ANNALES
POLONICI MATHEMATICI
90.2 (2007)

Invisible obstacles

by A. G. Ramm (Manhattan, Kansas)

Abstract. It is proved that one can choose a control function on an arbitrarilly small open subset of the boundary of an obstacle so that the total radiation from this obstacle for a fixed direction of the incident plane wave and for a fixed wave number will be as small as one wishes. The obstacle is called "invisible" in this case.

1. Introduction. Consider a bounded domain $D \subset \mathbb{R}^n$, $n = 3$, with a connected Lipschitz boundary S. Let F be an arbitrarily small, fixed, open subset of S, $F' = S \setminus F$, and N be the outer unit normal to S. The domain D is the *obstacle*. Consider the scattering problem:

$$\begin{aligned} &\nabla^2 u + k^2 u = 0 \quad \text{in } D' := \mathbb{R}^3 \setminus D, \\ &u = w \quad \text{on } F, \qquad u_N + hu = 0 \quad \text{on } F'. \end{aligned} \tag{1}$$

Here w is the function we can set up at will (the *control function*), h is a piecewise-continuous function with $\operatorname{Im} h \geq 0$, and $k > 0$ is a fixed constant, and u_N is the normal derivative of u. The function u satisfies the following condition:

$$u = u_0 + v, \qquad u_0 = e^{ik\alpha\cdot x}, \tag{2}$$

and

$$v = \frac{e^{ikr}}{r} A(\beta, \alpha) + o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty,\ \beta := \frac{x}{r}. \tag{3}$$

The function $A(\beta, \alpha)$ is called the *scattering amplitude*, $\alpha, \beta \in S^2$ are the unit vectors, S^2 is the unit sphere, α, the direction of the incident wave u_0, is assumed fixed, so $A(\beta, \alpha) = A(\beta)$. Problem (1)–(3) has a unique solution ([1]).

Define the cross-section σ, or the *total radiation* from the obstacle, as

$$\sigma = \int_{S^2} |A(\beta)|^2 \, d\beta. \tag{4}$$

2000 *Mathematics Subject Classification*: 35J05, 35R30, 74J20, 74J25.
Key words and phrases: wave scattering, inverse problems, invisible obstacles.

The problem is:

Given an arbitrary small $\varepsilon > 0$, can one choose w so that $\sigma < \varepsilon$?

If this choice is possible, we call the obstacle "invisible" for the fixed α and k.

Our basic result is the following theorem:

THEOREM 1. *Given an arbitrarily small $\varepsilon > 0$ and an arbitrarily small open subset $F \in S$, one can find $w \in C_0^\infty(F)$ such that $\sigma < \varepsilon$. The same result holds for the boundary conditions $u|_F = w$, $u|_{F'} = 0$.*

A similar problem was first posed and solved in [2], where the Neumann boundary condition was assumed and the control function was not u on F, but u_N on F. The boundary conditions in this paper allow one to consider impedance obstacles, so it broadens the possible applications of our theory. Inverse problems for scattering by obstacles are considered in [1] and [3].

2. Proof of Theorem 1. By Green's formula we get

$$v(x) = \int_{F'} G(x,s)(u_{0N} + hu_0)\,ds + \int_F G_N(x,s)v\,ds, \tag{5}$$

where G is the Green's function:

$$\nabla^2 G + k^2 G = -\delta(x-y) \quad \text{in } D', \qquad \lim_{|x|\to\infty} |x|\left(\frac{\partial G}{\partial |x|} - ikG\right) = 0, \tag{6}$$

and

$$G_N + hG = 0 \quad \text{on } F', \qquad G = 0 \quad \text{on } F. \tag{7}$$

By Ramm's lemma ([1, p. 46]), one gets

$$G(x,y) = \frac{e^{ikr}}{4\pi r}\,\psi(y,\nu) + o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty, \qquad \frac{x}{r} = -\nu. \tag{8}$$

Here $\psi := \psi(y,\nu) = \psi(y,\nu,k)$ is the scattering solution:

$$\nabla^2\psi + k^2\psi = 0 \quad \text{in } D', \quad \psi_N + h\psi = 0 \quad \text{on } F', \quad \psi = 0 \quad \text{on } F, \tag{9}$$

and

$$\psi = e^{ik\nu\cdot x} + \eta, \qquad \lim_{|x|\to\infty} |x|(\eta_r - ik\eta) = 0. \tag{10}$$

