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Cutting diagram method for systems of plane curves with base points

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Abstract. We develop a new method of proving non-speciality of a linear system with base fat points in general position. Using this method we show that the Hirschowitz–Harbourne conjecture holds for systems with base points of equal multiplicity bounded by 42.

1. Introduction. Let $\mathbb{K}$ be a field of characteristic zero, and $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{N}^* = \{1,2,\ldots\}$.

Definition 1. Let $D \subset \mathbb{N}^2$ be finite (any such set will be called a *diagram*), and let $m_1,\ldots,m_r \in \mathbb{N}^*$ and $p_1,\ldots,p_r \in \mathbb{K}^2$. Define the $\mathbb{K}$-vector space $\mathcal{L}_D(m_1,p_1,\ldots,m_r,p_r) \subset \mathbb{K}[X,Y]$ by

$$\mathcal{L}_D(m_1,p_1,\ldots,m_r,p_r) := \Big\{ f = \sum_{(\beta_1,\beta_2)\in D} c_{(\beta_1,\beta_2)} X^{\beta_1} Y^{\beta_2} \;\Big|\; c_{(\beta_1,\beta_2)} \in \mathbb{K},\ \frac{\partial^{\alpha_1+\alpha_2} f}{\partial X^{\alpha_1}\partial Y^{\alpha_2}}(p_j) = 0,\ \alpha_1+\alpha_2 < m_j,\ j=1,\ldots,r \Big\}.$$

Definition 2. Let $D \subset \mathbb{N}^2$ be a diagram, let $m_1,\ldots,m_r \in \mathbb{N}^*$. Define the *system of curves* $\mathcal{L}_D(m_1,\ldots,m_r)$ to be the projective space of all plane curves (that is, non-zero polynomials) generated by monomials with exponents from D having multiplicities at least $m_1,\ldots,m_r$ at r general points (see Remark 4). More formally, $\mathcal{L}_D(m_1,\ldots,m_r)$ can be viewed as a map which with any sequence of points $p_1,\ldots,p_r$ associates the space $\mathcal{L}_D(m_1,p_1,\ldots,m_r,p_r)$.

Definition 3. Let $L = \mathcal{L}_D(m_1,\ldots,m_r)$ be a system of curves. Define the *virtual dimension* vdim, *expected dimension* edim and *dimension* dim

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of L by

$$\begin{aligned} \operatorname{vdim} L &:= \#D - 1 - \sum_{j=1}^{r} \binom{m_j+1}{2}, \\ \operatorname{edim} L &:= \max\{\operatorname{vdim} L, -1\}, \\ \dim L &:= \min_{\{p_j\}_{j=1}^r \in (\mathbb{K}^2)^r} \dim_{\mathbb{K}} \mathcal{L}_D(m_1, p_1, \ldots, m_r, p_r) - 1. \end{aligned}$$

REMARK 4. If points $p_1, \ldots, p_r$ are in general position we have

$$\dim \mathcal{L}_D(m_1, \ldots, m_r) = \dim_{\mathbb{K}} \mathcal{L}_D(m_1, p_1, \ldots, m_r, p_r) - 1.$$

We can look at the space $\mathcal{L}_D(m_1, \ldots, m_r)$ as being equal to the space $\mathcal{L}_D(m_1, p_1, \ldots, m_r, p_r) \setminus \{0\}$ (for $p_1, \ldots, p_r$ in general position) modulo the equivalence relation: $f \sim g \Leftrightarrow \exists_{c \in \mathbb{K},\, c \neq 0}\, f = cg$.

Intuitively, we should have $\dim L = \operatorname{edim} L$.

DEFINITION 5. We say that a system of curves L is *special* if

$$\dim L > \operatorname{edim} L.$$

Otherwise we say that L is *non-special*.

Observe that by linear algebra we always have $\dim L \geq \operatorname{edim} L$ since multiplicity m imposes $\binom{m+1}{2}$ conditions.

2. The Hirschowitz–Harbourne conjecture. For systems of the form $\mathcal{L}_d(m_1, \ldots, m_r) := \mathcal{L}_D(m_1, \ldots, m_r)$, $D = \{\alpha \in \mathbb{N}^2 \mid |\alpha| \leq d\}$, the well known Hirschowitz–Harbourne conjecture giving a geometrical description of the speciality of a system was formulated in [9]. To formulate this conjecture consider the blowing-up $\pi : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ at r general points with exceptional divisors $E_1, \ldots, E_r$.

DEFINITION 6. A curve $C \subset \mathbb{P}^2$ is said to be a -1-*curve* if it is irreducible and the self-intersection $\widetilde{C}^2$ of its proper transform $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ is equal to -1.

CONJECTURE 7 (Hirschowitz–Harbourne). *A system $L = \mathcal{L}_d(m_1, \ldots, m_r)$ is special if and only if there exists a -1-curve $C \subset \mathbb{P}^2$ such that*

$$\widetilde{L}.\widetilde{C} \leq -2,$$

where $\widetilde{L} := |d\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \sum_{j=1}^r m_j E_j|$.

