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# Boundary cross theorem in dimension 1

by Peter Pflug (Oldenburg) and Viêt-Anh Nguyên (Trieste)

**Abstract.** Let $X$, $Y$ be two complex manifolds of dimension 1 which are countable at infinity, let $D \subset X$, $G \subset Y$ be two open sets, let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$), and let $W$ be the 2-fold cross $((D \cup A) \times B) \cup (A \times (B \cup G))$. Suppose in addition that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. We determine the "envelope of holomorphy" $\widehat{W}$ of $W$ in the sense that any function locally bounded on $W$, measurable on $A \times B$, and separately holomorphic on $(A \times G) \cup (D \times B)$ "extends" to a function holomorphic on the interior of $\widehat{W}$.

**1. Introduction.** In this paper we consider a boundary version of the cross theorem in the spirit of the pioneer work of Malgrange–Zerner [16]. Epstein's survey article [3] gives a historical discussion and motivation for this kind of theorems.

The first results in this direction were obtained by Komatsu [8] and Drużkowski [2], but only for some special cases. Recently, Gonchar [5, 6] has proved a more general result for the one-dimensional case. In recent works [10, 11] of the authors Gonchar's result has been generalized to the higher dimensional case.

However, in all cases considered so far in the literature the hypotheses on the function being extended and its domain of definition are, in some sense, rather restrictive. Therefore, the main goal of this work is to establish some boundary cross theorems in more general (one-dimensional) cases with more optimal hypotheses. This will perhaps be a first step towards understanding the higher dimensional case in its full generality.

Our approach here is based on the previous work [10], the Gonchar–Carleman operator developed in [5, 6], a new result of Zeriahi [15] and a thorough geometric study of harmonic measures.

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**2. Preliminaries.** In order to recall the classical versions of the boundary cross theorem and to discuss our motivation in more detail, we need to introduce some notation and terminology. In fact, we keep the main notation from [10]. In particular $E$ denotes the open unit disc in $\mathbb{C}$ and mes the linear measure (i.e. the one-dimensional Hausdorff measure). Throughout the paper, for a topological space $M$, $\mathcal{C}(M)$ denotes the space of all continuous functions $f : M \to \mathbb{C}$ equipped with the "sup-norm" $|f|_M := \sup_M |f| \in [0, \infty]$. Moreover, a function $f : M \to \mathbb{C}$ is said to be *locally bounded* on $M$ if, for any point $z \in M$, there are an open neighborhood $U$ of $z$ and a positive number $K = K_z$ such that $|f|_U < K$. Finally, for a complex manifold $\Omega$, $\mathcal{SH}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) denotes the set of all subharmonic (resp. holomorphic) functions on $\Omega$.

In this work all complex manifolds are supposed to be countable at infinity.

**2.1.** *Open sets with partly Jordan-curve-like boundary.* Let $X$ be a complex manifold of dimension 1. A *Jordan curve* in $X$ is the image $\mathcal{C} := \{\gamma(t) : t \in [a, b]\}$ of a continuous one-to-one map $\gamma : [a, b] \to X$, where $a, b \in \mathbb{R}$, $a < b$. The set $\{\gamma(t) : t \in (a, b)\}$ is said to be the *interior* of the Jordan curve. A *Jordan domain* is the image $\{\Gamma(t) : t \in E\}$ of a one-to-one continuous map $\Gamma : \overline{E} \to X$. A *closed Jordan curve* is the boundary of a Jordan domain.

Consider an open set $D \subset X$. Then $D$ is said to be *Jordan-curve-like at a point* $\zeta \in \partial D$ if there is a Jordan domain $U \subset X$ such that $\zeta \in U$ and $U \cap \partial D$ is the interior of a Jordan curve. Then $\zeta$ is said to be *of type 1* if there is a neighborhood $V$ of $\zeta$ such that $V \cap D$ is a Jordan domain. Otherwise, $\zeta$ is said to be *of type 2*. We see easily that if $\zeta$ is of type 2, then there are an open neighborhood $V$ of $\zeta$ and two Jordan domains $V_1$, $V_2$ such that $V \cap D = V_1 \cup V_2$. Moreover, $D$ is said to be *Jordan-curve-like on a subset* $A$ of $\partial D$ if $D$ is Jordan-curve-like at all points of $A$.

Now let $D \subset X$ be an open set which is Jordan-curve-like on a set $A \subset \partial D$. In the remaining part of this subsection we will introduce various notions. We point out that they are all intrinsic, i.e., do not depend on any choice (of open neighborhoods, Jordan domains, conformal mappings etc.) we made in their definitions.

$A$ is said to be *Jordan-measurable* if for every $\zeta \in A$ the following condition is fulfilled:

• *if $\zeta$ is of type 1*: there are an open neighborhood $U = U_\zeta$ of $\zeta$ such that $U \cap D$ is a Jordan domain and a conformal mapping $\Phi = \Phi_\zeta$ from $U \cap D$

onto the unit disc $E$ which extends homeomorphically from $\overline{U \cap D}$ onto $\overline{E}$ such that $\Phi(\overline{U \cap D} \cap A)$ is Lebesgue measurable on $\partial E$;

• *if $\zeta$ is of type 2*: there are an open neighborhood $U = U_\zeta$ of $\zeta$ such that $U \cap D = U_1 \cup U_2$ with Jordan domains $U_1 = U_{1,\zeta}$, $U_2 = U_{2,\zeta}$, and conformal mappings $\Phi_j = \Phi_{j,\zeta}$ ($j = 1, 2$) from $U_j$ onto $E$ which extend homeomorphically from $\overline{U}_j$ onto $\overline{E}$ such that $\Phi_j(\overline{U}_j \cap A)$ is Lebesgue measurable (on $\partial E$).

A Jordan-measurable set $A \subset \partial D$ is said to be *of zero length* if for all $\zeta \in A$, if one takes $U_\zeta$, $\Phi_\zeta$ when $\zeta$ is of type 1 (resp. $U_\zeta$, $\Phi_{j,\zeta}$ when $\zeta$ is of type 2), then $\mathrm{mes}(\Phi_\zeta(\overline{U_\zeta \cap D} \cap A)) = 0$ (resp. $\mathrm{mes}(\Phi_{j,\zeta}(\overline{U}_{j,\zeta} \cap A)) = 0$, $j = 1, 2$).

A Jordan-measurable set $A \subset \partial D$ is said to be *of positive length* if it is not of zero length.

Suppose that $D$ is Jordan-curve-like at a point $\zeta \in \partial D$. We define the concept of angular approach regions at $\zeta$ as follows. For any $0 < \alpha < \pi/2$, the *Stolz region* or *angular approach region* $\mathcal{A}_\alpha(\zeta)$ is given by:

• *if $\zeta$ is of type 1*:

$$\mathcal{A}_\alpha(\zeta) := \Phi^{-1}\left\{t \in E : \left|\arg\left(\frac{\Phi(\zeta) - t}{\Phi(\zeta)}\right)\right| < \alpha\right\},$$

where $\arg : \mathbb{C} \to (-\pi, \pi]$ is the usual argument function;

• *if $\zeta$ is of type 2*:

$$\mathcal{A}_\alpha(\zeta) := \bigcup_{j=1,2} \Phi_j^{-1}\left\{t \in E : \left|\arg\left(\frac{\Phi_j(\zeta) - t}{\Phi_j(\zeta)}\right)\right| < \alpha\right\}.$$

Geometrically, $\mathcal{A}_\alpha(\zeta)$ is the intersection of $D$ with one or two "cones" of aperture $2\alpha$ and vertex $\zeta$ according to the type of $\zeta$.

Let $\zeta \in \partial D$ be a point at which $D$ is Jordan-curve-like and let $U$ be an open neighborhood of $\zeta$. We say that a function $f$ defined on $U \cap D$ has *angular limit* $\lambda$ at $\zeta$ if

$$\lim_{z \in \mathcal{A}_\alpha(\zeta),\, z \to \zeta} f(z) = \lambda \quad \text{for all } 0 < \alpha < \pi/2.$$

Let $A \subset \partial D$ be a Jordan-measurable set and $f : D \to \mathbb{C}$, $g : A \to \mathbb{C}$ be two functions. Then $f$ is said to *have angular limit $g(a)$ for Jordan a.e. $a \in A$* if the set

$$\{a \in A : f \text{ does not have angular limit } g(a) \text{ at } a\}$$

is of zero length. For simplicity, we only write "a.e." instead of "Jordan a.e.".

We conclude this subsection with a simple example which may clarify the above definitions. Let $G$ be the open square in $\mathbb{C}$ with vertices $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$. Define the domain

$$D := G \setminus [-1/2, 1/2].$$

Then $D$ is Jordan-curve-like on $\partial G\cup(-1/2,1/2)$. Every point of $\partial G$ is of type 1 and every point of $(-1/2,1/2)$ is of type 2.

**2.2.** *Harmonic measure.* Let $X$ be a complex manifold of dimension 1, let $D$ be an open subset of $X$ and let $A\subset\partial D$. Consider the characteristic function

$$1_{\partial D\setminus A}(\zeta):=\begin{cases}1, & \zeta\in\partial D\setminus A,\\ 0, & \zeta\in A.\end{cases}$$

Then the *harmonic measure* of the set $\partial D\setminus A$ (denoted by $\omega(\cdot,A,D)$) is the Perron solution of the generalized Dirichlet problem with boundary data $1_{\partial D\setminus A}$. In other words,

$$\omega(\cdot,A,D):=\sup_{u\in\mathcal{U}}u,$$

where $\mathcal{U}=\mathcal{U}(A,D)$ denotes the family of all subharmonic functions $u$ on $D$ such that $\limsup_{D\ni z\to\zeta}u(z)\le 1_{\partial D\setminus A}(\zeta)$ for each $\zeta\in\partial D$.

It is well known (see, for example, the book of Ransford [13] for the case $X=\mathbb{C}$) that $\omega(\cdot,A,D)$ is harmonic on $D$.

