider letting $p = q = 4$, and take $\mu = \nu$. The requirement that

$$\mu(Q_j)^{1/q} \cdot l(Q_j)^{n/2+1} \Big(\int_{Q_j} (d\nu/dx)^{-2/(p-2)} dx \Big)^{(p-2)/2p}$$

be bounded by $Cl(Q_j)^{n+1}$ turns out to be equivalent to the following A^2-type condition:

$$\frac{1}{|Q_j|} \int_{Q_j} (d\mu/dx)\, dx \cdot \Big(\frac{1}{|Q_j|} \int_{2Q_j} (d\mu/dx)^{-1}\, dx \Big) \le C.$$

(In fact we know that μ is an A^2 measure by the reverse Hölder condition on μ.) The second condition, that

$$\mu(Q_j)^{1/q} \cdot \Big(\int_{Q_0} (1 + |y - x_{Q_j}|/l(Q_j))^{-np'/2} d\sigma(y) \Big)^{1/p'} \le Cl(Q_j)^{n+1},$$

does not have an exact interpretation as a well known measure condition, but it too can be viewed as a weighted version of an A^p-type condition with vanishing trace.

To prove Theorem 2 we start by dividing the integral $\|\nabla u\|^q_{L^q(\Omega, d\mu)}$ into a sum of integrals over Whitney cubes from $\mathcal{W}$:

$$\int_{\Omega} |\nabla u(x)|^q \, d\mu(x) = \sum_{Q_j \in \mathcal{W}} \int_{Q_j} |\nabla u(x)|^q \, d\mu(x)$$
$$\leq \sum_{Q_j \in \mathcal{W}} \Big(\int_{Q_j} |\nabla u(x)|^{q+\varepsilon} \, dx \Big)^{q/(q+\varepsilon)} \bigg(\int_{Q_j} \Big(\frac{d\mu}{dx} \Big)^{((q+\varepsilon)/q)'} \, dx \bigg)^{\varepsilon/(q+\varepsilon)}.$$

Now we can use the reverse Hölder conditions on both $d\mu/dx$ and $|\nabla u(x)|^{q+\varepsilon}$ if ε is sufficiently small (see [GM], [A] for reverse Hölder inequalities for ∇u), to bound the last sum by

$$C \sum_{Q_j \in \mathcal{W}} |Q_j| \bigg(\Big(\frac{1}{|Q_j|} \int_{2Q_j} |\nabla u(x)|^2 \, dx \Big)^{q/2} + \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\vec{f}(x)|^{q+\varepsilon} \Big)^{q/(q+\varepsilon)} \bigg) \times \bigg(\frac{1}{|Q_j|} \int_{Q_j} \Big(\frac{d\mu}{dx} \Big) \, dx \bigg).$$

Assuming that $\vec{f}$ lies in $H^1(\Omega)$, the Sobolev inequality allows us to replace $(|Q_j|^{-1} \int_{2Q_j} |\vec{f}(x)|^{q+\varepsilon})^{q/(q+\varepsilon)}$ by

$$C \bigg(\Big(\frac{1}{|Q_j|} \int_{2Q_j} |\vec{f}(x)|^2 \Big)^{q/2} + \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\operatorname{div} \vec{f}(x)|^2 \Big)^{q/2} \bigg).$$

Simplifying gives

$$\int_{\Omega} |\nabla u(x)|^q \, d\mu(x) \leq C \sum_{Q_j \in \mathcal{W}} \mu(Q_j) \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\nabla u(x)|^2 dx \Big)^{q/2}$$
$$+ C \sum_{Q_j \in \mathcal{W}} \mu(Q_j) \bigg(\Big(\frac{1}{|Q_j|} \int_{2Q_j} |\vec{f}(x)|^2 \Big)^{q/2} + \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\operatorname{div} \vec{f}(x)|^2 \Big)^{q/2} \bigg).$$

By duality it will suffice to bound the three expressions:

$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j \in \mathcal{W}} g(Q_j) \mu(Q_j) \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\nabla u(x)|^2 \, dx \Big)^{1/2},$$
$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j \in \mathcal{W}} g(Q_j) \mu(Q_j) \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\vec{f}(x)|^2 \Big)^{1/2},$$
$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j \in \mathcal{W}} g(Q_j) \mu(Q_j) \Big(\frac{1}{|Q_j|} \int_{2Q_j} |\operatorname{div} \vec{f}(x)|^2 \Big)^{1/2}.$$

Moreover we can assume $\{g(Q_j)\}$ is a finite sequence. The second and third sums are handled in the same way, so we only write out the details of bounding the second sum. By Hölder's inequality

$$\Big(\int_{2Q_j} |\vec{f}(x)|^2\Big)^{1/2} \le \Big(\int_{2Q_j} |\vec{f}(x)|^p \, d\nu(x)\Big)^{1/p} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-(2/p)(p/2)'} dx\Big)^{1/2(p/2)'}.$$

Using Hölder's inequality on the sum now gives

$$\sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\vec{f}(x)|^2\Big)^{1/2} \le \Big(\sum_{Q_j\in\mathcal{W}} \int_{2Q_j} |\vec{f}(x)|^p d\nu(x)\Big)^{1/p}$$
$$\times \Big(\sum_{Q_j\in\mathcal{W}} \Big(\frac{g(Q_j)\mu(Q_j)}{|Q_j|^{1/2}}\Big)^{p'} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{p'/2(p/2)'}\Big)^{1/p'}.$$

So we need to bound the second term by

$$C\Big(\sum_{Q_j\in\mathcal{W}} g(Q_j)^{q'}\mu(Q_j)\Big)^{1/q'}.$$

It is enough show that

$$\Big(\sum_{Q_j\in\mathcal{W}} \Big(\frac{g(Q_j)\mu(Q_j)}{|Q_j|^{1/2}}\Big)^{p'} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{p'/2(p/2)'}\Big)^{q'/p'} \le C \sum_{Q_j\in\mathcal{W}} g(Q_j)^{q'}\mu(Q_j).$$

Now, $p \le q$ so $q' \le p'$ and $q'/p' \le 1$. Consequently, the left hand side of the last inequality is less than or equal to

$$\sum_{Q_j\in\mathcal{W}} \Big(\frac{g(Q_j)\mu(Q_j)}{|Q_j|^{1/2}}\Big)^{q'} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{q'/2(p/2)'}.$$

If we compare the last two expressions term by term we will have a sufficient condition to obtain the desired inequality. So we need to have

$$\Big(\frac{g(Q_j)\mu(Q_j)}{|Q_j|^{1/2}}\Big)^{q'} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{q'/2(p/2)'} \lesssim g(Q_j)^{q'}\mu(Q_j)$$

or

$$\frac{\mu(Q_j)^{q'-1}}{|Q_j|^{q'/2}} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{q'/2(p/2)'} \le C_0'.$$

Taking q'th roots gives

$$\mu(Q_j)^{1/q} \Big(\int_{2Q_j} \Big(\frac{d\nu}{dx}\Big)^{-2/(p-2)} dx\Big)^{(p-2)/2p} \le C_0|Q_j|^{1/2}.$$

This condition also implies that

$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\operatorname{div}\vec{f}(x)|^2\Big)^{1/2} \le C\Big(\int_{\Omega} |\operatorname{div}\vec{f}(x)|^p\,d\nu(x)\Big)^{1/p}.$$

To handle the first sum,

$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla u(x)|^2\,dx\Big)^{1/2},$$

we note that

$$\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla u(x)|^2\,dx\Big)^{1/2} \le \Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla u(x)-\nabla\widetilde{u}(x)|^2\,dx\Big)^{1/2} + \Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla\widetilde{u}(x)|^2\,dx\Big)^{1/2} = I + II,$$

with $\widetilde{u}(x)=\int_{4Q_j}(\operatorname{div}\vec{f}(y))\widetilde{G}(x,y)\,dy$, where $\widetilde{G}(x,y)$ is the Green function for L on the domain $4Q_j$. Since

$$\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla\widetilde{u}(x)|^2\,dx\Big)^{1/2} \le C\Big(\frac{1}{|Q_j|}\int_{4Q_j} |\vec{f}(x)|^2\,dx\Big)^{1/2}$$

by standard results, the sum containing II can be handled as above. We are left with estimating

$$\sup_{\|\{g(Q_j)\}\|_{l^{q'}(\Omega,\mu)}=1} \sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla(u-\widetilde{u})(x)|^2\,dx\Big)^{1/2}.$$