This conjecture was studied by many authors; we refer only to the recent results. For homogenous systems ($m_1 = \cdots = m_r =: m$), the above conjecture holds for $m \leq 20$ (see [4, 5]). In the general case the conjecture holds for multiplicities bounded by 11 (see [8]). Both these results were obtained with the help of computers.

For further information about the above conjecture see for example [3, 1, 2].

In this paper we develop a new method (called "diagram cutting") based on some properties of matrices assigned to systems of curves. This method provides easy proofs of non-speciality for a large family of systems. Moreover, these proofs can often be found algorithmically with a computer program. Sometimes looking for a proof needs a lot of computations, but then the proof itself can easily be checked "by hand".

As a result of the method we show that in order to check non-speciality of all homogeneous systems of bounded multiplicity m it is enough to check a finite number of cases. This result was obtained in a purely theoretical way.

The second result is Theorem 32 stating that Conjecture 7 holds for homogeneous multiplicities bounded by 42. This result was obtained by using a computer program.

3. Diagram cutting method. Before introducing the method we establish the notation and describe when a system is non-special in the language of matrices.

DEFINITION 8. Let $j \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}^2$. Define the mapping

$$\varphi_{j,\alpha} : \mathbb{K}[X,Y] \ni f \mapsto \frac{\partial^{|\alpha|} f}{\partial X^\alpha}(P_{j,X}, P_{j,Y}) \in \mathbb{K}[P_{j,X}, P_{j,Y}],$$

where $P_{j,X}, P_{j,Y}$ are new indeterminates used instead of X, Y.

$\mathcal{M}(n,k;R)$ will denote the set of $n \times r$ matrices with coefficients from R (a ring or a field). For $M \in \mathcal{M}(n,k;R)$ we write $M_{[j,\ell]}$ for the element of M in the jth row and ℓth column.

DEFINITION 9. Let $L = \mathcal{L}_D(m_1, \ldots, m_r)$ be a system of curves, and let $D = \{(\alpha_{1,X}, \alpha_{1,Y}), \ldots, (\alpha_{n,X}, \alpha_{n,Y})\}$, $\alpha_{i,X}, \alpha_{i,Y} \in \mathbb{N}$ for $i = 1, \ldots, n$. Let $\mathcal{A} = \{(j, \beta) \in \mathbb{N} \times \mathbb{N}^2 \mid |\beta| < m_j,\ j = 1, \ldots, r\} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_c\}$. Define the matrix $M(L) \in \mathcal{M}(c, n; \mathbb{K}[P_{1,X}, P_{1,Y}, \ldots, P_{r,X}, P_{r,Y}])$ by

$$M(L)_{[j,k]} := \varphi_{\mathfrak{a}_j}(X^{\alpha_{k,X}} Y^{\alpha_{k,Y}}).$$

For given points $p_1 = (p_{1,X}, p_{1,Y}), \ldots, p_r = (p_{r,X}, p_{r,Y}) \in \mathbb{K}^2$ we will use the natural evaluation mapping

$$\nu_{p_1,\ldots,p_r} : \mathbb{K}[P_{1,X}, P_{1,Y}, \ldots, P_{r,X}, P_{r,Y}] \ni f \mapsto f|_{P_{i,X} \mapsto p_{i,X}, P_{i,Y} \mapsto p_{i,Y}} \in \mathbb{K}.$$

PROPOSITION 10. *Let $L = \mathcal{L}_D(m_1, \ldots, m_r)$ be a system of curves. Then $\dim L = \#D - \operatorname{rank} M(L) - 1$.*

Proof. Let $p_1, \ldots, p_r \in \mathbb{K}^2$ be points in general position. Consider the linear mapping

$$\varPhi : \Big\{ f = \sum_{(\alpha_X, \alpha_Y) \in D} c_{(\alpha_X, \alpha_Y)} X^{\alpha_X} Y^{\alpha_Y} \Big\} \ni f \mapsto (\nu_{p_1,\ldots,p_r} \circ \varphi_{\mathfrak{a}_j}(f))_{j=1}^c \in \mathbb{K}^c.$$

We have $\mathcal{L}_D(m_1, p_1, \ldots, m_r, p_r) = \ker \Phi$. Let M denote the matrix of Φ in the bases $\{(\alpha_{1,X}, \alpha_{1,Y}), \ldots, (\alpha_{n,X}, \alpha_{n,Y})\}$, $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_c\}$. We have

$$\nu_{p_1,\ldots,p_r}(M(L)_{[j,k]}) = M_{[j,k]},$$

hence $\operatorname{rank} M(L) = \operatorname{rank} M$ (we use the facts that $\operatorname{char} \mathbb{K} = 0$ and $p_1, \ldots, p_r$ are in general position). Now we compute

$$\dim L = n - \operatorname{rank} M - 1 = \#D - \operatorname{rank} M(L) - 1. \blacksquare$$

Definition 11. Define the bidegree $\operatorname{bdeg} : \mathbb{K}[P_{1,X}, P_{1,Y}, \ldots, P_{r,X}, P_{r,Y}] \to \mathbb{N}^2$ by setting $\operatorname{bdeg}(P_{i,X}) := (1, 0), \operatorname{bdeg}(P_{i,Y}) := (0, 1)$ for $i = 1, \ldots, r$.