For a point $\zeta\in\partial D$ at which $D$ is Jordan-curve-like, we say that is a *locally regular point relative to* $A$ if

$$\lim_{z\to\zeta,\,z\in\mathcal{A}_\alpha(\zeta)}\omega(z,A\cap U,D\cap U)=0$$

for any $0<\alpha<\pi/2$ and any open neighborhood $U$ of $\zeta$. Obviously, $\zeta\in\overline{A}$. If, moreover, $\zeta\in A$, then $\zeta$ is said to be a *locally regular point of* $A$. The set of all locally regular points relative to $A$ is denoted by $A^*$. Observe that, in general, $A^*\not\subset A$ and $A\not\subset A^*$. However, if $A$ is open in $\partial D$ and $D$ is Jordan-curve-like on $A$, then $A\subset A^*$.

As an immediate consequence of the subordination principle for the harmonic measure (see Corollary 4.3.9 in [13]), one gets

$$\lim_{z\to\zeta,\,z\in\mathcal{A}_\alpha(\zeta)}\omega(z,A,D)=0,\qquad \zeta\in A^*,\ 0<\alpha<\pi/2. \tag{2.1}$$

We extend the function $\omega(\cdot,A,D)$ to $D\cup A^*$ by simply setting

$$\omega(z,A,D):=0,\qquad z\in A^*.$$

Geometric properties of the harmonic measure will be discussed in Section 4 below. By Theorem 4.6, if either $A$ is a Borel set or $D\subset\mathbb{C}$, then $\omega(\cdot,A,D)\equiv\omega(\cdot,A^*,D)$.

**2.3.** *Cross and separate holomorphy.* Let $X$, $Y$ be two complex manifolds of dimension 1, let $D\subset X$, $G\subset Y$ be two open sets, let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. We define a 2-*fold* *cross* $W$, its *regular part* $W^*$, and its *interior* $W^{\mathrm{o}}$ as

$$\begin{aligned} W &:= \mathbb{X}(A,B;D,G) := ((D\cup A)\times B)\cup(A\times(B\cup G)),\\ W^* &:= \mathbb{X}(A^*,B^*;D,G),\\ W^{\mathrm{o}} &:= \mathbb{X}^{\mathrm{o}}(A,B;D,G) := (A\times G)\cup(D\times B). \end{aligned}$$

Moreover, put

$$\omega(z,w) := \omega(z,A^*,D)+\omega(w,B^*,G), \qquad (z,w)\in(D\cup A^*)\times(G\cup B^*).$$

It is clear that $\omega|_{D\times G}$ is harmonic.

For a 2-fold cross $W := \mathbb{X}(A,B;D,G)$ define *its wedge*

$$\widehat{W} = \widehat{\mathbb{X}}(A,B;D,G) := \{(z,w)\in(D\cup A^*)\times(G\cup B^*) : \omega(z,w)<1\}.$$

Then the set of all interior points of the wedge $\widehat{W}$ is given by

$$\widehat{W}^{\mathrm{o}} := \widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D,G) := \{(z,w)\in D\times G : \omega(z,w)<1\}.$$

In particular, if $A$ (resp. $B$) is an open set of $\partial D$ (resp. $\partial G$), one has $A\times B\subset A^*\times B^*$ and $W\subset W^*\subset\widehat{W}$.

We say that a function $f : W\to\mathbb{C}$ is *separately holomorphic on* $W^{\mathrm{o}}$, and write $f\in\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$, if for any $a\in A$ (resp. $b\in B$) the function $f(a,\cdot)|_G$ (resp. $f(\cdot,b)|_D$) is holomorphic on $G$ (resp. on $D$).

We say that $f : W\to\mathbb{C}$ (resp. $f : A\times B\to\mathbb{C}$) is *separately continuous on* $W$ (resp. *on* $A\times B$), and write $f\in\mathcal{C}_{\mathrm{s}}(W)$ (resp. $f\in\mathcal{C}_{\mathrm{s}}(A\times B)$), if it is continuous with respect to any variable when the remaining variable is fixed.

In the remaining part of this subsection we introduce two notions. As in Subsection 2.1 we point out that these notions are intrinsic, i.e., they do not depend on any choice we made in their definitions.

We say that a function $f : A\times B\to\mathbb{C}$ is *Jordan-measurable on* $A\times B$ if for every point $\zeta\in A$ of type $n$ (resp. $\eta\in B$ of type $m$) there is an open neighborhood $U$ of $\zeta$ (resp. $V$ of $\eta$) such that $U\cap D=\bigcup_{1\le j\le n}U_j$ (resp. $V\cap G=\bigcup_{1\le k\le m}V_k$) with Jordan domains $U_j$, $V_k$, and conformal mappings $\Phi_j$ (resp. $\Psi_k$) from $U_j$ (resp. $V_k$) onto $E$ which extend homeomorphically from $\overline{U}_j$ (resp. $\overline{V}_k$) onto $\overline{E}$ such that $f(\Phi_j^{-1}(\cdot),\Psi_k^{-1}(\cdot)) : \Phi_j(\overline{U}_j\cap A)\times\Psi_k(\overline{V}_k\cap B)\to\mathbb{C}$ is Lebesgue measurable.

Two Jordan-measurable functions $f,g : A\times B\to\mathbb{C}$ are said to be *equal a.e. on* $A\times B$ if for every point $\zeta\in A$ of type $n$ (resp. $\eta\in B$ of type $m$), the functions

$$f(\Phi_j^{-1}(\cdot),\Psi_k^{-1}(\cdot)),\, g(\Phi_j^{-1}(\cdot),\Psi_k^{-1}(\cdot)) : \Phi_j(\overline{U}_j\cap A)\times\Psi_k(\overline{V}_k\cap B)\to\mathbb{C}$$

are equal a.e. (we keep the previous notation).

We say that $f : \widehat{W}^{\mathrm{o}}\to\mathbb{C}$ has *angular limit* $\lambda\in\mathbb{C}$ at $(a,b)\in\widehat{W}$ if the following limit relation holds:

- *if* $a \in D$ *and* $b \in G$:

$$\lim_{z\to a,\, w\to b} f(z,w) = \lambda;$$

- *if* $a \in A^*$ *and* $b \in G$:

$$\lim_{z\to a,\, z\in\mathcal{A}_\alpha(a),\, w\to b} f(z,w) = \lambda, \qquad 0<\alpha<\pi/2;$$

- *if* $a \in D$ *and* $b \in B^*$:

$$\lim_{z\to a,\, w\to b,\, w\in\mathcal{A}_\alpha(b)} f(z,w) = \lambda, \qquad 0<\alpha<\pi/2;$$

- *if* $a \in A^*$ *and* $b \in B^*$:

$$\lim_{z\to a,\, z\in\mathcal{A}_\alpha(a),\, w\to b,\, w\in\mathcal{A}_\alpha(b)} f(z,w) = \lambda, \qquad 0<\alpha<\pi/2.$$

**2.4.** *Motivations for our work.* We are now able to formulate what we will refer to as the *classical version of the boundary cross theorem.*

THEOREM 1 (Gonchar [5, 6]). *Let $D, G \subset \mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a nonempty open set of the boundary $\partial D$ (resp. $\partial G$). Then, for any $f \in \mathcal{C}(W)\cap\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$, there is a unique function $\widehat{f} \in \mathcal{C}(\widehat{W})\cap\mathcal{O}(\widehat{W}^{\mathrm{o}})$ such that $\widehat{f} = f$ on $W$. Moreover, if $|f|_W < \infty$ then*

$$|\widehat{f}(z,w)| \le |f|_{A\times B}^{1-\omega(z,w)} |f|_W^{\omega(z,w)}, \qquad (z,w)\in\widehat{W},$$

*where $W$, $W^{\mathrm{o}}$, and $\widehat{W}$ denote the 2-fold cross, its interior and its wedge, respectively, associated to $A$, $B$, $D$, $G$.*

Theorem 1 admits various generalizations. The following result was announced by Gonchar in [5].

THEOREM 2. *Let $D, G \subset \mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a nonempty open and rectifiable subset of $\partial D$ (resp. $\partial G$). Let $f$ be a function defined on the 2-fold cross $W$ with the following properties:*

(i) $f|_{W^{\mathrm{o}}} \in \mathcal{C}(W^{\mathrm{o}})\cap\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$;
(ii) *$f$ is locally bounded on $W$;*
(iii) *for any $a\in A$ (resp. $b \in B$), the holomorphic function $f(a,\cdot)|_G$ (resp. $f(\cdot,b)|_D$) has angular limit $f_1(a,b)$ at $b$ for a.e. $b\in B$ (resp. $f_2(a,b)$ at $a$ for a.e. $a \in A$) and $f_1 = f_2 = f$ a.e. on $A\times B$.*

*Then:*

(1) *There is a unique function $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ such that*

$$\lim_{(z,w)\in\widehat{W}^{\mathrm{o}},\,(z,w)\to(\zeta,\eta)} \widehat{f}(z,w) = f(\zeta,\eta), \qquad (\zeta,\eta)\in W^{\mathrm{o}}.$$

(2) *If, moreover, $|f|_W < \infty$, then*

$$|\widehat{f}(z,w)| \le |f|_{A\times B}^{1-\omega(z,w)} |f|_W^{\omega(z,w)}, \qquad (z,w)\in\widehat{W}^{\mathrm{o}}.$$

(3) *If, moreover, $f$ is continuous at a point $(a,b)\in A\times B$, then*

$$\lim_{(z,w)\in\widehat{W}^{\mathrm{o}},\,(z,w)\to(a,b)}\widehat{f}(z,w)=f(a,b).$$

On the other hand, the following result due to Drużkowski [2] has a different flavor.

THEOREM 3. *Let $D,G\subset\mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a non-empty open connected subset of $\partial D$ (resp. $\partial G$). Let $f$ be a function defined on $W$ with the following properties*:

(i) $f\in\mathcal{C}_{\mathrm{s}}(W)\cap\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$;
(ii) *$f$ is locally bounded on $W$*;
(iii) *$f|_{A\times B}$ is continuous on $A\times B$.*

*Then all conclusions of Theorem 1 still hold.*

Observe that all these theorems require the following very strong hypothesis: $D$ and $G$ are Jordan domains in $\mathbb{C}$ and $A\times B$ is an open subset of $\partial D\times\partial G$. Moreover, the boundedness and continuity assumptions on $f$ are rather restrictive.