To do this, notice that

$$\begin{aligned}\Big(\frac{1}{|Q_j|}&\int_{2Q_j} |\nabla(u-\widetilde{u})(x)|^2\,dx\Big)^{1/2}\\ &= \Big(\frac{1}{|Q_j|}\int_{2Q_j} \Big|\int_{\Omega}(\operatorname{div}\vec{f}(y))(\nabla_x G(x,y)-\nabla_x\widetilde{G}(x,y))\,dy\Big|^2\,dx\Big)^{1/2}\\ &\le \int_{\Omega} |\operatorname{div}\vec{f}(y)|\Big(\frac{1}{|Q_j|}\int_{2Q_j} |\nabla_x(G(x,y)-\widetilde{G}(x,y))|^2\,dx\Big)^{1/2}\,dy.\end{aligned}$$

Now for each $y\in 4Q_j$, $G(x,y)-\widetilde{G}(x,y)$ is a solution to $Lv=0$ in $\operatorname{supp}(\widetilde{G}(\cdot,y))\subset 4Q_j$ (see [K, pp. 87–88]), and we have, by the Caccioppoli inequality,

$$\left(\frac{1}{|Q_j|}\int_{2Q_j}|\nabla_x(G(x,y)-\widetilde{G}(x,y))|^2\,dx\right)^{1/2} \le \frac{1}{l(Q_j)}\left(\frac{C}{|Q_j|}\int_{3Q_j}|(G(x,y)-\widetilde{G}(x,y))|^2\,dx\right)^{1/2}.$$

Since $G(x,y)-\widetilde{G}(x,y)\ge 0$ by the maximum principle, we can use the Harnack inequality to obtain

$$\left(\frac{C}{|Q_j|}\int_{3Q_j}|G(x,y)-\widetilde{G}(x,y)|^2\,dx\right)^{1/2} \le \frac{C}{|Q_j|}\int_{3Q_j}(G(x,y)-\widetilde{G}(x,y))\,dx.$$

If $y\in\Omega\setminus 4Q_j$, then $\widetilde{G}(x,y)=0$ for $x\in 2Q_j$, and $G(x,y)$ is a positive solution to $L_x v=0$ in $3.5Q_j$. In this case

$$\left(\frac{1}{|Q_j|}\int_{2Q_j}|\nabla_x(G(x,y)-\widetilde{G}(x,y))|^2\,dx\right)^{1/2} = \left(\frac{1}{|Q_j|}\int_{2Q_j}|\nabla_x G(x,y)|^2\,dx\right)^{1/2}$$

$$\le \frac{1}{l(Q_j)}\left(\frac{C}{|Q_j|}\int_{3Q_j}|G(x,y)|^2\,dx\right)^{1/2} \le \frac{1}{l(Q_j)}\,\frac{C}{|Q_j|}\int_{3Q_j}G(x,y)\,dx$$

$$= \frac{1}{l(Q_j)}\,\frac{C}{|Q_j|}\int_{3Q_j}(G(x,y)-\widetilde{G}(x,y))\,dx$$

for the same reasons. Putting all this together we see that

$$\sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\left(\frac{1}{|Q_j|}\int_{2Q_j}|\nabla(u-\widetilde{u})(x)|^2\,dx\right)^{1/2}$$

$$\le C\sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\int_{\Omega}|\operatorname{div}\vec{f}(y)|\,\frac{1}{l(Q_j)}\left(\frac{1}{|Q_j|}\int_{3Q_j}(G(x,y)-\widetilde{G}(x,y))\,dx\right)dy.$$

The last expression is then dominated by

$$C\int_{\Omega}|\operatorname{div}\vec{f}(y)|\sum_{Q_j\in\mathcal{W}} g(Q_j)\mu(Q_j)\,\frac{1}{l(Q_j)}\left(\frac{1}{|Q_j|}\int_{3Q_j}(G(x,y)-\widetilde{G}(x,y))\,dx\right)dy$$

$$= C\int_{\Omega}|\operatorname{div}\vec{f}(y)|\sum_{Q_j\in\mathcal{W}}\lambda_j\varphi_{(Q_j)}(y)\,dy.$$

The constants λ_j are taken to be

$$\lambda_j = \frac{g(Q_j)\mu(Q_j)}{l(Q_j)\sqrt{|Q_j|}}$$

and the functions $\varphi_{(Q_j)}$ are defined by

$$\varphi_{(Q_j)}(y) = \frac{1}{\sqrt{|Q_j|}} \int_{3Q_j} (G(x,y) - \widetilde{G}(x,y))\,dx.$$

Assuming for now that the $\varphi_{(Q_j)}(y)$ satisfy (a), (a′), (b), and (c), Hölder's inequality followed by the application of Theorem 1 to the function $h(y) = \sum_{Q_j \in \mathcal{W}} \lambda_j \varphi_{(Q_j)}(y)$ gives

$$\begin{aligned}\int_\Omega |\operatorname{div} \vec{f}(y)| \sum_{Q_j \in \mathcal{W}} \lambda_j \varphi_{(Q_j)}(y)\,dy \\ &\le \Big(\int_\Omega |\operatorname{div} \vec{f}(y)|^p\,d\nu(y)\Big)^{1/p} \Big(\int_\Omega |h(y)|^{p'}\,d\sigma(y)\Big)^{1/p'} \\ &\le C|\operatorname{div} \vec{f}\|_{L^p(\Omega,d\nu)} \|g^*(h)\|_{L^{p'}(\Omega,d\sigma)}.\end{aligned}$$

It will suffice to show that $\|g^*(h)\|^{p'}_{L^{p'}(\Omega,d\sigma)} \le C\|\{g(Q_j)\}\|^{p'}_{l^{q'}(\Omega,\mu)}$. Using $p > 2$ implies $p'/2 < 1$. Recall that the sum defining $h(y)$ is finite. We have

$$\begin{aligned}\|g^*(h)\|^{p'}_{L^{p'}(\Omega,d\sigma)} &= \int_\Omega \Big(\sum \frac{\lambda_j^2}{|Q_j|} \Big(1 + \frac{|y-x_j|}{l(Q_j)}\Big)^{-n}\Big)^{p'/2} d\sigma(y) \\ &\le \sum \frac{\lambda_j^{p'}}{|Q_j|^{p'/2}} \int_\Omega \Big(1 + \frac{|y-x_j|}{l(Q_j)}\Big)^{-np'/2} d\sigma(y).\end{aligned}$$

So it suffices to show that the last sum is dominated by $(\sum g(Q_j)^{q'} \times \mu(Q_j))^{p'/q'}$. Once again, taking advantage of the fact that $q'/p' \le 1$, this is equivalent to showing that

$$\sum \frac{\lambda_j^{q'}}{|Q_j|^{q'/2}} \Big(\int_\Omega \Big(1 + \frac{|y-x_j|}{l(Q_j)}\Big)^{-np'/2} d\sigma(y)\Big)^{q'/p'} \le \sum g(Q_j)^{q'} \mu(Q_j).$$

So if

$$\frac{\lambda_j^{q'}}{|Q_j|^{q'/2}} \Big(\int_\Omega \Big(1 + \frac{|y-x_j|}{l(Q_j)}\Big)^{-np'/2} d\sigma(y)\Big)^{q'/p'} \le C g(Q_j)^{q'} \mu(Q_j)$$

we will have the desired result. But this is the same as requiring that

$$\mu(Q_j)^{1/q} \Big(\int_\Omega \Big(1 + \frac{|y-x_j|}{l(Q_j)}\Big)^{-np'/2} d\sigma(y)\Big)^{1/p'} \le C l(Q_j)^{n+1}.$$