Proposition 12. *Let $\mathcal{L}_D(m_1, \ldots, m_r)$ be a system of curves. Let M be a square submatrix of $M(L)$ of size $s \in \mathbb{N}^*$. Renumbering columns and rows if necessary we can assume that M is given by columns $(\alpha_{1,X}, \alpha_{1,Y}), \ldots$ $\ldots, (\alpha_{s,X}, \alpha_{s,Y})$ and rows $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$. Then $\det M$ is a bihomogeneous (w.r.t. bdeg) polynomial of bidegree $(\sum_{i=1}^s \alpha_{i,X}, \sum_{i=1}^s \alpha_{i,Y}) - \gamma$, where $\gamma \in \mathbb{N}^2$ depends only on the choice of rows.*

Proof. We have

$$\det M = \sum_{\sigma \in S_s} \operatorname{sgn}(\sigma) M_{[1,\sigma(1)]} \cdots M_{[s,\sigma(s)]}.$$

For $M_{[j,k]} \neq 0$ we have $\operatorname{bdeg} M_{[j,k]} = (\alpha_{k,X}, \alpha_{k,Y}) - \beta_j$, where $\mathfrak{a}_j = (\ell_j, \beta_j)$ for some $\ell_j \in \mathbb{N}$, $\beta_j \in \mathbb{N}^2$. Hence

$$\operatorname{bdeg} M_{[1,\sigma(1)]} \cdots M_{[s,\sigma(s)]} = \Big(\sum_{i=1}^s \alpha_{i,X}, \sum_{i=1}^s \alpha_{i,Y}\Big) - \sum_{i=1}^s \beta_i.$$

We finish the proof by taking $\gamma = \sum_{i=1}^s \beta_i$. $\blacksquare$

Proposition 13. *Let $m \in \mathbb{N}^*$ and $D \subset \mathbb{N}^2$ with $\#D = \binom{m+1}{2}$. Then $L = \mathcal{L}_D(m)$ is non-special if and only if D does not lie on a curve of degree $m - 1$.*

Proof. From the previous proof we can see that $\det M(L) = cf$, where f is a monomial and $c \in \mathbb{K}$. Let $D = \{(\alpha_{1,X}, \alpha_{1,Y}), \ldots, (\alpha_{n,X}, \alpha_{n,Y})\}$. For $\beta = (\beta_X, \beta_Y)$ with $|\beta| < m$ we have

$$\begin{aligned} M(L)_{[(1,\beta),j]} &= \varphi_{(1,\beta)}(X^{\alpha_{j,X}} Y^{\alpha_{j,Y}}) \\ &= \prod_{k=1}^{\beta_X} (\alpha_{j,X} - k + 1) \cdot \prod_{k=1}^{\beta_Y} (\alpha_{j,Y} - k + 1) \cdot P_{1,X}^{\alpha_{j,X} - \beta_X} P_{1,Y}^{\alpha_{j,Y} - \beta_Y}. \end{aligned}$$

Since we are only interested in the value of c we compute the determinant of $M = M(L)_{X \mapsto 1, Y \mapsto 1}$. By row operations we can change M into M' where

$$M'_{[(1,\beta),j]} = \alpha_{j,X}^{\beta_X} \alpha_{j,Y}^{\beta_Y}, \qquad \det M \neq 0 \Leftrightarrow \det M' \neq 0.$$

Now $\det M' = 0$ if and only if the rows of M' are linearly dependent, but this happens if and only if D lies on a curve of degree $m-1$. ■

Now we can present the diagram cutting method and prove that it can be used to bound the dimension of a system of curves.

THEOREM 14. *Let $m_1, \dots, m_r, m_{r+1}, \dots, m_p \in \mathbb{N}^*$, let $D \subset \mathbb{N}^2$ be a diagram, and let $F : \mathbb{N}^2 \ni (\alpha_1, \alpha_2) \mapsto r_1\alpha_1 + r_2\alpha_2 + r_0 \in \mathbb{R}$ be an affine function with $r_0, r_1, r_2 \in \mathbb{R}$. Let*

$$D_1 := \{(\alpha_1, \alpha_2) \in D \mid F(\alpha_1, \alpha_2) < 0\},$$
$$D_2 := \{(\alpha_1, \alpha_2) \in D \mid F(\alpha_1, \alpha_2) > 0\}.$$

If $L_2 := \mathcal{L}_{D_2}(m_{r+1}, \dots, m_p)$ is non-special and $\operatorname{vdim} L_2 = -1$ *then*

$$\dim \mathcal{L}_D(m_1, \dots, m_p) \le \dim \mathcal{L}_{D_1}(m_1, \dots, m_r).$$

Proof. Put $L_1 := \mathcal{L}_{D_1}(m_1, \dots, m_r)$. We can compute the dimension of the system $L := \mathcal{L}_D(m_1, \dots, m_p)$ as $\dim L = \#D - \operatorname{rank} M(L) - 1$. As $D = D_1 \cup D_2$, in an appropriate basis the matrix $M(L)$ has the form

$$M(L) = \left[\begin{array}{c|c} M(L_1) & K_1 \\ \hline K_2 & M(L_2) \end{array}\right].$$