A natural question is whether Theorems 1–3 are still true if $D,G$ are open sets in complex manifolds of dimension 1 and $A$ (resp. $B$) is a not necessarily open subset of $\partial D$ (resp. $\partial G$). In addition, if one drops the hypothesis on the local boundedness and continuity of $f$, can one obtain a holomorphic extension of $f$ and what are its properties? These matters seem to be of interest, especially when one seeks to generalize Theorems 1–3 to higher dimensions.

The present paper is motivated by these questions. Our first purpose is to generalize Gonchar's theorems to a very general situation, where $D$, $G$ are, in some sense, almost general open subsets of complex manifolds of dimension 1 and where the boundary sets $A$, $B$ are almost general subsets of $\partial D$, $\partial G$. Our second goal is to establish, in this general context, an extension theorem analogous to Drużkowski's theorem with minimal hypotheses on $f$.

**3. Statement of the main results and outline of the proofs.** We are now ready to state our main result.

THEOREM A. *Let $X$, $Y$ be complex manifolds of dimension 1, let $D\subset X$, $G\subset Y$ be open sets and $A$ (resp. $B$) a subset of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and both $A$ and $B$ are of positive length. Let $f:W\to\mathbb{C}$ be such that*:

(i) *$f$ is locally bounded on $W$ and $f\in\mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$*;
(ii) *$f|_{A\times B}$ is Jordan-measurable*;

(iii) *for any* $a \in A$ (*resp.* $b \in B$), *the holomorphic function* $f(a,\cdot)|_G$ (*resp.* $f(\cdot,b)|_D$) *has angular limit* $f_1(a,b)$ *at* $b$ *for a.e.* $b \in B$ (*resp.* $f_2(a,b)$ *at* $a$ *for a.e.* $a \in A$) *and* $f_1 = f_2 = f$ *a.e. on* $A \times B$.

*Then there exists a unique function* $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ *with the following property*:

(1) *There are subsets* $\widetilde{A} \subset A \cap A^*$ *and* $\widetilde{B} \subset B \cap B^*$ *such that the sets* $A \setminus \widetilde{A}$ *and* $B \setminus \widetilde{B}$ *are of zero length* ([1]) *and* $\widehat{f}$ *has angular limit* $f(\zeta,\eta)$ *at every point* $(\zeta,\eta) \in \mathbb{X}^{\mathrm{o}}(\widetilde{A},\widetilde{B};D,G)$.

*In addition*, $\widehat{f}$ *enjoys the following properties*:

(2) *If* $|f|_W < \infty$, *then*

$$|\widehat{f}(z,w)| \le |f|_{A\times B}^{1-\omega(z,w)} |f|_W^{\omega(z,w)}, \qquad (z,w) \in \widehat{W}^{\mathrm{o}}.$$

(3) *For any* $(a_0,w_0) \in A^* \times G$ (*resp.* $(z_0,b_0) \in D \times B^*$) *if*

$$\lim_{(z,w)\to(a_0,w_0),\,(z,w)\in W} f(z,w) \ (\textit{resp.} \lim_{(z,w)\to(z_0,b_0),\,(z,w)\in W} f(z,w))\ (=: \lambda)$$

*exists*, *then* $\widehat{f}$ *has angular limit* $\lambda$ *at* $(a_0,w_0)$ (*resp. at* $(z_0,b_0)$).

(4) *For any* $(a_0,b_0) \in A^* \times B^*$, *if*

$$\lim_{(a,b)\to(a_0,b_0),\,(a,b)\in A\times B} f(a,b) \quad (=: \lambda)$$

*exists*, *then* $\widehat{f}$ *has angular limit* $\lambda$ *at* $(a_0,b_0)$.

(5) *If* $f|_{A\times B}$ *can be extended to a continuous function defined on* $A^* \times B^*$, *then* $f$ *can be extended to a unique continuous function* (*still denoted by*) $f$ *defined on* $W^* := \mathbb{X}(A^*,B^*;D,G)$ *and* $\widehat{f}$ *has angular limit* $f(\zeta,\eta)$ *at every* $(\zeta,\eta) \in W^*$ *and* $f_1 = f_2 = f$ *on* $(A\cap A^*) \times (B \cap B^*)$.

Theorem A has an immediate consequence.

Corollary A. *Under the hypotheses and notation of Theorem* A, *suppose in addition that* $f \in \mathcal{C}(W^{\mathrm{o}})$. *Then the function* $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ *provided by Theorem* A *has angular limit* $f(\zeta,\eta)$ *at every* $(\zeta,\eta) \in ((A\cap A^*) \times G) \cup (D \times (B \cap B^*))$.

It is worth noting that Theorem A and Corollary A generalize, in some sense, Theorems 1–3.

Now we drop the local boundedness and continuity hypotheses on $f$. The examples of Drużkowski in [2] (see Section 10 below) show that, without these conditions, the extended function $\widehat{f}$ (if it exists) is, in general, not continuous on $\widehat{W}$. However, our second main result partially solves this problem.

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([1]) Under this condition it follows from Theorem 4.6(1) below that $\widetilde{A} \subset \widetilde{A}^*$ and $\widetilde{B} \subset \widetilde{B}^*$.

THEOREM B. *Let $X$, $Y$ be complex manifolds of dimension* 1, *let $D \subset X$, $G \subset Y$ be open sets, and let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. Let $f : W \to \mathbb{C}$ have the following properties*:

(i) $f|_{A\times B} \in \mathcal{C}_{\mathrm{s}}(A \times B)$ *and* $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$;
(ii) *for any $a \in A$ (resp. $b \in B$), the function $f(a,\cdot)$ (resp. $f(\cdot,b)$) is locally bounded on $G \cup B$ (resp. $D \cup A$) and the (holomorphic) restriction $f(a,\cdot)|_G$ (resp. $f(\cdot,b)|_D$) has angular limit $f(a,b)$ at $b$ for every $b \in B$ (resp. at $a$ for every $a \in A$).*

*Then there are subsets $\widetilde{A} \subset A \cap A^*$ and $\widetilde{B} \subset B \cap B^*$, and a unique function $\widehat{f} \in \mathcal{O}(\widehat{W^{\mathrm{o}}})$ with the following properties*:

(1) *$A \setminus \widetilde{A}$ and $B \setminus \widetilde{B}$ are of zero length*;
(2) *$\widehat{f}$ has angular limit $f(\zeta,\eta)$ at every $(\zeta,\eta) \in \mathbb{X}(\widetilde{A},\widetilde{B}; D, G)$.*

Observe that if $f \in \mathcal{C}_{\mathrm{s}}(W) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$, then conditions (i)–(ii) above are fulfilled. Although our results have been stated only for the case of a 2-fold cross, they can be formulated for an $N$-fold cross with any $N \geq 2$ (see also [9, 10]).

Now we present the main ideas of the proof of Theorems A and B.

Our method consists of two steps: first, we suppose that $D$ and $G$ are Jordan domains in $\mathbb{C}$, and then we treat the general case. The key technique is to use *level sets* of the harmonic measure. More precisely, we exhaust $D$ (resp. $G$) by the level sets of the harmonic measure $\omega(\cdot, A, D)$ (resp. $\omega(\cdot, B, G)$), i.e. by the sets $D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta\}$ (resp. $G_\delta := \{w \in G : \omega(w, B, G) < 1 - \delta\}$) for $0 < \delta < 1$.

In the first step, we improve Gonchar's method [5, 6] and make intensive use of Carleman's formula and of the geometric properties of the level sets of harmonic measures.

The main ingredient for the second step is a mixed cross type theorem (see also [10]) valid for measurable boundary sets on complex manifolds of dimension 1. We prove it using a recent work of Zeriahi (see [15]) and the classical method of doubly orthogonal bases of Bergman type.

We apply the mixed cross type theorem to prove Theorems A and B with $D$ (resp. $G$) replaced by $D_\delta$ (resp. $G_\delta$). Then we construct the solution for the original open sets $D$ and $G$ by a gluing procedure (see also [9]).

**4. Properties of the harmonic measure and its level sets.** In this section $X$ is a complex manifold of dimension 1, $D \subset X$ an open set, and $A$ a nonempty Jordan-measurable subset of $\partial D$. Observe that $\partial D$ is then nonpolar.

Let $\mathcal{P}_D$ be the generalized Poisson integral of $D$. If, in addition, $A$ is a Borel set, then, by Theorem 4.3.3 of [13], the harmonic measure of $\partial D \setminus A$ is given by

$$\omega(\cdot, A, D) = \mathcal{P}_D[1_{\partial D\setminus A}]. \tag{4.1}$$

The following elementary lemma will be useful.