The verification of (a), (a′), and (b) for the functions

$$\varphi_{(Q_j)}(y) = \frac{1}{\sqrt{|Q_j|}} \int_{3Q_j} (G(x,y) - \widetilde{G}(x,y))\,dx$$

follows directly from the estimates of Grüter and Widman [GW] for the Green function on any bounded domain satisfying an exterior cone condition. To see that (c) is also true, we may assume that $\lambda_j \geq 0$ and that $\varphi_{(Q_j)}(y) \geq 0$ on Ω. We can write

$$\begin{aligned}
\int_\Omega |h(y)|^2\,dy &= \int_\Omega \Big| \sum_{Q_j\in\mathcal{F}} \lambda_j \varphi_{(Q_j)}(y)\Big|^2\,dy = \sum_{Q_j\in\mathcal{F}} \lambda_j \int_\Omega h(y)\varphi_{(Q_j)}(y)\,dy \\
&= \sum_{Q_j\in\mathcal{F}} \lambda_j \int_\Omega h(y)\bigg(\frac{1}{\sqrt{|Q_j|}} \int_{3Q_j} (G(x,y) - \widetilde{G}(x,y))\,dx\bigg)\,dy \\
&\leq \sum_{Q_j\in\mathcal{F}} \lambda_j \frac{1}{\sqrt{|Q_j|}} \int_{3Q_j} v(x)\,dx \\
&\leq \Big(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\Big)^{1/2} \bigg(\sum_{Q_j\in\mathcal{F}} \bigg(\frac{1}{\sqrt{|Q_j|}} \int_{3Q_j} v(x)\,dx\bigg)^2\bigg)^{1/2} \\
&\leq \Big(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\Big)^{1/2} \Big(\sum_{Q_j\in\mathcal{F}} \int_{3Q_j} v(x)^2\,dx\Big)^{1/2} \\
&\leq C\Big(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\Big)^{1/2} \Big(\int_\Omega v(x)^2\,dx\Big)^{1/2} \leq C'\Big(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\Big)^{1/2} \Big(\int_\Omega h(x)^2\,dx\Big)^{1/2}.
\end{aligned}$$

We have taken $v(x) = \int_\Omega G(x,y)h(y)\,dy$ to be the solution to $Lv = h$ in Ω. Dividing by $(\int_\Omega h(x)^2\,dx)^{1/2}$ gives the property of almost orthogonality.

In contrast to Theorem 2 the condition on the weights for the inequality with the local Hölder norm replacing $|\nabla u|$ is much simpler. The other obvious advantage of using Hölder norms is that one obtains results for a larger range of exponents p and q. Suppose Ω is a bounded domain in $\mathbb{R}^n$ that satisfies an exterior cone condition. Then we have

THEOREM 3. *Let u be a solution to $Lu = \operatorname{div}\vec{f}$ on Ω, $u|_{\partial\Omega} = 0$, and let μ and ν be Borel measures on Ω, with ν finite and absolutely continuous with respect to Lebesgue measure, and $d\sigma(x) = (d\nu/dx)^{1-p'}\,dx$. If there is a constant $C > 0$ so that*

$$\mu(Q_j)^{1/q} \sup_{w\in 2Q_j} \bigg(\int_\Omega \frac{1}{|w-y|^{(n+\alpha-2)p'}}\,d\sigma(y)\bigg)^{1/p'} \leq C|Q_j|^{1/q}$$

for all dyadic cubes Q_j in $\mathcal{W}$, then for any $0 < q < \infty$, $1 < p < \infty$, there is a constant $C' = C'(C, n, \lambda, \alpha, q, \Omega)$ so that

$$\Big(\int_\Omega \|u(x)\|_{H^\alpha}^q\,d\mu(x)\Big)^{1/q} \leq C'\Big(\int_\Omega |\operatorname{div}\vec{f}(x)|^p\,d\nu(x)\Big)^{1/p}, \tag{2}$$

with $\|u(x)\|_{H^\alpha}$ defined as above, $\alpha = \alpha(n, \lambda, \partial\Omega)$.

Proof. As in the proof of Theorem 2 we start by subdividing Ω into Whitney cubes Q_j:

$$\begin{aligned}\int_\Omega \|u(x)\|^q_{H^\alpha}\,d\mu(x) &= \sum_{Q_j\in\mathcal{W}}\int_{Q_j}\|u(x)\|^q_{H^\alpha}\,d\mu(x)\\ &= \sum_{Q_j\in\mathcal{W}}\int_{Q_j}\Big(\sup_{w\in P(x),\,w\neq x}\frac{|u(x)-u(w)|}{|x-w|^\alpha}\Big)^q\,d\mu(x)\\ &\le \sum_{Q_j\in\mathcal{W}}\int_{Q_j}\Big(\sup_{\substack{x\in Q_j,\,x\neq w\\ w\in 2Q_j}}\frac{|u(x)-u(w)|}{|x-w|^\alpha}\Big)^q\,d\mu(x)\\ &= \sum_{Q_j\in\mathcal{W}}\Big(\sup_{\substack{x\in Q_j,\,x\neq w\\ w\in 2Q_j}}\frac{|u(x)-u(w)|}{|x-w|^\alpha}\Big)^q\int_{Q_j}d\mu(x).\end{aligned}$$

Now the integral representation of $u(x)$ and the fact that the Green function is Hölder continuous imply the last expression is dominated by

$$\begin{aligned}&C\sum_{Q_j\in\mathcal{W}}\mu(Q_j)\Big(\sup_{\substack{x\in Q_j,\,x\neq w\\ w\in 2Q_j}}\frac{1}{|x-w|^\alpha}\int_\Omega|\mathrm{div}\,\vec f(y)|\,|G(x,y)-G(w,y)|\,dy\Big)^q\\ &\le C\sum_{Q_j\in\mathcal{W}}\mu(Q_j)\Big(\sup_{\substack{x\in Q_j,\,x\neq w\\ w\in 2Q_j}}\frac{1}{|x-w|^\alpha}\\ &\qquad\times\int_\Omega|\mathrm{div}\,\vec f(y)|\,|x-w|^\alpha\Big(\frac{1}{|x-y|^{n-2+\alpha}}+\frac{1}{|w-y|^{n-2+\alpha}}\Big)\,dy\Big)^q\\ &\le C\sum_{Q_j\in\mathcal{W}}\mu(Q_j)\Big(\sup_{w\in 2Q_j}\int_\Omega|\mathrm{div}\,\vec f(y)|\,|w-y|^{2-n-\alpha}\,dy\Big)^q.\end{aligned}$$

The next to last inequality was obtained from the result of Theorem 1.9 in [GW]. The constant C has changed from one line to the next, but is independent of u, f and Q_j. The last sum is less than or equal to

$$\begin{aligned}&C\sum_{Q_j\in\mathcal{W}}\mu(Q_j)\Big(\int_\Omega|\mathrm{div}\,\vec f(x)|^p\,d\nu(x)\Big)^{q/p}\Big(\sup_{w\in 2Q_j}\Big(\int_\Omega|w-y|^{(2-n-\alpha)p'}\,d\sigma(y)\Big)^{1/p'}\Big)\\ &= C\Big(\int_\Omega|\mathrm{div}\,\vec f(x)|^p d\nu(x)\Big)^{q/p}\\ &\qquad\sum_{Q_j\in\mathcal{W}}\mu(Q_j)\Big(\sup_{w\in 2Q_j}\Big(\int_\Omega|w-y|^{(2-n-\alpha)p'}\,d\sigma(y)\Big)^{1/p'}\Big).\end{aligned}$$

The qth root of this last expression will be bounded by

$$C|\Omega|^{1/q}\Big(\int_\Omega|\mathrm{div}\,\vec f(x)|^p\,d\nu(x)\Big)^{1/p}$$

if

$$\mu(Q_j)^{1/q} \sup_{w\in 2Q_j} \Big(\int_{\Omega} |w-y|^{(2-n-\alpha)p'}\, d\sigma(y)\Big)^{1/p'} \le |Q_j|^{1/q}$$

for every $Q_j \in \mathcal{W}$.