Using (4), (5) and (8), we get

$$A(\beta) = \frac{1}{4\pi}\int_{F'} \psi(s,-\beta)(u_{0N} + hu_0)\,ds + \frac{1}{4\pi}\int_F (w - u_0)\psi_N(s,-\beta)\,ds, \tag{11}$$

and

$$\sigma = \int_{S^2} |A_0(\beta) - A_1(\beta)|^2\,d\beta, \tag{12}$$

where

$$A_0(\beta) := \frac{1}{4\pi}\int\limits_{F'} \psi(s,-\beta)(u_{0N}+hu_0)\,ds - \frac{1}{4\pi}\int\limits_{F} u_0\psi_N(s,-\beta)\,ds, \tag{13}$$

and

$$A_1(\beta) := -\frac{1}{4\pi}\int\limits_{F} w(s)\psi_N(s,-\beta)\,ds. \tag{14}$$

The conclusion of Theorem 1 follows immediately from Lemma 1.

LEMMA 1. *Given an arbitrary function $f \in L^2(S^2)$ and an arbitrarily small $\varepsilon > 0$, one can find $w \in C_0^\infty(F)$ such that $\|f(\beta) - A_1(\beta)\| < \varepsilon$, where $\|\cdot\| := \|\cdot\|_{L^2(S^2)}$.*

Indeed, one can take $f(\beta) = A_0(\beta)$ and use Lemma 1.

Let us prove Lemma 1. If this lemma is false, then there is an $f \in L^2(S^2)$, $f \neq 0$, such that

$$\int\limits_{S^2} d\beta\, f(\beta)\int\limits_{F} ds\, w(s)\psi_N(s,-\beta) = 0 \quad \forall w \in C_0^\infty(F). \tag{15}$$

This implies

$$\int\limits_{S^2} d\beta\, f(\beta)\psi_N(s,-\beta) = 0 \quad \forall s \in F. \tag{16}$$

Define the function

$$z(x) := \int\limits_{S^2} d\beta\, f(\beta)\psi(x,-\beta). \tag{17}$$

This function solves the equation

$$\nabla^2 z + k^2 z = 0 \quad \text{in } D'$$

and satisfies the boundary conditions

$$z = z_N = 0 \quad \text{on } F.$$

By the uniqueness of the solution to the Cauchy problem for elliptic equations, this implies

$$z(x) = 0 \quad \text{in } D'. \tag{18}$$

It follows from (18) that $f = 0$. This contradiction proves Lemma 1 and, consequently, Theorem 1.

To complete the proof, let us derive from (18) that $f = 0$. The function ψ satisfies

$$\psi(x,\beta) = Te^{ik\beta\cdot x},$$

where T is a linear boundedly invertible operator, acting on the x variable only (see [1]). The specific form of T is not important for our argument.

Applying the inverse operator T^{-1} to (17) and taking into account (18), one gets

$$\int_{S^2} d\beta\, f(\beta) e^{-ik\beta\cdot x} = 0 \quad \forall x \in D'. \tag{19}$$

The left-hand side in (19) is an entire function of x. Therefore (19) implies

$$\int_{S^2} d\beta\, f(\beta) e^{-ik\beta\cdot x} = 0 \quad \forall x \in \mathbb{R}^3. \tag{20}$$

Equation (20) means that the Fourier transform of the distribution

$$f(\beta)\,\frac{\delta(|\xi| - k)}{|\xi|^2}$$

is zero. Here $\xi = |\xi|\beta$ is the (dual to x) Fourier transform variable. By the injectivity of the Fourier transform, it follows that this distribution is zero, so $f = 0$, and the proof is completed. The last statement of Theorem 1 is proved similarly. ∎

3. Conclusion. The basic result of this note is the proof of the following statement:

By choosing a suitable control function on an arbitrarily small open subset of the boundary of a bounded obstacle, *one can make the total radiation from this obstacle*, *although positive*, *but as small as one wishes*, *for a fixed wave number and a fixed direction of the incident wave. Thus*, *the obstacle can be made practically invisible.*

References

[1] A. G. Ramm, *Scattering by Obstacles*, Reidel, Dordrecht, 1986.
[2] —, *Minimization of the total radiation from an obstacle by a control function on a part of the boundary*, J. Inverse Ill-Posed Probl. 4 (1996), 531–534.
[3] —, *Inverse Problems*, Springer, New York, 2005.

Mathematics Department
Kansas State University
Manhattan, KS 66506-2602, U.S.A.
E-mail: ramm@math.ksu.edu

Received 26.9.2006
and in final form 6.12.2006 (1700)