LEMMA 4.1. *Let $E$ be the unit disc and $A$ a measurable subset of $\partial E$.*

(1) *Let $u$ be a subharmonic function on $E$ with $u \le 1$ and let $\alpha \in (0, \pi/2)$ be such that*

$$\limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z) \le 0 \quad \textit{for a.e. } \zeta \in A.$$

*Then $u \le \omega(\cdot, A, E)$ on $E$.*

(2) *For all density points $\zeta$ of $A$,*

$$\lim_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} \omega(z, A, E) = 0, \qquad 0 < \alpha < \pi/2.$$

*In particular, all density points of $A$ are contained in $A^*$.*

(3) *For all interior points $\zeta$ of $A$,*

$$\lim_{z\to\zeta} \omega(z, A, E) = 0.$$

*Proof.* This follows almost immediately from the explicit formula for the Poisson integral $\mathcal{P}_E$. ∎

PROPOSITION 4.2 (Maximum principle). *Let $u \in \mathcal{SH}(D)$ be bounded from above and*

$$\limsup_{z\to\zeta} u(z) \le 0, \qquad \zeta \in \partial D \setminus A,$$

$$\limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z) \le 0, \qquad \zeta \in A,\ 0 < \alpha < \pi/2.$$

*Then $u \le 0$ on $D$.*

*Proof.* Suppose that $u < M$ for some $M$. Let $\zeta_0 \in A$. Fix a Jordan domain $U \subset D$ such that $\partial U \cap \partial D$ is a closed arc which is a neighborhood of $\zeta_0$ in $\partial D$. Let $B$ be an open arc in $\partial U \cap \partial D$ which contains $\zeta_0$. Lemma 4.1(1), (3) applied to $u|_U$ yields

$$\limsup_{z\to\zeta,\, z\in U} u(z) \le M \limsup_{z\to\zeta,\, z\in U} \omega(z, B, U) = 0, \qquad \zeta \in B.$$

Hence

$$\limsup_{z\to\zeta,\, z\in D} u(z) \le 0, \qquad \zeta \in A.$$

Combining this with the hypothesis, we obtain the desired conclusion from the classical maximum principle (see Theorem 2.3.1 in [13]). ∎

We formulate an important stability property of the harmonic measure. Let $\phi : \partial D \to \mathbb{R}$ be a bounded function. The associated *Perron function* $H_{D,A} : D \to \mathbb{R}$ is defined by

$$H_{D,A}[\phi] := \sup_{u \in \widehat{\mathcal{U}}} u, \tag{4.2}$$

where $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(\phi, A, D)$ denotes the family of all subharmonic functions $u$ on $D$ such that

$$\limsup_{z \to \zeta} u(z) \le \phi(\zeta), \quad \zeta \in \partial D \setminus A,$$
$$\limsup_{z \to \zeta,\, z \in \mathcal{A}_\alpha(\zeta)} u(z) \le \phi(\zeta), \quad \zeta \in A,\ 0 < \alpha < \pi/2.$$

In the following, $\widehat{\mathcal{U}}(A, D)$ will stand for $\widehat{\mathcal{U}}(1_{\partial D \setminus A}, A, D)$.

By the above proposition, several results in Sections 4.1 and 4.2 of [13] are still valid with $H_{D,A}$ in place of $H_D$ upon making the obvious changes. In particular, we have the following (see Corollary 4.2.6 in [13]):

PROPOSITION 4.3. *Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\phi : \partial D \to \mathbb{R}$ a bounded function which is continuous nearly everywhere* (2) *on $\partial D$. Then there exists a unique bounded harmonic function $h$ on $D$ such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for nearly every $\zeta \in \partial D$. Moreover, $h = H_D[\phi] = H_{D,A}[\phi]$.*

In view of this result, Theorem 4.3.3 in [13] is still valid in the context of $H_{D,A}$. More precisely,

PROPOSITION 4.4. *Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\phi : \partial D \to \mathbb{R}$ a bounded Borel function. Then $H_D[\phi] = H_{D,A}[\phi] = \mathcal{P}_D[\phi]$.*

In the special case $X = \mathbb{C}$ we can say even more.

PROPOSITION 4.5. *Let $D$ be a proper open subset of $\mathbb{C}$. Let $A$ be a nonempty Borel subset of $\partial D$ such that $D$ is Jordan-curve-like on $A$ and $A$ is of zero length. Then $\mathcal{P}_D[1_A] \equiv 0$ on $D$.*

*Proof.* Suppose without loss of generality that $D$ is Jordan-curve-like on the interval $[0,1] \subset \partial D$ and that $A$ is a Borel subset of $[0,1]$ with $\operatorname{mes}(A) = 0$. Since $D \subset \mathbb{C} \setminus [0,1]$, it follows from the subordination principle that

$$\mathcal{P}_D[1_A] \le \mathcal{P}_{\mathbb{C} \setminus [0,1]}[1_A] \quad \text{on } D.$$

Therefore, it suffices to show that $\mathcal{P}_{\mathbb{C} \setminus [0,1]}[1_A] \equiv 0$ on $\mathbb{C} \setminus [0,1]$. To this end consider the conformal mapping $\Phi(z) := \sqrt{1/z - 1}$ from $\mathbb{C} \cup \{\infty\} \setminus [0,1]$

(2) A property is said to hold *nearly everywhere* on $\partial D$ if it holds everywhere on $\partial D \setminus \mathcal{N}$ for some Borel polar set $\mathcal{N}$.

onto $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. It is not difficult to show that

$$\mathcal{P}_{\mathbb{C}\setminus[0,1]}[1_A] = \mathcal{P}_{\mathbb{H}}[1_{\Phi(A)}] \circ \Phi^{-1} \equiv 0. \ \blacksquare$$

Now we arrive at one of the main results of this section.

THEOREM 4.6. *Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\mathcal{N}$ a Jordan-measurable subset of $\partial D$ which is of zero length.*

(1) *Then $A^*$ is a Borel set, $(A^*)^* = A^*$, $(A \setminus \mathcal{N})^* = A^*$, and $A \setminus A^*$ is of zero length.*
(2) *If $A$ is a Borel set then $\omega(z, A, D) = H_{D,A}[1_{\partial D\setminus A}]$ for $z \in D$. In particular,*

$$\begin{aligned}\omega(z, A^*, D) = H_{D,A^*}[1_{\partial D\setminus A^*}] &= H_{D,(A\cap A^*)\setminus\mathcal{N}}[1_{\partial D\setminus[(A\cap A^*)\setminus\mathcal{N}]}] \\ &= \omega(z, A, D), \qquad z \in D.\end{aligned}$$

(3) *If $X = \mathbb{C}$ then $\omega(z, A, D) = \omega(z, A \setminus \mathcal{N}, D) = \omega(z, A^*, D)$.*

*Proof.* (1) can be checked using the definition and Lemma 4.1, and (2) is an immediate consequence of Proposition 4.4 and (1).

Now we turn to (3). Choose Borel sets $A_1$, $A_2$ so that $A_1 \subset A \setminus \mathcal{N}$, $A \subset A_2 \subset \partial D$ and $A_2 \setminus A_1$ is of zero length. Then the subordination principle and Proposition 4.5 show that

$$\begin{aligned}\omega(z, A_2, D) \le \omega(z, A, D) \le \omega(z, A \setminus \mathcal{N}, D) &\le \omega(z, A_1, D) \\ &= \omega(z, A_2, D), \qquad z \in D.\end{aligned}$$

This proves the first identity.

Since $A^*$ is, by (1), a Borel set, (2) gives that

$$\omega(z, A^*, D) = H_{D,A^*}[1_{\partial D\setminus A^*}].$$

Consequently, $\omega(z, A, D) \le \omega(z, A^*, D)$, $z \in D$. On the other hand, let $B$ be a Borel set such that $B \subset A \cap A^*$ and $A \setminus B$ is of zero length. Then

$$\begin{aligned}\omega(\cdot, A, D) = \omega(\cdot, B, D) &= H_{D,B}[1_{\partial D\setminus B}] \\ &\ge H_{D,A^*}[1_{\partial D\setminus A^*}] = \omega(\cdot, A^*, D) \quad \text{on } D.\end{aligned}$$

Combining the above estimates yields the last identity in (3). $\blacksquare$

PROPOSITION 4.7. *Let $D$ be an open subset of $X$ and $A$ a nonempty Jordan-measurable subset of $\partial D$. Let $(D_k)_{k=1}^\infty$ be a sequence of open subsets of $D$ and $(A_k)_{k=1}^\infty$ a sequence of Jordan-measurable subsets of $A$ such that:*

(i) *$D_k \subset D_{k+1}$ and $\bigcup_{k=1}^\infty D_k = D$;*
(ii) *$A_k \subset A_{k+1}$, $A_k \subset \partial D \cap \partial D_k$, $D_k$ is Jordan-curve-like on $A_k$, and $\bigcup_{k=1}^\infty A_k = A$;*
(iii) *for any $\zeta \in A$ there is an open neighborhood $V = V_\zeta$ of $\zeta$ in $\mathbb{C}$ such that $V \cap D = V \cap D_k$ for some $k$.*

*Then*

$$\omega(z, A^*, D) = \lim_{k\to\infty} \omega(z, A_k^*, D_k), \qquad z \in D.$$

*Proof.* Using the subordination principle it is easy to see that the sequence $(\omega(\cdot, A_k^*, D_k))_{k=1}^\infty$ is decreasing and the limit

$$u := \lim_{k\to\infty} \omega(\cdot, A_k^*, D_k)$$

exists and defines a subharmonic function in $D$. By the subordination principle again, we have $u \ge \omega(\cdot, A^*, D)$. Therefore, it remains to establish the opposite inequality. In view of (i)–(iii), we conclude that

$$\sup_{0<\alpha<\pi/2} \ \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u = 0, \qquad \zeta \in B, \tag{4.3}$$

where $B := \bigcup_{k=1}^\infty A_k^*$.

On the other hand, since $(A \cap A^*) \setminus B \subset \bigcup_{k=1}^\infty (A_k \setminus A_k^*)$, Theorem 4.6(1) implies that $(A\cap A^*)\setminus B$ is of zero length. Consequently, we deduce from (4.3) and Theorem 4.6(2) that $u(z) \le \omega(z, A^*, D)$, $z \in D$. ∎

Next, we introduce a notion which will be relevant for our further study.

Definition 4.8. Let $D, G \subset X$ be open sets such that $G \subset D$ and let $\zeta \in \partial D$ be such that $D$ is Jordan-curve-like at $\zeta$. Then $\zeta$ is said to be an *end-point* of $G$ in $D$ if, for every $0 < \alpha < \pi/2$, there is an open neighborhood $U = U_\alpha$ of $\zeta$ such that $U \cap \mathcal{A}_\alpha(\zeta) \subset G$. The set of all end-points of $G$ in $D$ is denoted by $G^D$.

Note that the above definition is intrinsic.

The remaining part of this section is devoted to the study of level sets of the harmonic measure. We begin with the following important properties of these sets.