3. Proof of Theorem 1. To prove Theorem 1 we follow the method of Wilson [W] in using the functions

$$F(I,x) = \sum_{J\in\mathcal{S}(I)} \lambda_J \varphi_{(J)}(x), \quad F(I) = F(I,x_I), \quad F^*(x) = \sup_{I\ni x} F(I)$$

and

$$G(I,x) = \Big(\sum_{J\in\mathcal{S}(I)} \frac{\lambda_J^2}{|J|}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n}\Big)^{1/2},$$

$$G(I) = G(I,x_I), \quad G^*(x) = \sup_{I\ni x} G(I).$$

They are always generated by a given function $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, where $\mathcal{F}$ is a finite family of dyadic cubes, $\mathcal{S}(I) = \{J\in\mathcal{F} : J \not\subset I\}$, and $l(I)$ is the side length of the dyadic cube I. $F(I,x)$ and $G(I,x)$ are only defined for $x\in I$. We note some special properties of the particular functions $\varphi_{(J)}(x)$ that were used in the proof of Theorem 2. These properties will be crucial in proving the estimates in Lemmas 1–7 and the Central Lemma. We have $\varphi_{(J)}(x) = 0$ whenever x lies outside Ω. Also each $\varphi_{(J)}$ is chosen so that $J\in\mathcal{W}$. As in [W] we obtain local estimates relating the functions $F(I,x)$, $G(I,x)$, etc. in order to use these functions to prove the crucial good-λ inequality of the Corollary to the Central Lemma. The good-λ inequality then yields the result of Theorem 1 by standard methods. The local estimates are established in Lemmas 1–7 below.

For the remaining part of the paper we take $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, where $\mathcal{F}$ is a finite family of dyadic cubes; the $\varphi_{(J)}$ satisfy properties (a), (a′), (b), and (c), and they have all the properties mentioned in the previous paragraph. We note that many of the constants obtained in Lemmas 1–7 depend on $\operatorname{diam}(\Omega)$. For the functions $\varphi_{(J)}$ that appeared in the proof of Theorem 2, i.e.

$$\varphi_{(J)}(y) = \frac{1}{\sqrt{|J|}}\Big(\int_{2J} (G(x,y) - \widetilde{G}(x,y))\, dx\Big),$$

the constants in (a′) and (b) also depend on $\operatorname{diam}(\Omega)$ and Ω (see [GW]), so this is no new restriction. We also note that having $\varphi_{(J)}(x) = 0$ whenever x lies outside Ω means that $F(I,x) = 0$ when $x\in\Omega^{\mathrm{c}}$. However, $F^*(x)$, $G(I,x)$, $G^*(x)$ are not necessarily zero for x outside Ω. Following Wilson [W] we start with

LEMMA 1. *$f(x) \le F^*(x)$ for a.e. $x \in Q_0$.*

Proof. This follows from the definition of $F^*(x)$, the fact that $\mathcal{F}$ is a finite family, and that (a) and (b) imply that f is continuous.

LEMMA 2. *There is a constant C so that $G^*(x) \le Cg^*(x)$.*

Proof. We have $G^*(x) = \sup_{Q \ni x} G(Q)$. If $x \in Q$ and $I \in \mathcal{S}(Q)$, then either $I \supsetneq Q$ or I lies outside Q. In both cases, $|x_I - x_Q| \ge cl(Q)$ and $|x - x_Q| \le c'l(Q)$. Therefore $|x - x_I| \le |x - x_Q| + |x_Q - x_I| \le C|x_Q - x_I|$. So $1 + |x - x_I|/l(I) \le C'(1 + |x_Q - x_I|/l(I))$ or

$$(1 + |x_Q - x_I|/l(I))^{-n} \le C''(1 + |x - x_I|/l(I))^{-n}.$$

For $I \in \mathcal{F}$, whenever the term on the left is in $G(Q)$, the term on the right appears in $g^*(x)$, $x \in Q$, multiplied by $1/C''$. This is true for all dyadic cubes Q with the same constant $C = \max(1, C'')$, so $G(Q) \le Cg^*(x)$.

LEMMA 3. *For any $0 < \eta < 1$, if $x \in \eta Q$, then there is a constant $C_1 = C(n, \eta)$ so that $C_1^{-1}G(Q) \le G(Q, x) \le C_1 G(Q)$.*

Proof. For any cube $I \in \mathcal{S}(Q)$, $|x - x_I|/|x_Q - x_I|$ is bounded above and below by constants that depend on η and n.

LEMMA 4. *For any $0 < \eta < 1$, if $x, y \in \eta Q$, then there is a constant $C_2 = C(n, \lambda, \eta, \operatorname{diam}(\Omega), \Omega, \alpha)$ so that $|F(Q, x) - F(Q, y)| \le C_2 G(Q)$.*

Proof. As in [W] we write

$$\begin{aligned}|F(Q,x) - F(Q,y)| &= \Big|\sum_{J \in \mathcal{S}(Q)} \lambda_J(\varphi_{(J)}(x) - \varphi_{(J)}(y))\Big| \\ &\le \sum_{J \in \mathcal{S}(Q),\, l(J) \ge l(Q)} |\lambda_J|\,|\varphi_{(J)}(x) - \varphi_{(J)}(y)| \\ &\quad + \sum_{J \in \mathcal{S}(Q),\, l(J) < l(Q)} |\lambda_J|\,|\varphi_{(J)}(x) - \varphi_{(J)}(y)| = I + II.\end{aligned}$$

When x and y both lie inside Ω, I will be shown to be bounded by $CG(Q)$ using the Hölder continuity of the $\varphi_{(J)}$'s (property (b)), and II should be bounded using Hölder continuity. When both x and y lie outside Ω, $F(Q, x)$ and $F(Q, y)$ are both 0, so the estimate of Lemma 4 is trivially valid. However, the situation when $x \in \Omega$ but $y \in \Omega^{\mathrm{c}}$ needs to be considered separately. We are not guaranteed that (b) is valid when one point, x or y, lies outside the domain Ω. In this case I and II should be estimated using (a′).

We start with the proof for $x, y \in \eta Q \subset \Omega$. Then by (b) and the Cauchy–Schwarz inequality (remember that $n \ge 3$),

$$
\begin{aligned}
I &= \sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} |\lambda_J|\,|\varphi_{(J)}(x)-\varphi_{(J)}(y)| \\
&\lesssim \sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} |\lambda_J|\,|x-y|^{\alpha} l(J)^{2-n/2-\alpha}\Big(1+\frac{|x-x_J|}{l(J)}+\frac{|y-x_J|}{l(J)}\Big)^{2-n-\alpha} \\
&\lesssim \sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} |\lambda_J|\Big(\frac{|x-y|}{l(J)}\Big)^{\alpha} l(J)^{2-n/2}\Big(1+\frac{|x-x_J|}{l(J)}+\frac{|y-x_J|}{l(J)}\Big)^{2-n-\alpha} \\
&\lesssim \Big(\sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} \frac{\lambda_J^2}{|J|}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n}\Big)^{1/2} \\
&\quad\times\Big(\sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} \Big(\frac{|x-y|}{l(J)}\Big)^{2\alpha} l(J)^4\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{4-n-2\alpha}\Big)^{1/2} \\
&\lesssim G(Q,x)\cdot C(d(\Omega))^2 \\
&\quad\times\Big(\sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} \Big(\frac{|x-y|}{l(J)}\Big)^{2\alpha}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha}\Big)^{1/2}.
\end{aligned}
$$