Pick a maximal non-zero minor M' of $M(L_1)$ and consider the following square submatrix of $M(L)$:

$$M := \left[\begin{array}{c|c} M' & K_1' \\ \hline K_2' & M(L_2) \end{array}\right],$$

where K_1' and K_2' are suitable submatrices of K_1 and K_2. It suffices to show that $\det M \ne 0$. The columns of M' are indexed by elements of some $D_1' \subset D_1$, hence the columns of M are indexed by $D' := D_1' \cup D_2$. Let $U = [M'\ K_1']$ and $L = [K_2'\ M(L_2)]$ be submatrices of M, and

$$\mathcal{C} := \{C \subset D' \mid \#C = \#D_2\}.$$

For $C \subset D'$ define L_C (respectively U_C) as the submatrix of L (resp. U) consisting of the columns indexed by elements of C. Now we can compute

$$\det M = \sum_{C \in \mathcal{C}} \varepsilon(C) \det L_C \det U_{D' \setminus C},$$

with $\varepsilon(C) = \pm 1$. Observe that $\det L_C \in \mathbb{K}[P_{r+1,X}, P_{r+1,Y}, \dots, P_{p,X}, P_{p,Y}]$ and $\det U_{D' \setminus C} \in \mathbb{K}[P_{1,X}, P_{1,Y}, \dots, P_{r,X}, P_{r,Y}]$, and consider $\det M$ as a polynomial of the indeterminates $\{P_{\ell,X}, P_{\ell,Y}\}_{\ell > r}$ over $\mathbb{K}[\{P_{\ell,X}, P_{\ell,Y}\}_{\ell \le r}]$. We have

$$\det M = \pm \det M' \det M(L_2) + f.$$

Assume that $\det M = 0$. As $\det M' \ne 0$ and $\det M(L_2) \ne 0$ by the assumptions, we must have another non-zero term $g \in \mathbb{K}[\{P_{\ell,X}, P_{\ell,Y}\}_{\ell > r}]$ appearing in f such that $\mathrm{bdeg}(g) = \mathrm{bdeg}(\det M(L_2))$. The term g is given by some $C \in \mathcal{C}$, $C \neq D_2$. Since

$$\mathrm{bdeg}(g) = \sum_{(\alpha_1,\alpha_2)\in C} (\alpha_1,\alpha_2) - \gamma, \quad \mathrm{bdeg}(\det M(L_2)) = \sum_{(\alpha_1,\alpha_2)\in D_2} (\alpha_1,\alpha_2) - \gamma$$

we have

$$F\Big(\sum_{(\alpha_1,\alpha_2)\in C} (\alpha_1,\alpha_2)\Big) = F\Big(\sum_{(\alpha_1,\alpha_2)\in D_2} (\alpha_1,\alpha_2)\Big).$$

As F is an affine form and $\#C = \#D_2$ we have

$$\sum_{(\alpha_1,\alpha_2)\in C} F(\alpha_1,\alpha_2) = \sum_{(\alpha_1,\alpha_2)\in D_2} F(\alpha_1,\alpha_2),$$

but from the definition of D_2 this is possible only when $C = D_2$, a contradiction. ∎

4. Reduction of homogeneous systems. We will use the following notation for a sequence of multiplicities:

Definition 15. Let $m_1,\dots,m_r \in \mathbb{N}^*$ and $k_1,\dots,k_r \in \mathbb{N}$. Define

$$(m_1^{\times k_1},\dots,m_r^{\times k_r}) = (\underbrace{m_1,\dots,m_1}_{k_1},\dots,\underbrace{m_r,\dots,m_r}_{k_r}).$$

We will use diagrams of the following form:

Definition 16. Let $a_1,\dots,a_n,u_1,\dots,u_n \in \mathbb{N}$. Define

$$(a_1^{\uparrow u_1},\dots,a_n^{\uparrow u_n}) := \bigcup_{i=1}^{n} \{i-1\} \times \{u_i,\dots,u_i+a_i-1\} \subset \mathbb{N}^2.$$

Example 17.

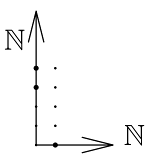


Fig. 1. Diagram $(2^{\uparrow 3}, 1^{\uparrow 0})$

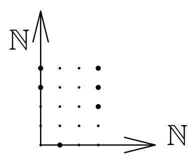


Fig. 2. Diagram $(2^{\uparrow 3}, 1^{\uparrow 0}, 0^{\uparrow 0}, 3^{\uparrow 2})$

Observe that $\#(a_1^{\uparrow u_1},\dots,a_n^{\uparrow u_n}) = \sum_{i=1}^n a_i$.

Definition 18. We say that two diagrams D_1, D_2 are *equivalent* if there exists $\alpha \in \mathbb{Z}^2$ such that $D_1 = D_2 + \alpha$.

Remark 19. Observe that the diagram $(0^{\uparrow 0},\dots,0^{\uparrow 0},a_1^{\uparrow u_1},\dots,a_n^{\uparrow u_n})$ is equivalent to $(a_1^{\uparrow u_1},\dots,a_n^{\uparrow u_n})$, which, in turn, is equivalent to $(a_1^{\uparrow u_1+u},\dots$
$\dots,a_n^{\uparrow u_n+u})$.