Theorem 4.9. *Let $D \subset X$ be an open set and $A$ a Jordan-measurable set of $\partial D$ such that $A$ is of positive length. Then, for any $0 < \varepsilon < 1$, the "$\varepsilon$-level set"*

$$D_\varepsilon := \{z \in D : \omega(z, A^*, D) < 1 - \varepsilon\}$$

*enjoys the following properties*:

(i) *Let $G_1$, $G_2$ be arbitrary distinct connected components of $D_\varepsilon$. Then $G_1^D \cap G_2^D = \emptyset$.*

(ii) *For any $\zeta \in A^*$, there is exactly one connected component $G$ of $D_\varepsilon$ such that $\zeta \in G^D$.*

(iii) *$G^D \cap A$ is Jordan-measurable* (*on $\partial D$*) *and of positive length for every connected component $G$ of $D_\varepsilon$.*

*Proof.* To prove (i), suppose for contradiction that $G_1^D \cap G_2^D \ne \emptyset$. Fix $\zeta_0 \in G_1^D \cap G_2^D$. Then, for every $0 < \alpha < \pi/2$, there is an open neighborhood

$U_\alpha$ of $\zeta_0$ such that $\mathcal{A}_\alpha(\zeta_0)\cap U_\alpha\subset G_1\cap G_2$. This implies that $G_1\cap G_2\neq\emptyset$. Hence, $G_1=G_2$, contrary to hypothesis.

Next, we turn to the proof of (ii). Fix $\zeta_0\in A^*$. By (i), it suffices to prove the existence of a connected component $G$ of $D_\varepsilon$ such that $\zeta_0\in G^D$. Since $\zeta\in A^*$, for every $0<\alpha<\pi/2$, there is an open neighborhood $U_\alpha$ of $\zeta_0$ such that

$$\mathcal{A}_\alpha(\zeta_0)\cap U_\alpha\subset D_\varepsilon. \tag{4.4}$$

Fix $0<\alpha_0<\pi/2$, and let $G$ be the connected component of $D_\varepsilon$ containing $\mathcal{A}_{\alpha_0}(\zeta_0)\cap U_{\alpha_0}$. Since

$$(\mathcal{A}_{\alpha_0}(\zeta_0)\cap U_{\alpha_0})\cap(\mathcal{A}_\alpha(\zeta_0)\cap U_\alpha)\neq\emptyset,\quad 0<\alpha<\pi/2,$$

we deduce from (4.4) that $G$ also contains $\mathcal{A}_\alpha(\zeta_0)\cap U_\alpha$ for every $0<\alpha<\pi/2$. Hence $\zeta_0\in G^D$. The proof of (ii) is finished.

Finally, we prove (iii). First, we find a sequence $(U_k)_{k=1}^\infty$ of open sets of $X$ such that $U_k\cap D$ is either a Jordan domain or the disjoint union of two Jordan domains and $A\subset\bigcup_{k=1}^\infty\partial(U_k\cap D)$. Since $A$ is Jordan-measurable, we see that to prove the Jordan-measurability of $G^D\cap A$, it is sufficient to check that $G^D\cap\partial(D\cap U_k)$ is Jordan-measurable for every $k\ge 1$. To see this, fix $k_0\ge 1$ and let $U:=U_{k_0}$. Let $\Phi$ be a conformal mapping from $D\cap U$ onto $E$ which extends to a homeomorphic mapping (still denoted by) $\Phi$ from $\overline{D\cap U}$ onto $\bar{E}$. It is clear that for any $\zeta\in\partial(D\cap U)$, $\zeta\in G^D$ if and only if $\Phi(\zeta)\in[\Phi(G\cap U)]^E$. We shall prove that $[\Phi(G\cap U)]^E$ is a Borel subset of $\partial E$. Taking this for granted, $G^D\cap\partial(D\cap U)$ is also a Borel set. Consequently, $G^D\cap A$ is Jordan-measurable.

To check that $[\Phi(G\cap U)]^E$ is a Borel set, put

$$\mathcal{A}_{n,m}(\eta):=\{w\in E\cap\mathcal{A}_{(1-1/n)\cdot\pi/2}(\eta):|w-\eta|<1/m\},\qquad n,m\ge 1,\ \eta\in\partial E. \tag{4.5}$$

For any $n,m,p\ge 1$, let

$$T_{nmp}:=\{\eta\in\partial E:\mathcal{A}_{n,m}(\eta)\subset\Phi(G\cap U)\text{ and }\omega(\Phi^{-1}(w),A^*,D)\le 1-\varepsilon-1/p,\ \forall w\in\mathcal{A}_{n,m}(\eta)\}. \tag{4.6}$$

We observe the following *geometric fact*:

*Let $\eta_0\in\partial E$ and $(\eta_q)_{q=1}^\infty\subset\partial E$ be such that* $\lim_{q\to\infty}\eta_q=\eta_0$. *Then*

$$\mathcal{A}_{n,m}(\eta_0)\subset\bigcup_{q=1}^\infty\mathcal{A}_{n,m}(\eta_q).$$

This follows immediately from the geometric shape of the cone $\mathcal{A}_{n,m}(\eta)$ given in (4.5).

Let $(\eta_q)_{q=1}^\infty\subset T_{nmp}$ be such that $\lim_{q\to\infty}\eta_q=\eta_0\in\partial E$. Using the above geometric fact, we see that $\mathcal{A}_{n,m}(\eta_0)\subset\Phi(G\cap U)$. This, combined with (4.6) and the continuity of $\omega(\Phi^{-1}(\cdot), A, D)|_E$, implies that $\eta_0 \in T_{nmp}$. Hence, the set $T_{nmp}$ is closed. Clearly, we have

$$[\Phi(G \cap U)]^E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} T_{nmp},$$

and so $[\Phi(G \cap U)]^E$ is a Borel set. Consequently, as already discussed before, $G^D \cap A$ is Jordan-measurable.

To finish the proof of (iii), it remains to show that $G^E \cap A$ is of positive length. Suppose otherwise and consider the function

$$u(z) := \begin{cases} \omega(z, A^*, D), & z \in D \setminus G, \\ 1 - \varepsilon, & z \in G. \end{cases}$$

Then clearly $u \in \mathcal{SH}(D)$ and $u \le 1$. By (i), (ii) and the definition of locally regular points, we have

$$\begin{aligned} \sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z) &= \sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} \omega(z, A^*, E) \\ &= 0, \qquad \zeta \in (A \cap A^*) \setminus (G^D \cap A). \end{aligned}$$

Consequently, in the notation of (4.2),

$$u \in \widehat{\mathcal{U}}((A \cap A^*) \setminus \mathcal{N}, D),$$

where $\mathcal{N} := G^D \cap A$. Since, by assumption, $\mathcal{N}$ is of zero length, it follows from Theorem 4.6 that $u \le \omega(\cdot, A^*, D)$. But on the other hand, $\omega(z, A^*, D) < 1 - \varepsilon = u(z)$ for $z \in G$. This is the desired contradiction. ∎

THEOREM 4.10. *Let $D \subset X$ be an open set and $A$ a Jordan-measurable subset of $\partial D$ such that $A$ is of positive length. For any $0 \le \varepsilon < 1$, let $D_\varepsilon := \{z \in D : \omega(z, A^*, D) < 1 - \varepsilon\}$.*

(1) *For any Jordan-measurable subset $\mathcal{N} \subset \partial D$ of zero length, let*

$$\mathcal{U}_\varepsilon(A, \mathcal{N}, D) := \{u \in \mathcal{SH}(D_\varepsilon) : u \le 1 \text{ and } \sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z) \le 0,\ \zeta \in (A \cap A^*) \setminus \mathcal{N}\}.$$

*Then $\mathcal{U}_\varepsilon(A, \mathcal{N}, D) = \mathcal{U}_\varepsilon(A, \emptyset, D)$.*

(2) *Define the "harmonic measure of the $\varepsilon$-level set" $\omega_\varepsilon(\cdot, A, D)$ as*

$$\omega_\varepsilon(z, A, D) := \begin{cases} \sup_{u\in\mathcal{U}_\varepsilon(A,\emptyset,D)} u(z), & z \in D_\varepsilon, \\ 0, & z \in A^*. \end{cases}$$

*Then*

$$\omega_\varepsilon(z, A, D) = \frac{\omega(z, A^*, D)}{1 - \varepsilon}, \qquad z \in D_\varepsilon \cup A^*.$$

*Proof.* Clearly, by definition, $\mathcal{U}_\varepsilon(A,\emptyset,D) \subset \mathcal{U}_\varepsilon(A,\mathcal{N},D)$. To prove the reverse inclusion, fix $u \in \mathcal{U}_\varepsilon(A,\mathcal{N},D)$ and consider the function

$$\widehat{u}(z) := \begin{cases} \max\{(1-\varepsilon)u(z), \omega(z,A^*,D)\}, & z \in D_\varepsilon, \\ \omega(z,A^*,D), & z \in D \setminus D_\varepsilon. \end{cases}$$

Then $\widehat{u} \in \mathcal{SH}(D)$ and $\widehat{u} \le 1$. Moreover, by Theorem 4.9(ii), we have $A^* \subset (D_\varepsilon)^D$. Consequently, for every $\zeta \in (A \cap A^*) \setminus \mathcal{N}$,

$$\begin{aligned} &\sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} \widehat{u}(z) \\ &\quad\le \max\{\sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z), \sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} \omega(z,A,D)\}. \end{aligned} \tag{4.7}$$

Observe that the first sup in the line above is 0 because $u \in \mathcal{U}_\varepsilon(A,\mathcal{N},D)$. In addition, the second sup is also 0. Hence, $\widehat{u} \in \widehat{\mathcal{U}}((A \cap A^*) \setminus \mathcal{N}, D)$. Consequently, by Theorem 4.6, $\widehat{u} \le \omega(\cdot,A^*,D)$. In particular,

$$u(z) \le \frac{\omega(z,A^*,D)}{1-\varepsilon}, \qquad z \in D,\ u \in \mathcal{U}_\varepsilon(A,\mathcal{N},D). \tag{4.8}$$

On the other hand, it is clear that $\omega(\cdot,A^*,D)/(1-\varepsilon) \in \mathcal{U}_\varepsilon(A,\emptyset,D) \subset \mathcal{U}_\varepsilon(A,\mathcal{N},D)$. This, combined with (4.8), implies (1) and (2). ■

An immediate consequence of Theorem 4.10 is the following two-constant theorem for level sets.