The last inequality follows from the fact that $l(J)(1+|x-x_J|/l(J)) \leq c\,\mathrm{diam}(\Omega) = cd(\Omega)$. From Lemma 3, $G(Q,x)\leq C(n,\eta)G(Q)$ because $x\in\eta Q$. So we need only show that

$$
\Big(\sum_{J\in\mathcal{S}(Q),\, l(J)\geq l(Q)} \Big(\frac{|x-y|}{l(J)}\Big)^{2\alpha}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha}\Big) \leq C.
$$

Since x and y lie in ηQ, the sum on the left can be written as

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\substack{l(J)=2^k l(Q)\\ J\in\mathcal{S}(Q)}} 2^{-2\alpha k}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha} \\
&\qquad= \sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j\geq k}\sum_{\substack{2^{j-1}l(Q)<l(J)+|x-x_J|\leq 2^j l(Q)\\ l(J)=2^k l(Q),\, J\in\mathcal{S}(Q)}}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha} \\
&\qquad\leq \sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j\geq k}\sum_{\substack{2^{j-k-1}<1+|x-x_J|/l(J)\leq 2^{j-k}\\ l(J)=2^k l(Q),\, J\in\mathcal{S}(Q)}}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha} \\
&\qquad\leq C(n)\sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j=k}^{\infty}2^{n(j-k)}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha}.
\end{aligned}
$$

The last estimate follows from counting the number of cubes J of side length $2^k l(Q)$ that can exist in the annular region $2^{j-k-1} < 1 + |x_Q - x_J|/l(J) \leq 2^{j-k}$ if $j \geq k$. It is easy to see that the last expression is bounded by

$$C(n)\sum_{k=0}^{\infty} 2^{-2\alpha k}\sum_{j=k}^{\infty} 2^{n(j-k)}(2^{j-k-1})^{-(n+2\alpha)} \leq C(n,\alpha)\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} 2^{-2\alpha j}$$
$$\leq C(n,\alpha)\sum_{k=0}^{\infty} 2^{-2\alpha k}\sum_{j=0}^{\infty} 2^{-2\alpha j} \leq C(n,\alpha).$$

Now to bound II, still keeping $x, y \in \eta Q \subset \Omega$, we have

$$II \leq \sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} |\lambda_J|\,|\varphi_{(J)}(x) - \varphi_{(J)}(y)|,$$

so by Lemma 3 it is enough to show this sum is $\leq CG(Q,x)$. Using (b) gives

$$\text{(A)} \qquad \sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} |\lambda_J|\,|\varphi_{(J)}(x) - \varphi_{(J)}(y)|$$
$$\lesssim \sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} |\lambda_J|\,|x-y|^{\alpha} l(J)^{2-n/2-\alpha}\left(1 + \frac{|x-x_J|}{l(J)} + \frac{|y-x_J|}{l(J)}\right)^{2-n-\alpha}$$
$$\lesssim \left(\sum_{J\in\mathcal{S}(Q)} \frac{\lambda_J^2}{|J|}\left(1+\frac{|x-x_J|}{l(J)}\right)^{-n}\right)^{1/2}$$
$$\times \left(\sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} l(Q)^{2\alpha} l(J)^{4-2\alpha}\left(1+\frac{|x-x_J|}{l(J)}\right)^{4-n-2\alpha}\right)^{1/2}$$
$$= CG(Q,x)\left(\sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} \left(\frac{l(Q)}{l(J)}\right)^{2\alpha} l(J)^4\left(1+\frac{|x-x_J|}{l(J)}\right)^{4-n-2\alpha}\right)^{1/2}$$
$$\leq CG(Q)\cdot H_{(Q)}(x).$$

Now,

$$H_Q(x) \leq C(\operatorname{diam}\Omega)^2\left(\sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} \left(\frac{l(Q)}{l(J)}\right)^{2\alpha}\left(1+\frac{|x-x_J|}{l(J)}\right)^{-n-2\alpha}\right)^{1/2}$$
$$\leq C(d(\Omega), n, \eta)$$
$$\times \left(\sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} \left(\frac{l(Q)}{l(J)}\right)^{-n-2\alpha+2\alpha}\left(1+\frac{|x_Q-x_J|}{l(Q)}\right)^{-n-2\alpha}\right)^{1/2}.$$

The last inequality follows from the fact that for $J \cap Q = \emptyset$, $J \in \mathcal{S}(Q)$, $x \in \eta Q$, we have $|x - x_J|/|x_Q - x_J| \sim C$, and since $|x_Q - x_J| \gtrsim l(Q)$, we have

$$1+\frac{|x-x_J|}{l(J)} \gtrsim \frac{l(Q)}{l(J)}\,\frac{|x_Q-x_J|}{l(Q)} \gtrsim \frac{l(Q)}{l(J)}\Big(1+\frac{|x_Q-x_J|}{l(Q)}\Big).$$

This means that

$$\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n-2\alpha} \lesssim \Big(\frac{l(Q)}{l(J)}\Big(1+\frac{|x_Q-x_J|}{l(Q)}\Big)\Big)^{-n-2\alpha}.$$

Finally, to estimate

$$\bigg(\sum_{J\in\mathcal{S}(Q),\,l(J)<l(Q)}\Big(\frac{l(Q)}{l(J)}\Big)^{-n}\Big(1+\frac{|x_Q-x_J|}{l(Q)}\Big)^{-n-2\alpha}\bigg)^{1/2}$$

we can proceed as in [W] to divide $Q_0\setminus Q$ into dyadic cubes Q' whose size is the same as that of Q. We write the sum as

$$\bigg(\sum_{Q'\subset Q_0\setminus Q}\ \sum_{J\in\mathcal{S}(Q),\,J\subset Q'}\Big(\frac{l(J)}{l(Q)}\Big)^{n}\Big(1+\frac{|x_Q-x_J|}{l(Q)}\Big)^{-n-2\alpha}\bigg)^{1/2}$$
$$\lesssim \bigg(\sum_{Q'\subset Q_0\setminus Q}\ \sum_{J\in\mathcal{S}(Q),\,J\subset Q'}\Big(\frac{l(J)}{l(Q)}\Big)^{n}\Big(1+\frac{|x_Q-x_{Q'}|}{l(Q')}\Big)^{-n-2\alpha}\bigg)^{1/2},$$

which is valid since $|x_Q-x_J|\gtrsim|x_Q-x_{Q'}|$. Now the J are Whitney cubes from $\mathcal{F}$, so they are disjoint. Consequently, for each Q',

$$\sum_{J\subset Q'}\Big(\frac{l(J)}{l(Q)}\Big)^{n}=\sum_{J\subset Q'}\frac{|J|}{|Q'|}\le 1.$$

Therefore we can write

$$H_Q(x)^2 \lesssim \sum_{k=0}^{\infty}\ \sum_{\substack{Q'\subset Q_0\setminus Q\\ 2^{k-1}l(Q)\le|x_Q-x_{Q'}|<2^k l(Q)}}\Big(1+\frac{|x_Q-x_{Q'}|}{l(Q')}\Big)^{-n-2\alpha}$$
$$\lesssim \sum_{k=0}^{\infty}2^{kn}2^{-k(n+2\alpha)}\le C(\alpha,n),$$

by counting the maximum number of cubes Q' that can lie inside the annular region $2^{k-1}l(Q)\le|x_Q-x_{Q'}|<2^k l(Q)$.