PROPOSITION 20. *Let* $m_1,\dots,m_r\in\mathbb{N}^*$, *and let* D_1, D_2 *be diagrams. If* D_1 *and* D_2 *are equivalent, then* $\dim\mathcal{L}_{D_1}(m_1,\dots,m_r)=\dim\mathcal{L}_{D_2}(m_1,\dots,m_r)$.

Proof. Let $D_1+(\alpha_1,\alpha_2)=D_2$, and let $p_1,\dots,p_r\in\mathbb{K}^2$ be points in general position. The maps

$$\mathcal{L}_{D_1}(m_1p_1,\dots,m_rp_r)\ni f\mapsto X^{\alpha_1}Y^{\alpha_2}f\in\mathcal{L}_{D_2}(m_1p_1,\dots,m_rp_r),$$
$$\mathcal{L}_{D_2}(m_1p_1,\dots,m_rp_r)\ni f\mapsto X^{-\alpha_1}Y^{-\alpha_2}f\in\mathcal{L}_{D_1}(m_1p_1,\dots,m_rp_r)$$

are well defined (we can assume that none of the coordinates of $p_1,\dots,p_r$ are zero), linear and inverse to each other. ∎

LEMMA 21. *Let* $m\in\mathbb{N}^*$ *and* $D=(1^{\uparrow m-1},2^{\uparrow m-2},\dots,(m-1)^{\uparrow 1},m^{\uparrow 0})$. *Then* $\mathcal{L}_D(m)$ *is non-special and* $\operatorname{vdim}\mathcal{L}_D(m)=-1$.

Proof. By Proposition 13 it is enough to show that D (Fig. 3 shows an example for $m=3$) does not lie on a curve C of degree $m-1$. Let $L_j=\{(x,y)\in\mathbb{R}^2\mid x+y+j=0\}$, $j=0,\dots,m-1$. Observe that $\#(D\cap L_j)=j+1$ so by the Bézout theorem and induction we have $\bigcup_{j=0}^{m-1}L_j\subset C$, a contradiction. ∎

REMARK 22. Observe that we can do the same for the diagram $(m^{\uparrow 0}, (m-1)^{\uparrow 0},\dots,1^{\uparrow 0})$.

LEMMA 23. *Let* $m\in\mathbb{N}^*$ *and* $D=(m^{\uparrow m},m^{\uparrow m-1},\dots,m^{\uparrow 0})$. *Then* $\mathcal{L}_D(m^{\times 2})$ *is non-special and* $\operatorname{vdim}\mathcal{L}_D(m^{\times 2})=-1$.

Proof. Let $F=y-m+1/2$. Observe that $D=D_1\cup D_2$ (Fig. 4 shows an example for $m=3$), where

$$D_1:=\{p\in D\mid F(p)<0\}=(0,1^{\uparrow m-1},2^{\uparrow m-2},\dots,(m-1)^{\uparrow 1},m^{\uparrow 0}),$$
$$D_2:=\{p\in D\mid F(p)>0\}=(m^{\uparrow m},(m-1)^{\uparrow m},\dots,1^{\uparrow m}).$$

The diagram D_1 is equivalent to $(1^{\uparrow m-1},2^{\uparrow m-2},\dots,(m-1)^{\uparrow 1},m^{\uparrow 0})$ hence from Lemma 21 the system $\mathcal{L}_{D_1}(m)$ is non-special. The diagram D_2 is equivalent to $(m^{\uparrow 0},(m-1)^{\uparrow 0},\dots,1^{\uparrow 0})$ so $\mathcal{L}_{D_2}(m)$ is non-special. As $\#D_2=\binom{m+1}{2}$, we can use Theorem 14 to obtain non-speciality of $\mathcal{L}_D(m^{\times 2})$. ∎

LEMMA 24. *Let* $m,k\in\mathbb{N}^*$ *and* $D=(m^{\uparrow k-1},m^{\uparrow k-2},\dots,m^{\uparrow 0})$. *If* $(m+1)\mid k$ *then* $L=\mathcal{L}_D(m^{\times 2k/(m+1)})$ *is non-special and* $\operatorname{vdim}L=-1$.

Proof. We proceed by induction on k. For $k=m+1$ we use the previous lemma. Let $k>m+1$. Put $F=x-(m+1)+1/2$. Observe that $D=D_1\cup D_2$ (Fig. 5 shows an example for $m=3$, $k=12$), where

$$D_1:=\{p\in D\mid F(p)<0\}=(m^{\uparrow k-1},\dots,m^{\uparrow k-1-m}),$$
$$D_2:=\{p\in D\mid F(p)>0\}=(0,\dots,0,m^{\uparrow k-(m+1)-1},\dots,m^{\uparrow 0}).$$

The diagram D_1 is equivalent to $(m^{\uparrow m}, \ldots, m^{\uparrow 0})$, hence from Lemma 23 the system $\mathcal{L}_{D_1}(m^{\times 2})$ is non-special. The diagram D_2 is equivalent to $(m^{\uparrow k-(m+1)-1}, \ldots, m^{\uparrow 0})$ and from the induction hypothesis we know that the system $\mathcal{L}_{D_2}(m^{\times 2(k-(m+1))/(m+1)})$ is non-special. Now, Theorem 14 finishes the proof. ∎

Lemma 25. *Let* $m, k, h \in \mathbb{N}^*$ *and* $D = (h^{\uparrow k-1}, h^{\uparrow k-2}, \ldots, h^{\uparrow 0})$. *If* $(m+1) \mid k$ *and* $m \mid h$ *then* $L = \mathcal{L}_D(m^{\times 2kh/m(m+1)})$ *is non-special and* $\operatorname{vdim} L = -1$.