COROLLARY 4.11. *Let $D \subset X$ be an open set and $A, \mathcal{N}$ two Jordan-measurable subsets of $\partial D$ such that $A$ is of positive length and $\mathcal{N}$ of zero length. Let $0 \le \varepsilon < 1$ and put $D_\varepsilon := \{z \in D : \omega(z,A^*,D) < 1-\varepsilon\}$. If $u \in \mathcal{SH}(D_\varepsilon)$ satisfies $u \le M$ on $D_\varepsilon$ and*

$$\sup_{0<\alpha<\pi/2} \limsup_{z\to\zeta,\, z\in\mathcal{A}_\alpha(\zeta)} u(z) \le m, \qquad \zeta \in (A \cap A^*) \setminus \mathcal{N},$$

*then*

$$u(z) \le m(1-\omega_\varepsilon(z,A,D)) + M\omega_\varepsilon(z,A,D).$$

**5. Boundary behavior of the Gonchar–Carleman operator.** Before recalling the Gonchar–Carleman operator and investigating its boundary behavior, we first introduce the following notion and study its properties.

**5.1.** *Angular Jordan domains.* Let $E$ be the unit disc. We begin with

DEFINITION 5.1. For every closed subset $F$ of $\partial E$ and any real number $h$ such that $\operatorname{mes}(F) > 0$ and $\sup_{x,y\in F} |x-y| < h < 1-\sqrt{2}/2$, the open set

$$\Omega = \Omega(F,h) := \bigcup_{\zeta\in F} \{z \in \mathcal{A}_{\pi/4}(\zeta) : |z| > 1-h\}$$

is called the *angular Jordan domain* with base $F$ and height $h$.

Now we list some properties of angular Jordan domains.

PROPOSITION 5.2. *Let $\Omega = \Omega(F, h)$ be an angular Jordan domain.*

(1) *There exist exactly two points $\zeta_1, \zeta_2 \in F$ such that $|\zeta_1 - \zeta_2| = \sup_{x,y\in F} |x - y|$ and $F \subset [\zeta_1, \zeta_2]$, where $[\zeta_1, \zeta_2]$ is the (small) closed arc of $\partial E$ from $\zeta_1$ to $\zeta_2$, oriented in the positive sense.*

(2) *Write the open set $[\zeta_1, \zeta_2] \setminus F$ as the union of disjoint open arcs*

$$[\zeta_1, \zeta_2] \setminus F = \bigcup_{j\in J} (a_j, b_j),$$

*where $(a_j, b_j)$ is the (small) open arc of $\partial E$ which goes from $a_j$ to $b_j$ and which is oriented in the positive sense, and the index set $J$ is finite or countable. For $j \in J$, we construct the isosceles triangle with vertices $a_j$, $b_j$ and $c_j$ whose base is the segment $[a_j, b_j]$ connecting $a_j$ to $b_j$, and $c_j$ satisfies*

$$\arg\left(\frac{c_j - a_j}{a_j}\right) = \frac{3\pi}{4} \quad \textit{and} \quad \arg\left(\frac{c_j - b_j}{b_j}\right) = \frac{-3\pi}{4}.$$

*Set*

$$F_0 := F \cup \bigcup_{j\in J} ([a_j, c_j] \cup [c_j, b_j]).$$

*Then $F_0$ is a rectifiable Jordan curve from $\zeta_1$ to $\zeta_2$.*

(3) *Let $\eta_1$ (resp. $\eta_2$) be the unique point in the circle $\partial\mathbb{B}(0, 1 - h)$ such that*

$$\arg\left(\frac{\eta_1 - \zeta_1}{\zeta_1}\right) = \frac{-3\pi}{4} \qquad \left(\textit{resp.}\ \arg\left(\frac{\eta_2 - \zeta_2}{\zeta_2}\right) = \frac{3\pi}{4}\right)$$

*and that $|\eta_1 - \zeta_1|$ (resp. $|\eta_2 - \zeta_2|$) is minimal. Let $F_1$ (resp. $F_2$) denote the segment from $\eta_1$ to $\zeta_1$ (resp. $\zeta_2$ to $\eta_2$). Let $F_3$ be the (small) closed arc of the circle $\partial\mathbb{B}(0, 1 - h)$ from $\eta_2$ to $\eta_1$, oriented in the negative sense. Then $\Omega$ is a rectifiable Jordan domain and its boundary $\Gamma$ consists of the rectifiable Jordan curve $F_0$, the two segments $F_1$, $F_2$ and the closed arc $F_3$.*

(4) *For every $\varepsilon \in (0, h/4)$ define the dilatation $\tau_\varepsilon : E \to E$ by*

$$\tau_\varepsilon(z) := (1 - \varepsilon)z, \qquad z \in E.$$

*Put*

$$\Omega_\varepsilon := \tau_\varepsilon(\Omega) \setminus \overline{\mathbb{B}(0, (1 + \varepsilon)(1 - h))}.$$

*Then $\Omega_\varepsilon$ is a rectifiable Jordan domain and its boundary $\Gamma_\varepsilon$ consists of the rectifiable Jordan curve $F_{0\varepsilon} := \tau_\varepsilon(F_0)$, a sub-segment $F_{1\varepsilon}$ of $\tau_\varepsilon(F_1)$, a sub-segment $F_{2\varepsilon}$ of $\tau_\varepsilon(F_2)$, and a closed arc $F_{3\varepsilon}$ of $\partial\mathbb{B}(0, (1 + \varepsilon)(1 - h))$.*

(5) *Consider the projection* $\tau : E \setminus \{0\} \to \partial E$ *given by* $\tau(z) := z/|z|$, $z \in E \setminus \{0\}$. *Notice that* $F_{0\varepsilon} \cup F_{1\varepsilon} \cup F_{2\varepsilon} = \Gamma_\varepsilon \setminus \partial\mathbb{B}(0,(1+\varepsilon)(1-h))$ *for every* $\varepsilon \in (0, h/4)$. *Then the two maps*

$$F_{0\varepsilon} \cup F_{1\varepsilon} \cup F_{2\varepsilon} \ni \zeta \mapsto \tau(\zeta) \in \partial E,$$
$$F_{3\varepsilon} \ni \zeta \mapsto \tau(\zeta) \in \partial E,$$

*are one-to-one. In addition, for any linearly measurable subset* $A$ *of* $\Gamma_\varepsilon$,

$$\operatorname{mes}(A) \le 10 \operatorname{mes}(\tau(A)).$$

(6) $\Omega_\varepsilon \nearrow \Omega$ *as* $\varepsilon \searrow 0$.
(7) *For any closed Jordan curve* $\mathcal{C}$ *contained in* $\Omega$ *there is an* $\varepsilon > 0$ *such that* $\mathcal{C} \subset \Omega_\varepsilon$.
(8) $\operatorname{mes}(F \setminus \Omega^E) = 0$.

*Proof.* All the assertions are proved by elementary geometric arguments. Therefore, we leave the details to the reader. However, we give the proof that $\Omega$ is a domain, which will clarify Definition 5.1.

In view of the condition on $F$ and $h$ given in Definition 5.1, we see that the set $\{z \in \mathcal{A}_{\pi/4}(\zeta) : |z| > 1-h\}$ is connected for any $\zeta \in \partial E$, and

$$\{z \in \mathcal{A}_{\pi/4}(\zeta) : |z| > 1-h\} \cap \{z \in \mathcal{A}_{\pi/4}(\eta) : |z| > 1-h\} \ne \emptyset,$$
$$\forall \zeta, \eta \in \partial E : |\zeta - \eta| < h < 1 - \sqrt{2}/2.$$

Hence, $\Omega$ is a domain. ∎

THEOREM 5.3. *Let* $X$ *be a complex manifold of dimension* 1, $D \subset X$ *an open set, and* $A$ *a Jordan-measurable subset of* $\partial D$ *such that* $A$ *is of positive length. Then, for any* $0 \le \varepsilon < 1$ *and any connected component* $G$ *of* $D_\varepsilon := \{z \in D : \omega(z, A^*, D) < 1-\varepsilon\}$, *there are an open set* $U \subset X$, *a conformal mapping* $\Phi : E \to X$, *and an angular Jordan domain* $\Omega = \Omega(F,h)$ *such that*

(i) $U \cap D$ *is either a Jordan domain or the disjoint union of two Jordan domains*;
(ii) $\Phi$ *maps* $E$ *conformally onto one connected component of* $U \cap D$ (*notice that, by* (i), $U \cap D$ *has at most two connected components*);
(iii) $\Phi(F) \subset A \cap A^* \cap G^D$ *and* $\Phi(\Omega) \subset G$.

*Proof.* We have already shown in the proof of Theorem 4.9(iii) that there is a sequence $(U_k)_{k=1}^\infty$ of open sets in $X$ such that $U_k \cap D$ is either a Jordan domain or the disjoint union of two Jordan domains, $A \subset \bigcup_{k=1}^\infty \partial(U_k \cap D)$, and $A \cap A^* \cap G^D$ is of positive length. Consequently, there is an index $k_0$ such that

$$A \cap A^* \cap G^D \cap \partial(D \cap U) \quad \text{is of positive length,} \tag{5.1}$$

where $U := U_{k_0}$. Suppose without loss of generality that $U \cap D$ is a Jordan domain. The case where $U \cap D$ is the disjoint union of two Jordan domains may be proved in the same way. Let $\Phi$ be a conformal mapping from $E$ onto $D\cap U$. By the Carathéodory theorem (see [4]), $\Phi$ extends to a homeomorphic map (still denoted by) $\Phi$ from $\overline{E}$ onto $\overline{D\cap U}$. Hence, (i) and (ii) are satisfied.

On the other hand, it follows from (5.1) that

$$\operatorname{mes}(\Phi^{-1}(A\cap A^*\cap G^D\cap\partial(D\cap U)))>0. \tag{5.2}$$

For any $m\geq 1$, let

$$A_m:=\{\eta\in\partial E:\mathcal{A}_{2,m}(\eta)\subset\Phi^{-1}(G)\}, \tag{5.3}$$

where $\mathcal{A}_{2,m}(\eta)$ is given by (4.5).