We have shown that $|F(Q,x)-F(Q,y)|\le C_2G(Q)$ when both x and y lie inside Ω, or when both points lie in $Q_0\setminus\Omega$. The remaining case is for one point lying inside Ω and the other outside Ω. This implies of course that the dyadic cube Q is such that $\eta Q\cap\Omega\ne\emptyset$ and $\eta Q\cap\Omega^{\mathrm{c}}\ne\emptyset$. Without loss of generality $x\in\Omega$ and $y\notin\Omega$. So $F(Q,y)=0$. Here we cannot use (b), since the decay in (b) is not necessarily valid for points outside Ω. However, we note that (a′) is useful. Since Q overlaps the boundary of Ω, and $x \in Q \cap \Omega$, we have $\delta(x) = \operatorname{dist}(x, \partial\Omega) \lesssim l(Q)$. So

$$\begin{aligned} |F(Q,x) - F(Q,y)| &= |F(Q,x)| \\ &\le \sum_{J\in\mathcal{S}(Q),\, l(J)\ge l(Q)} |\lambda_J|\,|\varphi_J(x)| + \sum_{J\in\mathcal{S}(Q),\, l(J)<l(Q)} |\lambda_J|\,|\varphi_J(x)| = I' + II'. \end{aligned}$$

Now,

$$(I')^2 \lesssim \sum_{J\in\mathcal{S}(Q),\, l(J)\ge l(Q)} |\lambda_J| \delta(x)^\alpha l(J)^{2-n/2-\alpha} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{2-n-\alpha}$$

from using (a′) on the $|\varphi_J(x)|$'s in I'. The last sum is bounded by

$$\begin{aligned} C(\operatorname{diam}(\Omega))^2 &\sum_{J\in\mathcal{S}(Q),\, l(J)\ge l(Q)} |\lambda_J| \left(\frac{l(Q)}{l(J)}\right)^\alpha l(J)^{-n/2} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{-n-\alpha} \\ &\le C(d(\Omega), n, \alpha, \lambda, \eta) G(Q) \\ &\quad \times \left(\sum_{J\in\mathcal{S}(Q),\, l(J)\ge l(Q)} \left(\frac{l(Q)}{l(J)}\right)^{2\alpha} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{-n-2\alpha} \right)^{1/2}. \end{aligned}$$

Dominating the last sum by a constant follows as before. Estimating II' follows from almost the same proof that gave the bound for II in the first case, in which x and y were both located inside Ω. Here the fact that $\delta(x) \lesssim l(Q)$ replaces the similar estimate for $|x - y|$ in (A). After that the calculations are identical.

For the next four lemmas we define

$$N(I) = \{I^* \in \mathfrak{D} : I^* \subset I \text{ and } l(I^*) = 0.5 l(I)\}$$

for any dyadic cube $I \in \mathfrak{D}$. We have

LEMMA 5. *$G(I) \le CG(I^*)$.*

Proof. We have $x_{I^*} \in \eta I$ if $0 < \eta < 1$ is sufficiently large, depending on n. By Lemma 3, $G(I) \le CG(I, x_{I^*})$, and by definition $G(I, x_{I^*}) \le G(I^*, x_{I^*}) = G(I^*)$.

LEMMA 6. *For $I^* \in N(I)$, $G(I^*) \le CG^*(x)$ whenever $x \in I$.*

Proof. By definition $G^*(x) = \sup_{J\ni x} G(J)$, so if $x \in I^*$, then $G(I^*) \le G^*(x)$. Suppose that x lies in $I \setminus I^*$. For any $J \subset I \setminus I^*$ such that $x \in J$, we have $G(I^*)^2 \le CG(J)^2 + B$, where

$$B = \sum_{K\subset J,\, K\in\mathcal{S}(I^*)} \frac{\lambda_K^2}{|K|} \left(1 + \frac{|x_{I^*} - x_K|}{l(K)}\right)^{-n}.$$

All the terms in B occur in $G(I^*)^2$. If $L \in \mathcal{S}(I^*)\backslash\{K \subset J : K \in \mathcal{S}(I^*)\}$, then $L \in \mathcal{S}(J)$. Moreover $|x_L - x_J| \le |x_L - x_{I^*}| + |x_J - x_{I^*}| \le |x_L - x_{I^*}| + cl(I) \le c'|x_L - x_{I^*}|$ since $|x_L - x_{I^*}| \gtrsim l(I)$. We may assume $c' \ge 1$; this implies that

$$\left(1 + \frac{|x_L - x_J|}{l(L)}\right)^{-n} \ge C'\left(1 + \frac{|x_L - x_{I^*}|}{l(L)}\right)^{-n}.$$

So each term in $G(I^*)$ that does not occur in B is less than or equal to a constant times a term that occurs in $G(J)$. Now $\mathcal{F}$ is a finite family, so for $|J|$ sufficiently small, the sum in B will be empty, and $G(J) \le G^*(x)$.

Lemma 7. *$|F(I^*) - F(I)| \le CG(I^*)$.*

Proof. Lemmas 4 and 5 imply that $|F(I, x_{I^*}) - F(I)| \le CG(I) \le C'G(I^*)$; consequently, it is enough to show that $|F(I, x_{I^*}) - F(I^*)| \le CG(I^*)$. If $x_{I^*} \in \Omega^{\mathrm{c}}$, then both functions on the left are zero, so we can assume that $x_{I^*} \in \Omega$. We have

$$\begin{aligned}|F(I, x_{I^*}) - F(I^*)| &= \Big|\sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)} \lambda_J \varphi_J(x_{I^*})\Big| \\ &\le \sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)} |\lambda_J| \cdot l(J)^{2-n/2} \cdot \left(1 + \frac{|x_{I^*} - x_J|}{l(J)}\right)^{2-n}\end{aligned}$$

from (a). The Cauchy–Schwarz inequality gives

$$\begin{aligned}|F(I, x_{I^*}) - F(I^*)| &\le \Big(\sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)} \frac{|\lambda_J|^2}{|J|}\left(1 + \frac{|x_{I^*} - x_J|}{l(J)}\right)^{-n}\Big)^{1/2} \\ &\quad \times \Big(\sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)} l(J)^4\left(1 + \frac{|x_{I^*} - x_J|}{l(J)}\right)^{4-n}\Big)^{1/2} \\ &\le CG(I^*) \cdot C(\mathrm{diam}(\Omega))^2 \cdot \Big(\sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(1 + \frac{|x_{I^*} - x_J|}{l(J)}\right)^{-n}\Big)^{1/2}.\end{aligned}$$

If we can show that $(\sum_{J \in \mathcal{S}(I^*)\backslash\mathcal{S}(I)}(1 + |x_{I^*} - x_J|/l(J))^{-n})^{1/2}$ is bounded by a constant, we will be done. Notice that

$$1 + \frac{|x_{I^*} - x_J|}{l(J)} \ge \frac{|x_{I^*} - x_J|}{l(J)} = \frac{l(I)}{l(J)} \cdot \frac{|x_{I^*} - x_J|}{l(I)},$$

and $|x_{I^*} - x_J| \sim l(I)$ because $J \in \mathcal{S}(I^*) \setminus \mathcal{S}(I)$. So

$$\frac{l(I)}{l(J)} \cdot \frac{|x_{I^*} - x_J|}{l(I)} \ge C\,\frac{l(I)}{l(J)}\left(1 + \frac{|x_{I^*} - x_J|}{l(I)}\right).$$

We have

$$\left(1+\frac{|x_{I^*}-x_J|}{l(J)}\right)^{-n} \le C\left(\frac{l(J)}{l(I)}\right)^{n}\left(1+\frac{|x_{I^*}-x_J|}{l(I)}\right)^{-n}.$$

This gives

$$\sum_{J\in\mathcal{S}(I^*)\setminus\mathcal{S}(I)}\left(1+\frac{|x_{I^*}-x_J|}{l(J)}\right)^{-n} \le C\sum_{J\in\mathcal{S}(I^*)\setminus\mathcal{S}(I)}\left(\frac{l(J)}{l(I)}\right)^{n}\left(1+\frac{|x_{I^*}-x_J|}{l(I)}\right)^{-n}.$$