Proof. We proceed by induction on h. The case $h = m$ was treated in the previous lemma. Let $h > m$. Set $F = y + x - (k-1+m) + 1/2$. Observe that $D = D_1 \cup D_2$ (Fig. 6 shows an example for $m = 3$, $k = 12$, $h = 9$), where

$$D_1 := \{p \in D \mid F(p) < 0\} = (m^{\uparrow k-1}, m^{\uparrow k-2}, \ldots, m^{\uparrow 0}),$$
$$D_2 := \{p \in D \mid F(p) > 0\} = ((h-m)^{\uparrow k-1+m}, \ldots, (h-m)^{\uparrow m}).$$

According to Lemma 24 the system $\mathcal{L}_{D_1}(m^{\times 2k/(m+1)})$ is non-special. The diagram D_2 is equivalent to $((h-m)^{\uparrow k-1}, (h-m)^{\uparrow k-2}, \ldots, (h-m)^{\uparrow 0})$ and by the induction hypothesis the system $\mathcal{L}_{D_2}(m^{\times 2k(h-m)/m(m+1)})$ is non-special. Again we finish the proof by using Theorem 14. ∎

Definition 26. Let $m \in \mathbb{N}^*$ and $h = m(m+1)$. Define the set (called the *end of layer systems*)

$$\operatorname{EoLS}(m) = \{\mathcal{L}_D(m^{\times 2k+h-1}) \mid D = (h^{\uparrow k-1}, \ldots, h^{\uparrow 0}, (h-1)^{\uparrow 0}, \ldots, 1^{\uparrow 0}), k = 1, \ldots, m+1\}.$$

Observe that for every $L \in \operatorname{EoLS}(m)$ we have $\operatorname{vdim} L = -1$.

Lemma 27. *Let* $m, k \in \mathbb{N}^*$, $h = m(m+1)$, $p = 2k + h - 1$ *and* $D = (h^{\uparrow k-1}, h^{\uparrow k-2}, \ldots, h^{\uparrow 0}, (h-1)^{\uparrow 0}, \ldots, 1^{\uparrow 0})$. *If* $\operatorname{EoLS}(m)$ *contains only non-special systems then the system* $L = \mathcal{L}_D(m^{\times p})$ *is non-special.*

Proof. Take $k_1, k_2 \in \mathbb{N}$ such that $k = k_1(m+1) + k_2$, $1 \le k_2 \le m+1$. Put $F = x - k_1(m+1) + 1/2$. Observe that $D = D_1 \cup D_2$ (Fig. 7 shows an example for $m = 3$, $k = 11$), where

$$D_1 := \{p \in D \mid F(p) < 0\} = (h^{\uparrow k-1}, h^{\uparrow k-2}, \ldots, h^{\uparrow k_2}),$$
$$D_2 := \{p \in D \mid F(p) > 0\} = (h^{\uparrow k_2-1}, \ldots, h^{\uparrow 0}, (h-1)^{\uparrow 0}, \ldots, 1^{\uparrow 0}).$$

The diagram D_1 is equivalent to the diagram $(h^{\uparrow k-k_2-1}, h^{\uparrow k-k_2-2}, \ldots, h^{\uparrow 0})$ and since $(m+1) \mid (k-k_2)$, it follows from Lemma 25 that $\mathcal{L}_{D_1}(m^{\times 2(k-k_2)h/m(m+1)})$ is non-special. The system $\mathcal{L}_{D_2}(m^{\times 2k_2+h-1})$ is in $\operatorname{EoLS}(m)$ and we can use Theorem 14 to complete the proof. ∎

THEOREM 28. *Let $m, d_L \in \mathbb{N}^*$. Assume that for $d = d_L, \ldots, d_L + m(m+1)$ every system $\mathcal{L}_d(m^{\times p})$, $p \in \mathbb{N}$, is non-special. Moreover, assume that* EoLS(m) *contains only non-special systems. Then for any $d \geq d_L$ and $p \in \mathbb{N}$ the system $\mathcal{L}_d(m^{\times p})$ is non-special.*