Using the geometric fact following (4.6), we see that $A_m$ is closed. On the other hand, it is clear that

$$\Phi^{-1}(A\cap A^*\cap G^D\cap\partial(D\cap U))\subset\bigcup_{m=1}^{\infty}A_m.$$

Therefore, by (5.2), there is an index $m_0$ such that

$$\operatorname{mes}(A_{m_0}\cap\Phi^{-1}(A\cap A^*\cap G^D\cap\partial(D\cap U)))>0.$$

Put $h:=1/2m_0$. By the last inequality one can find a closed set $F$ contained in $A_{m_0}\cap\Phi^{-1}(A\cap A^*\cap G^D\cap\partial(D\cap U))$ such that $\operatorname{mes}(F)>0$ and $\sup_{x,y\in F}|x-y|<h$. Since $h=1/2m_0$, a geometric argument shows that

$$\{z\in\mathcal{A}_{\pi/4}(\zeta):|z|>1-h\}\subset\mathcal{A}_{2,m_0}(\zeta),\qquad \zeta\in\partial E.$$

This together with (5.3) implies that $\Omega=\Omega(F,h)\subset\Phi^{-1}(G)$. Hence, (iii) is verified. ∎

The following uniqueness theorem will play a vital role.

Theorem 5.4. *Let $X$ be a complex manifold of dimension* 1, $D\subset X$ *an open set*, *and* $A$, $\mathcal{N}$ *two Jordan-measurable subsets of* $\partial D$ *such that* $A$ *is of positive length and* $\mathcal{N}$ *is of zero length. Let* $0\leq\varepsilon<1$ *and* $G$ *a connected component of* $D_\varepsilon:=\{z\in D:\omega(z,A^*,D)<1-\varepsilon\}$. *If* $f\in\mathcal{O}(G)$ *has angular limit* 0 *at every point of* $(A\cap A^*\cap G^D)\setminus\mathcal{N}$, *then* $f\equiv 0$.

*Proof.* Applying Theorem 5.3 we obtain an open set $U$ in $X$, a conformal mapping $\Phi$ from $E$ onto $D\cap U$ which extends homeomorphically to $\overline{E}$, and an angular Jordan domain $\Omega:=\Omega(F,h)$ satisfying assertions (i)–(iii) of that theorem.

By hypothesis, $f\circ\Phi\in\mathcal{O}(\Omega)$ has angular limit 0 at a.e. point in $F$. Since $\operatorname{mes}(F)>0$, Privalov's uniqueness theorem (see [4]) shows that $f\circ\Phi\equiv 0$ on $\Omega$. Hence, $f\equiv 0$ on the subdomain $\Phi(\Omega)$ of $G$, and so $f\equiv 0$. ∎

**5.2.** *Main result of this section.* Let $D,G\subset\mathbb{C}$ be open discs and let $A$ (resp. $B$) be a measurable subset of $\partial D$ (resp. $\partial G$) with $\operatorname{mes}(A)>0$ (resp. $\operatorname{mes}(B)>0$). Let $f$ be a function defined on $W:=\mathbb{X}(A,B;D,G)$ with the following properties:

(i) $f|_{A\times B}$ is measurable and there is a finite constant $C$ with $|f|_W < C$;
(ii) $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$;
(iii) there exist functions $f_1, f_2 : A \times B \to \mathbb{C}$ such that for any $a \in A$ (resp. $b \in B$), $f(a, \cdot)$ (resp. $f(\cdot, b)$) has angular limit $f_1(a, b)$ at $b$ for a.e. $b \in B$ (resp. $f_2(a, b)$ at $a$ for a.e. $a \in A$), and $f_1 = f_2 = f$ a.e. on $A \times B$.

Let $\widetilde{\omega}(\cdot, A, D)$ (resp. $\widetilde{\omega}(\cdot, B, G)$) be the conjugate harmonic function of $\omega(\cdot, A, D)$ (resp. $\omega(\cdot, B, G)$) such that $\widetilde{\omega}(z_0, A, D) = 0$ (resp. $\widetilde{\omega}(w_0, B, G) = 0$) for a certain fixed point $z_0 \in D$ (resp. $w_0 \in G$). Then we define the holomorphic functions $g_1(z) := \omega(z, A, D) + i\widetilde{\omega}(z, A, D)$, $g_2(w) := \omega(w, B, G) + i\widetilde{\omega}(w, B, G)$, and

$$g(z, w) := g_1(z) + g_2(w), \qquad (z, w) \in D \times G.$$

The function $e^{-g_1}$ (resp. $e^{-g_2}$) is bounded on $D$ (resp. on $G$). Therefore, in view of [4, p. 439], we may define $e^{-g_1(a)}$ (resp. $e^{-g_2(b)}$) for a.e. $a \in A$ (resp. $b \in B$) to be the angular boundary limit of $e^{-g_1}$ at $a$ (resp. $e^{-g_2}$ at $b$).

In view of (i), for each positive integer $N$, we define the *Gonchar–Carleman operator* as follows:

$$\begin{aligned} K_N(z, w) &= K_N[f](z, w) \\ &:= \frac{1}{(2\pi i)^2} \int\limits_{A\times B} e^{-N(g(a,b)-g(z,w))} \frac{f(a, b)\, da\, db}{(a - z)(b - w)}, \qquad (z, w) \in D \times G. \end{aligned} \tag{5.4}$$

We recall from Gonchar's work [6] that the limit

$$K(z, w) = K[f](z, w) := \lim_{N\to\infty} K_N(z, w) \tag{5.5}$$

exists for all $(z, w) \in \widehat{W}^{\mathrm{o}}$, and it is uniform on compact subsets of $\widehat{W}^{\mathrm{o}}$.

The boundary behavior of the Gonchar–Carleman operator is described below.

Theorem 5.5. *Under the above hypothesis and notation, let* $0 < \delta < 1$ *and* $w \in G$ *be such that* $\omega(w, B, G) < \delta$, *and let* $U$ *be any connected component of*

$$D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta\}.$$

*Then there is an angular Jordan domain* $\Omega = \Omega(F, h)$ *such that* $\Omega \subset U$, $F \subset A \cap A^* \cap U^D$, *and the Gonchar–Carleman operator* $K[f]$ (*see formula* (5.4)–(5.5) *above*) *satisfies*

$$\lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} K[f](z, w) = f(a, w), \qquad 0 < \alpha < \pi/2,$$

*for a.e.* $a \in F$.

The proof of this theorem will be given in Subsection 5.4 below.

**5.3.** *Preparatory results.* For the proof of Theorem 5.5 we need the following results.

For every $f \in L^1(\partial E, |d\zeta|)$, let $\mathcal{C}[f]$ denote the Cauchy integral

$$\mathcal{C}[f](z) := \frac{1}{2\pi i} \int_{\partial E} \frac{f(\zeta)}{z - \zeta}\, d\zeta, \qquad z \in E.$$

For a function $F : E \to \mathbb{C}$, the radial maximal function $M_{\mathrm{rad}}F : \partial E \to [0, \infty]$ is defined by

$$(M_{\mathrm{rad}}F)(\zeta) := \sup_{0 \le r < 1} |F(r\zeta)|, \qquad \zeta \in \partial E.$$

Now we are able to state the following classical result (see Theorem 6.3.1 in Rudin's book [14]):

THEOREM 5.6 (Korányi–Vági type theorem). *There is a constant $C > 0$ such that*

$$\int_{\partial E} |M_{\mathrm{rad}}\mathcal{C}[f](\zeta)|^2\, |d\zeta| \le C \int_{\partial E} |f(\zeta)|^2\, |d\zeta|$$

*for every $f \in L^2(\partial E, |d\zeta|)$.*

We recall the definition of the Smirnov class $E^p$, $p > 0$, on rectifiable Jordan domains.

DEFINITION 5.7. Let $p > 0$ and $\Omega$ a rectifiable Jordan domain. A function $f \in \mathcal{O}(\Omega)$ is said to belong to the *Smirnov class* $E^p(\Omega)$ if there exists a sequence $(\mathcal{C}_n)_{n=1}^{\infty}$ of rectifiable closed Jordan curves in $\Omega$, tending to the boundary in the sense that $\mathcal{C}_n$ eventually surrounds each compact subdomain of $\Omega$, such that

$$\int_{\mathcal{C}_n} |f(z)|^p\, |dz| \le M < \infty, \qquad n \ge 1.$$

Next, we rephrase some facts concerning the Smirnov class $E^p$, $p > 0$, on rectifiable Jordan domains in the context of angular Jordan domains $\Omega(F, h)$.

THEOREM 5.8.

(1) *Let $\Omega$ be a rectifiable Jordan domain. Then every $f \in E^p(\Omega)$ $(p > 0)$ has an angular limit $f^*$ a.e. on $\partial\Omega$.*

(2) *Let $\Omega := \Omega(F, h)$ be an angular Jordan domain and let $\Gamma := \partial\Omega$. For any $0 < \varepsilon < h/4$, let $\Gamma_\varepsilon$ be the rectifiable closed Jordan curve defined in Proposition* 5.2(4). *Then $f \in E^p(\Omega)$ if $\sup_{0<\varepsilon<h/4} \int_{\Gamma_\varepsilon} |f(z)|^p\, |dz| < \infty$. In addition, for every $f \in E^p(\Omega)$, $p > 0$,*

$$\int_{\Gamma} |f^*(z)|^p\, |dz| \le \sup_{0<\varepsilon<h/4} \int_{\Gamma_\varepsilon} |f(z)|^p\, |dz|.$$

(3) *Every* $f \in E^1(E)$ *has a Cauchy representation* $f := \mathcal{C}[f^*]$. *Conversely, if* $g \in L^1(\partial E, |dz|)$ *and*

$$\int_{\partial E} z^n g(z)\,dz = 0, \qquad n = 0, 1, 2, \ldots,$$

*then* $f := \mathcal{C}[g] \in E^1(E)$ *and* $g$ *coincides with* $f^*$ *a.e. on* $\partial E$.