Now remember that the cubes J originally came from $\mathcal{F}$ so they are disjoint. Also $J\in\mathcal{S}(I^*)\setminus\mathcal{S}(I)$ means that either $J=I$ or $J\subset I\setminus I^*$. As a result

$$\sum_{J\in\mathcal{S}(I^*)\setminus\mathcal{S}(I)}\left(\frac{l(J)}{l(I)}\right)^{n}\left(1+\frac{|x_{I^*}-x_J|}{l(I)}\right)^{-n} \le \sum_{J\in\mathcal{S}(I^*)\setminus\mathcal{S}(I)}\left(\frac{l(J)}{l(I)}\right)^{n} \le 1.$$

The purpose of establishing Lemmas 1–7 is to prove the following

Central Lemma. *Let* $f(x)=\sum_{I\in\mathcal{F}}\lambda_I\varphi_{(I)}(x)$, *where* $\mathcal{F}$ *is a finite family of cubes from* $\mathcal{W}$, *the* $\varphi_{(I)}$ *satisfy* (a), (a′), (b) *and* (c), *and* $\lambda_I=0$ *for any* $I\not\subset I_0$, *where* I_0 *is a fixed cube in* $\mathfrak{D}$. *For any* $0<\beta<1$, *there is a* $\gamma=\gamma(\beta,n,\lambda,\alpha,\Omega,\eta)$ *such that*

$$|\{x\in I_0: F^*(x)>1 \text{ and } G^*(x)\le\gamma\}|\le\beta|I_0|.$$

Proof. Let I_j be the dyadic cubes for which one of the subcubes $I_j^*\in N(I_j)$ is a maximal dyadic cube in I_0 so that $G(I_j^*)>A\gamma$ for A large enough so $AC^{-1}>1$, C being the constant in Lemma 6. Notice that $G(I_0)=0$ (and so $F(I_0,x)=0$ for any $x\in I_0$), and $I_j\subset I_0$. We see that $G(I_j)\le A\gamma$, $x\in I_j$ implies that $G^*(x)>AC^{-1}\gamma>\gamma$ from Lemma 6, and $G^*(x)\le A\gamma$ for all $x\in I_0\setminus\bigcup I_j$.

Let $E=\{x\in I_0: F^*(x)>1 \text{ and } G^*(x)\le\gamma\}$. For any $x\in E$ there is a maximal dyadic cube Q_i such that $F(Q_i)>1$. We have $Q_i\subset I_0$ and $Q_i\not\subset I_j$ for any of the maximal cubes defined in the previous paragraph, because $G^*(x)\le\gamma$ means that x cannot lie in I_j. Following the argument in the proof of the Main Lemma in [W], we create the family $\mathcal{G}=\{P_k\}$ of dyadic cubes which consists of the maximal disjoint cubes that result from combining the I_j and the Q_i. So $E\subset\bigcup_k P_k$. In fact $x\in E$ implies that $x\in P_{k'}$ for some maximal cube in $\mathcal{G}$ for which $F(P_{k'})>1$. It is also true that $G(P_{k'})\le\gamma$, since $G^*(x)\le\gamma$. We proceed to divide the cubes in $\mathcal{F}$ into two sets, $\mathcal{F}_1=\{J: J\not\subset P_k \text{ for any } P_k\in\mathcal{G}\}$ and $\mathcal{F}_2=\{J: J\subset P_k$ for some $P_k\in\mathcal{G}\}$. Writing $f(x)=\sum_{J\in\mathcal{F}_1}\lambda_J\varphi_{(J)}(x)+\sum_{J\in\mathcal{F}_2}\lambda_J\varphi_{(J)}(x)=f_1(x)+f_2(x)$, we can define $F_i(Q,x)$, $F_i^*(Q)$, $G_i(Q,x)$, $G_i^*(x)$ for $i=1,2$ just as we did for $f(x)$. We have $F(Q,x)=F_1(Q,x)+F_2(Q,x)$, while $G_i(Q,x)\le G_1(Q,x)+G_2(Q,x)=G(Q,x)$.

The facts that $E\subset\bigcup P_{k'}$ and that Lebesgue measure is a doubling measure mean $|E|\le C(n)\sum_{k'}|c(P_{k'})|$, where $c(P_k)=\{x\in P_k: x\in\frac{1}{10}P_k\}$.

For $x \in P_{k'}$, we must have either $F_1(P_{k'}) > 0.5$ or $F_2(P_{k'}) > 0.5$. For $x \in c(P_{k'})$, Lemma 4 says that either $F_1(P_{k'}, x) > 0.25$ or $F_2(P_{k'}, x) > 0.25$ whenever γ is small enough. Also

$$\sum_{k'} |c(P_{k'})| \leq \sum_{F_1(P_{k'})>0.5} |c(P_{k'})| + \sum_{F_2(P_{k'})>0.5} |c(P_{k'})|$$
$$\leq \sum_k |\{x \in c(P_k) : F_1(P_k, x) > 0.25\}| + \sum_k |\{x \in c(P_k) : F_2(P_k, x) > 0.25\}|.$$

Using Chebyshev's inequality we can say we only need to estimate

$$\sum_k 16 \int_{c(P_k)} |F_1(P_k, x)|^2 dx \quad \text{and} \quad \sum_k 16 \int_{c(P_k)} |F_2(P_k, x)|^2 \, dx.$$

In fact, for the second sum we will estimate each integral taken over a smaller set than $c(P_k)$. This will be explained after we obtain a bound for the first sum. Notice that the definition of $\mathcal{F}_1$ gives that $F_1(P_k, x) = f_1(x)$ for any $x \in P_k$. Then

$$\sum_k 16 \int_{c(P_k)} |F_1(P_k, x)|^2 \, dx = \sum_k 16 \int_{c(P_k)} |f_1(x)|^2 \, dx \leq C \int_{I_0} |f_1(x)|^2 \, dx.$$

By almost orthogonality, property (c) for the $\varphi_{(I)}$'s,

$$\int_{I_0} |f_1(x)|^2 \, dx \leq \sum_{J \in \mathcal{F}_1} \lambda_J^2 = \int_{I_0} \sum_{\substack{J \in \mathcal{F}_1 \\ J \ni x}} \frac{\lambda_J^2}{|J|} \, dx \leq (A\gamma)^2 |I_0| \leq \frac{\beta}{3} |I_0|$$

for γ sufficiently small. The second to the last estimate follows from the fact that for $x \in I_0 \setminus \bigcup I_j$ (I_j are the maximal cubes defined at the beginning of the proof), $\sum_{J \ni x} \lambda_J^2/|J| \leq G^*(x)^2 \leq (A\gamma)^2$, and for $x \in I_j$, the sum $\sum_{J \ni x, J \in \mathcal{F}_1}$ is empty.

Next we bound $\sum_k |\{x \in c(P_k) : F_2(P_k, x) > 0.25\}|$. As in [W] we cut out a thin annular region around each of the P_k's to handle edge effects. Choosing $\tau > 1$ so that $|\tau P_k \setminus P_k| \leq (\beta/3)|P_k|$, and letting $D = \bigcup\{\tau P_k \setminus P_k\}$, we have $|D| \leq (\beta/3)|I_0|$ (remember the P_k are disjoint). Also

$$\sum_k |\{x \in c(P_k) : F_2(P_k, x) > 0.25\}| \leq |D| + \sum_k 16 \int_{c(P_k) \setminus D} |F_2(P_k, x)|^2 \, dx.$$

We need only prove that

$$\sum_k 16 \int_{c(P_k) \setminus D} |F_2(P_k, x)|^2 \, dx \leq C'(A\gamma)^2 |I_0|,$$

and take γ small enough so that $C'(A\gamma)^2 \leq \beta/3$.