Proof. We proceed by induction on d. For $d_L \leq d \leq d_L + m(m+1)$ the conclusion is obvious. Choose $d > d_L + m(m+1)$. We want to show that the system $\mathcal{L}_D(m^{\times p})$ is non-special, where $D = ((d+1)^{\uparrow 0}, \ldots, 1^{\uparrow 0})$. Let

$$h = m(m+1) \quad \text{and} \quad F = y + x - (d-h) + 1/2.$$

Observe that $D = D_1 \cup D_2$ (Fig. 8 shows an example for $m = 3$, $d_L = 3$, $d = 16$), where

$$D_1 := \{p \in D \mid F(p) < 0\} = ((d+1-h)^{\uparrow 0}, \ldots, 1^{\uparrow 0}),$$
$$D_2 := \{p \in D \mid F(p) > 0\} = (h^{\uparrow d+1-h}, \ldots, h^0, (h-1)^{\uparrow 0}, \ldots, 1^{\uparrow 0}).$$

As $d + 1 - h \geq d_L$ we may use the induction hypothesis for the system $\mathcal{L}_{D_1}(m^{\times p-(2d-h+3)})$. By Lemma 27 the system $\mathcal{L}_{D_2}(m^{\times 2d-h+3})$ is non-special. Again we finish the proof by using Theorem 14. ∎

DEFINITION 29. Let $m, d_0 \in \mathbb{N}^*$. Put

$$S(m, d_0) := \mathrm{EoLS}(m) \cup \{\mathcal{L}_d(m^{\times r}) \mid \operatorname{vdim} \mathcal{L}_d(m^{\times r}) \geq -2m^2,\ d_0 \leq d \leq d_0 + m(m+1),\ r \in \mathbb{N}^*\}.$$

THEOREM 30. *Let $m, d_0 \in \mathbb{N}$. If $S(m, d_0)$ contains only non-special systems then every system $\mathcal{L}_d(m^{\times r})$ for $d \geq d_0$, $r \in \mathbb{N}$ is non-special.*

Proof. By Theorem 28 it suffices to show that every system $L = \mathcal{L}_d(m^{\times r})$, $r \in \mathbb{N}$, $d = d_0, \ldots, d_0 + m(m+1)$, is non-special. If $\operatorname{vdim} L \geq -2m^2$ then L is non-special by the assumptions. Let $\operatorname{vdim} L < -2m^2$. From now on we use the notations and theory of reductions introduced in [7, 8]. We want to apply a sequence of r weak m-reductions to the diagram $D = \{\alpha \in \mathbb{N}^2 \mid |\alpha| \leq d\}$ to end with the empty diagram,

$$D \xrightarrow{m\ \mathrm{w}} D_1 \xrightarrow{m\ \mathrm{w}} D_2 \xrightarrow{m\ \mathrm{w}} D_3 \xrightarrow{m\ \mathrm{w}} \cdots \xrightarrow{m\ \mathrm{w}} \emptyset.$$

Following the notations of [8] consider a diagram $D = (a_1, \ldots, a_n)$. Observe that an m-reduction is not possible only if $a_i = a_{i+1} < m$. As D is the result of a sequence of weak m-reductions this can only happen for $i \leq 2m$. While performing an m-weak reduction we use at most m additional points for each a_i, $i = 1, \ldots, 2m$, and for each i it is sufficient to do it only once. So we use at most $2m^2$ additional points to reduce D to the empty diagram, hence if $\operatorname{vdim} L < -2m^2$ then L is non-special. ∎

Example 31.

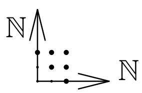


Fig. 3

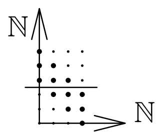


Fig. 4

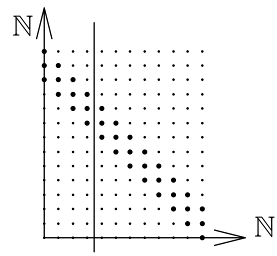


Fig. 5

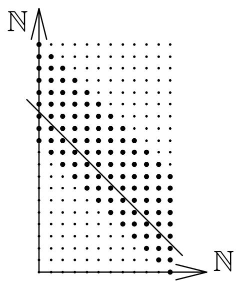


Fig. 6

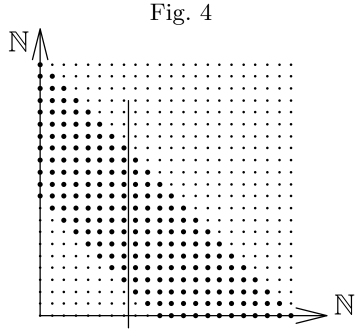


Fig. 7

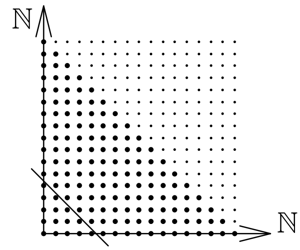


Fig. 8

5. Homogeneous systems with bounded multiplicity

Theorem 32. *The Hirschowitz–Harbourne conjecture holds for homogeneous systems with multiplicities bounded by* 42.

Proof. For $m < 20$ the result can be found in [5]. For $m = 20, \dots, 42$ we choose $d_0(m) = 3m$. To check the conjecture we have to do the following:

1. We have to find all non-special systems among $\mathcal{L}_d(m^{\times r})$ for $d \leq d_0(m)$ (there are only finitely many of them). Next, for every such system we must show that it satisfies the Hirschowitz–Harbourne conjecture. This was done with the help of computer programs. By the proof of Theorem 30 the maximal size of a matrix (for $m = 42$) can be 8128×11656, but in most cases the combination of the reduction method and Cremona transformation gives an immediate answer.
2. For every system in $S(m, d_0(m))$ we must prove its non-speciality.