If k is temporarily fixed and $x \in c(P_k) \setminus D$, then $F_2(P_k,x) = \sum_{J\in\mathcal{F}_2, J\not\subset P_k} \lambda_J \varphi_{(J)}(x)$, so

$$|F_2(P_k,x)|^2 \le \Big|\sum_{J\in\mathcal{F}_2,\, J\not\subset P_k} \lambda_J\varphi_{(J)}(x)\Big|^2 \le \bigg(\sum_{J\in\mathcal{F}_2,\, J\not\subset P_k} |\lambda_J| l(J)^{2-n/2}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{2-n}\bigg)^2$$

by (a). Again, the Cauchy–Schwarz inequality gives

$$\begin{aligned}|F_2(P_k,x)|^2 &\le \bigg(\sum_{J\subset P_j,\, j\ne k} \frac{\lambda_J^2}{|J|}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n}\bigg)\cdot\bigg(\sum_{J\subset P_j,\, j\ne k} l(J)^4\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{4-n}\bigg)\\ &\le CG(P_k)^2\cdot C(\mathrm{diam}(\Omega))^4\cdot\bigg(\sum_{J\subset P_j,\, j\ne k}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n}\bigg).\end{aligned}$$

To bound the last sum by a constant we note that $|x-x_J| \ge C|x-x_{P_j}|$ whenever $x\in c(P_k)\setminus D$ and $J\subset P_j$, $j\ne k$. So as above

$$1+\frac{|x-x_J|}{l(J)} \ge \frac{|x-x_J|}{l(J)} \ge C\,\frac{|x-x_{P_j}|}{l(P_j)}\cdot\frac{l(P_j)}{l(J)} \ge C'\,\frac{l(P_j)}{l(J)}\Big(1+\frac{|x-x_{P_j}|}{l(P_j)}\Big),$$

since also $|x-x_{P_j}| \ge C'' l(P_j)$. We have

$$\sum_{J\subset P_j,\, j\ne k}\Big(1+\frac{|x-x_J|}{l(J)}\Big)^{-n} \le C\sum_{J\subset P_j,\, j\ne k}\Big(\frac{l(P_j)}{l(J)}\Big)^{-n}\Big(1+\frac{|x-x_{P_j}|}{l(P_j)}\Big)^{-n}.$$

This means that

$$\begin{aligned}\sum_k \int_{c(P_k)\setminus D} |F_2(P_k,x)|^2\,dx &\le C\sum_k\int_{c(P_k)\setminus D}\sum_{j\ne k}\sum_{J\subset P_j}\frac{|J|}{|P_j|}\Big(1+\frac{|x-x_{P_j}|}{l(P_j)}\Big)^{-n}dx\\ &\le C\int_{I_0}\sum_j\Big(1+\frac{|x-x_{P_j}|}{l(P_j)}\Big)^{-n}dx\\ &\le C\sum_j\int_{I_0}\Big(1+\frac{|x-x_{P_j}|}{l(P_j)}\Big)^{-n} l(P_j)^n\,d\Big(\frac{(x-x_{P_j})}{l(P_j)}\Big)\\ &\le C\sum_j |P_j|\int\frac{r^{n-1}}{(1+r)^n}\,dr\,d\omega_{n-1} \le C|I_0|\log(1+\mathrm{diam}(\Omega)),\end{aligned}$$

using polar coordinates and the fact that the P_j's are disjoint in I_0. The Central Lemma is proved.

Corollary. *Suppose* $\sigma \in A^\infty(Q_0, dx)$ *and* $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$ *with* $\mathcal{F}$ *a finite family of cubes from* $\mathcal{W}$ *and the* $\varphi_{(J)}$ *satisfying* (a), (a′), (b) *and* (c). *Then for any* $\beta > 0$ *there exists a* $\gamma = \gamma(n, \lambda, \varepsilon, \Omega, \alpha, \beta)$ *so that, for every* $\xi > 0$,

$$\sigma(\{x \in Q_0 : F^*(x) > 2\xi,\, G^*(x) \le \gamma\xi\}) \le \beta\sigma(\{x \in Q_0 : F^*(x) > \xi\}).$$

Proof. Let $\{I_j\}$ be the maximal dyadic cubes in Q_0 such that $F(I_j) > \xi$. We need only show that

$$|\{x \in I_j : F^*(x) > 2\xi,\, G^*(x) \le \gamma\xi\}| \le \widehat{\beta}|\{x \in I_j : F^*(x) > \xi\}|$$

for some $\widehat{\beta}$ such that $(C_0\widehat{\beta})^{1/\kappa} \le \beta$, because $\sigma \in A^\infty(dx)$. Notice that $\{x \in Q_0 : F^*(x) > \xi\} = \bigcup I_j$. Once again we cut out a small annular region for each cube I_j, but here the region lies inside I_j. We take $\varepsilon > 0$ so small that $|\{x \in I_j : \operatorname{dist}(x, I_j^{\mathrm{c}}) \le \varepsilon\}| \le (\widehat{\beta}/3)|I_j|$. For $x \in (1-\varepsilon)I_j$ we have $|F(I_j) - F(I_j, x)| \le CG(I_j)$, by Lemma 4. It is also true that for $\widehat{I_j} \supset I_j$ with $l(I_j) = 0.5l(\widehat{I_j})$, Lemma 7 implies that $|F(\widehat{I_j}) - F(I_j)| \le C'G(I_j)$. By maximality $F(\widehat{I_j}) \le \xi$. We also have

$$\begin{aligned} E_j &= \{x \in I_j : F^*(x) > 2\xi,\, G^*(x) \le \gamma\xi\} \\ &\subset \{x \in (1-\varepsilon)I_j : F^*(x) > 2\xi,\, G^*(x) \le \gamma\xi\} \\ &\quad \cup \{x \in I_j : \operatorname{dist}(x, I_j^{\mathrm{c}}) \le \varepsilon\}. \end{aligned}$$

For any I_j such that $E_j \ne \emptyset$, we have $G(I_j) \le \gamma\xi$. From the previous calculations we obtain $|F(I_j, x)| \le F(\widehat{I_j}) + cG(I_j)$ for any $x \in \eta I_j$. So if γ is small enough then $|F(I_j, x)| \le 1.2\xi$. Writing

$$f(x) = \sum_{J\not\subset I_j,\, J\in\mathcal{F}} \lambda_J\varphi_{(J)}(x) + \sum_{J\subset I_j, J\in\mathcal{F}} \lambda_J\varphi_{(J)}(x) = F(I_j, x) + h(x),$$

we get

$$F^*(x) - 1.2\xi \le H^*(x) \quad \text{with} \quad H^*(x) = \sup_{I\ni x} H(I, x_I).$$

This happens since I_j is maximal so that $F(I_j) > \xi$; consequently, any dyadic cube $Q \ni x$ such that $F(Q) > 2\xi$ must be contained in the I_j that contains x. Setting $F_j(x) = F(I_j, x)$, we have

$$\sup_{\substack{J\ni x\\ J\subset I_j}} F_j(J, x_J) = \sup_{\substack{J\ni x\\ J\subset I_j}} F_j(x_J).$$

Also, $x \in (1-\varepsilon)I_j$ means that for any dyadic $J \subset I_j$ such that $x \in J$, $\operatorname{dist}(x_J, I_j^{\mathrm{c}}) \ge (\varepsilon/2)l(I_j)$. Taking $\eta = 1-\varepsilon/2$, we have $F_j(x_J) \le 1.2\xi$ for any such J, so

$$\{x \in (1-\varepsilon)I_j : F^*(x) > 2\xi,\, G^*(x) \le \gamma\xi\} \subset \{x \in (1-\varepsilon)I_j : H^*(x) > 0.8\xi,\, G^*(x) \le \gamma\xi\}.$$

After rescaling, the Central Lemma can be applied to the function $h(x)$.

The full result of Theorem 1 follows from the Corollary by a standard argument because Ω is bounded and $f(x)$ being a finite sum, means that $F^* \in L^p(\Omega, d\sigma)$. To prove Theorem 1 for infinite sums we can use Fatou's lemma on $|f_n(x)|^p$, for $f_n(x) = \sum_{J\in\mathcal{F},\, l(J)\ge 1/n} \lambda_J \varphi_{(J)}(x)$, taking $\mathcal{F}$ to be an infinite family of dyadic cubes from $\mathcal{W}$.

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