As $S(m, d_0(m))$ contains systems with diagrams of big size, this cannot be done without preparations. For a system $L = \mathcal{L}_D(m^{\times r}) \in \mathrm{EoLS}(m)$ we use the reduction method described in [8] to reduce the problem to the question of non-speciality of $L' = \mathcal{L}_{D'}(m^{\times 5})$ for some diagram D'. For $m = 42$ this forces us to compute the determinants of 4515×4515 matrices 43 times. For the other systems from $S(m, d_0(m))$ we use the following fact (see Proposition 28 in [8]).

Theorem 33. *Let $m_1, \ldots, m_r \in \mathbb{N}^*$. There exists a diagram D with the property: if $\mathcal{L}_D(m_1, \ldots, m_r)$ is non-special then for all $d \in \mathbb{N}$ the system $\mathcal{L}_d(m_1, \ldots, m_r)$ is non-special.*

So for each r such that $\mathcal{L}_d(m^{\times r}) \in S(m, d_0(m))$ we have to check only one system $\mathcal{L}_D(m^{\times r})$ for some diagram D depending on r. We also reduce this system to $\mathcal{L}_{D'}(m^{\times 9})$. In [6] the reader can find a table with the actual number of cases to be checked, as well as all necessary software with instructions on how to perform the tests. ∎

Remark 34. The test decribed above can also be performed for greater values of m, but for each $m \geq 43$ this will take at least several days of computation. It seems that one should reorganize the method.

6. Closing remarks

Remark 35. There exists another method of proving non-speciality of a given system (or a family of systems) based on blowing-up the projective space introduced by C. Ciliberto and R. Miranda ([5]). It seems that the diagram cutting method is different from the blowing-up method and sometimes works better. Moreover, all definitions and results of Section 3 can be easily carried over to the higher-dimensional case of the systems of polynomials in n variables vanishing (with multiplicities) at points in general position. This is not known for the method of C. Ciliberto and R. Miranda.

Remark 36. Observe that Theorem 28 can be reformulated as follows.

Theorem 37. *If the set $\mathrm{EoLS}(m)$ contains only non-special systems and $\mathcal{L}_d(m_1, \ldots, m_r)$ is non-special then*

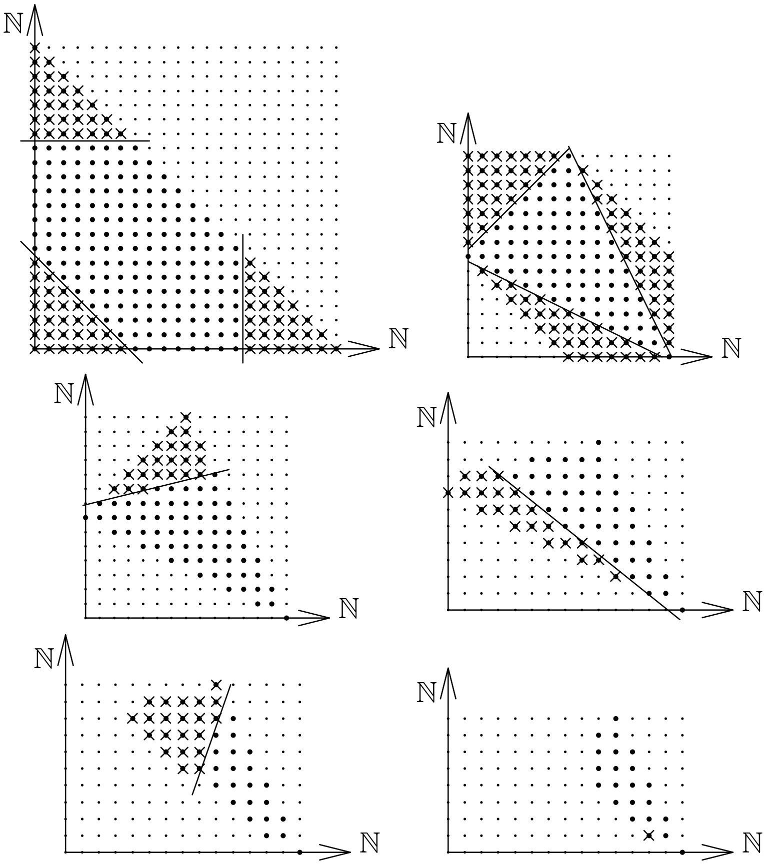
$$\mathcal{L}_{d+m(m+1)}(m_1, \ldots, m_r, m^{\times p})$$

is non-special, where

$$p = 2d + m(m+1) + 1.$$

This shows that in order to find all non-special systems of the form $\mathcal{L}_d(m_1, \ldots, m_r)$ with $m_i \leq M$, $i = 1, \ldots, r$, it is sufficient to check a finite number of cases.

Example 38. We show that $L = \mathcal{L}_{21}(7^{\times 6}, 6^{\times 4}, 1)$ is non-special by the diagram cutting method. The proof (found by computer) can be easily read off from the picture. The system L was studied in [10].



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