*Proof.* For the proof of (1) and (3), see [4, pp. 438–441]. Taking into account Proposition 5.2(6), (7), assertion (2) also follows from the results in [4, pp. 438–441]. ∎

**5.4.** *Proof of Theorem 5.5.* We fix $w_0 \in G$ and $0 < \delta_0 < \delta$ with $\omega(w_0, B, G) < \delta_0$ and a connected component $U$ of $D_\delta$. Applying Theorem 5.3, we find an angular Jordan domain $\Omega := \Omega(F, h) \subset U$ such that $F \subset A \cap A^* \cap U^D$. In the course of the proof, the letter $C$ will denote a positive constant that is not necessarily the same at each step.

Applying the Carleman theorem (see, for example, [1, p. 2]), we have

$$f(z,b) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_A e^{-N(g_1(a)-g_1(z))} \frac{f(a,b)}{a-z}\,da, \qquad z \in D,\ b \in B,$$

$$f(a,b) = \lim_{r\to 1-} f(ra, b), \qquad a \in \partial D,\ b \in B.$$

Consequently, $f|_{\partial D\times B}$ is measurable. In addition, by (iii) it is bounded. Therefore, for every $N \in \mathbb{N}$ we can define $K_{\infty,N}(\cdot, w_0) : \partial D \to \mathbb{C}$ by

$$K_{\infty,N}(a, w_0) := \frac{1}{2\pi i} \int_B e^{N(g_2(w_0)-g_2(b))} \frac{f(a,b)}{b-w_0}\,db, \qquad a \in \partial D. \tag{5.6}$$

Since, in view of (ii)–(iii), $f(a,\cdot) \in \mathcal{O}(G)$ and $|f(a,\cdot)|_G < C$ for $a \in A$, it follows from the Carleman theorem that

$$\lim_{N\to\infty} K_{\infty,N}(a, w_0) = f(a, w_0), \qquad a \in A, \tag{5.7}$$

and the above convergence is uniform with respect to $a \in A$.

On the other hand, by (5.6) we see that $K_{\infty,N}(\cdot, w_0)$ is measurable and bounded. In addition, for any $n = 0, 1, 2, \ldots,$ taking (ii) into account, we have

$$\int_{\partial D} K_{\infty,N}(a, w_0) a^n\,da = \frac{1}{2\pi i} \int_B \Big( \int_{\partial D} f(a,b) a^n\,da \Big) \frac{e^{N(g_2(w_0)-g_2(b))}}{b-w_0}\,db = 0,$$

where the first equality follows from Fubini's theorem and the second from an application of Theorem 5.8(3) to $f(\cdot, b)$, $b \in B$. Consequently, by Theorem 5.8(3), we can extend $K_{\infty,N}(\cdot,w_0)$ to $\overline{D}$ by setting

$$(5.8)\qquad K_{\infty,N}(z,w_0) := \mathcal{C}[K_{\infty,N}(\cdot,w_0)](z) = \frac{1}{2\pi i}\int\limits_{\partial D}\frac{K_{\infty,N}(a,w_0)}{a-z}\,da, \qquad z\in D.$$

Then

$$(5.9)\qquad \lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} K_{\infty,N}(z,w_0) = K_{\infty,N}(a,w_0), \qquad 0<\alpha<\pi/2,$$

for a.e. $a\in\partial D$.

Now we return to the angular Jordan domain $\Omega$. We keep the notation introduced in Proposition 5.2. For any $0<\varepsilon<h/4$ and any $z\in\Gamma_\varepsilon$, applying the Cauchy integral formula, we obtain

$$(5.10)\qquad K_{\infty,N}(z,w_0)-K_N(z,w_0) = \frac{1}{(2\pi i)^2}\int\limits_{\partial D\setminus A}\int\limits_{B} e^{N(g_1(z)-g_1(a))+N(g_2(w_0)-g_2(b))}\frac{f(a,b)}{(a-z)(b-w_0)}\,da\,db = e^{N(g_1(z)-(1-\delta))}\int\limits_{\partial D}\frac{p_N(a)}{a-z}\,da.$$

Using the choice of $U$ and the hypothesis on $\delta$ and $\delta_0$, it can be checked that

$$(5.11)\qquad |e^{N(g_1(\cdot)-(1-\delta))}|_U \le 1, \qquad |p_N|_{\partial D}\le Ce^{-N(\delta-\delta_0)}.$$

Therefore, recalling the projection $\tau: E\setminus\{0\}\to\partial E$ (see Proposition 5.2(5)), we estimate

$$(5.12)\qquad \int\limits_{\Gamma_\varepsilon}|K_{\infty,N}(z,w_0)-K_N(z,w_0)|^2\,|dz| \le C\int\limits_{\Gamma_\varepsilon}|M_{\mathrm{rad}}\mathcal{C}[p_N](\tau(z))|^2\,|dz|$$
$$\le 10C\int\limits_{\tau(F_{0\varepsilon}\cup F_{1\varepsilon}\cup F_{2\varepsilon})}|M_{\mathrm{rad}}\mathcal{C}[p_N](a)|^2\,|da| + 10C\int\limits_{\tau(F_{3\varepsilon})}|M_{\mathrm{rad}}\mathcal{C}[p_N](a)|^2\,|da|$$
$$\le 20C\int\limits_{\partial E}|M_{\mathrm{rad}}\mathcal{C}[p_N](a)|^2\,|da| \le C\int\limits_{\partial E}|p_N(a)|^2\,|da| \le Ce^{-N(\delta-\delta_0)}.$$

Here the first estimate follows from (5.10)–(5.11) and the definition of the radial maximal function, the second and third are consequences of Proposition 5.2(5), the fourth holds by Theorem 5.6, and the last one follows from (5.11).

On the other hand, for any $0<\varepsilon<h/4$,

$$(5.13)\qquad \int\limits_{\Gamma_\varepsilon}|K_{N+1}(z,w_0)-K_N(z,w_0)|^2\,|dz| \le 2\int\limits_{\Gamma_\varepsilon}|A_N(z,w_0)|^2\,|dz| + 2\int\limits_{\Gamma_\varepsilon}|B_N(z,w_0)|^2\,|dz| \le Ce^{-N(\delta-\delta_0)},$$

where $A_N$ and $B_N$ are given by formula (6) in [6] and the latter estimate follows from the same argument as in the proof of (5.10)–(5.12). We recall from (5.5) that

$$\lim_{N\to\infty} K_N(z,w_0) = K(z,w_0), \qquad z \in \Gamma_\varepsilon.$$

This, combined with (5.12)–(5.13), implies that

$$(5.14) \qquad \int_{\Gamma_\varepsilon} |K_{\infty,N}(z,w_0) - K(z,w_0)|^2\,|dz| \le Ce^{-N(\delta-\delta_0)}, \qquad 0<\varepsilon<h/4.$$

Since we have already shown that $|K_{\infty,N}(\cdot,w_0)|_D < \infty$, in view of Theorem 5.8(2), we deduce from (5.14) that $K(\cdot,w_0)|_\Omega \in E^2(\Omega)$. For every $a\in\partial D$, let $K(a,w_0)$ denote the angular limit of $K(\cdot,w_0)|_\Omega$ at $a$ (if the limit exists). It follows from (5.14) and Theorem 5.8(2) that

$$\lim_{N\to\infty} \int_{\Gamma} |K_{\infty,N}(a,w_0) - K(a,w_0)|^2\,|da|$$
$$\le \sup_{0<\varepsilon<h/4} \int_{\Gamma_\varepsilon} |K_{\infty,N}(z,w_0) - K(z,w_0)|^2\,|dz| \le \lim_{N\to\infty} Ce^{-N(\delta-\delta_0)} = 0.$$

This, combined with (5.7) and 5.2(8), implies finally that

$$K(a,w_0) = f(a,w_0) \qquad \text{for a.e. } a\in F.$$

Hence, Theorem 5.5 has been proved. ∎

**6. Proof of Theorem A for the case where $D$ and $G$ are Jordan domains.** Using an exhaustion argument, a compactness argument and conformal mappings, the case where $D$ and $G$ are Jordan domains can be reduced to the following case:

(∗) *We assume that $D=G=E$, and $|f|_W<1$.*

Using hypotheses (i)–(iii) and (∗), we apply Theorem 5.5 to obtain a function $K[f]\in\mathcal{O}(\widehat{W}^{\mathrm{o}})$. Consequently, we define the desired extension by

$$\widehat{f} := K[f].$$

In this section we will use repeatedly Theorem 4.6(3):

$$\omega(\cdot,A,\Omega) = \omega(\cdot,A^*,\Omega),$$

where $\Omega\subset\mathbb{C}$ is an open set and $A$ is a Jordan measurable subset of $\partial\Omega$.

The remaining part of the proof is divided into several steps.

STEP 1: *Proof of the estimate $|\widehat{f}|_{\widehat{W}^{\mathrm{o}}} \le |f|_W$.* Let $(z_0,w_0)\in\widehat{W}^{\mathrm{o}}$. Then we can find $\delta\in(0,1)$ such that $0<\omega(w_0,B,G)<\delta<1-\omega(z_0,A,D)$. Let $U$ be the connected component of $D_\delta$ that contains $z_0$. By Theorem 5.3 we find an angular Jordan domain $\Omega := \Omega(F,h)\subset U$ such that $F\subset A\cap A^*\cap U^D$.

In addition, for every $N \in \mathbb{N}$, applying Theorem 5.5 to $f^N$, we obtain $K[f^N] \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ with

$$\begin{aligned}\lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} K[f^N](z,w_0) &= f(a,w_0)^N\\ &= \lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} (K[f](z,w_0))^N, \qquad 0<\alpha<\pi/2,\end{aligned}$$

for a.e. $a \in F$. Consequently, an application of Theorem 5.4 gives

$$K[f^N](z_0,w_0) = (K[f](z_0,w_0))^N, \qquad N\in\mathbb{N},$$

It follows that

$$K[f^N](z,w) = (K[f](z,w))^N, \qquad N\in\mathbb{N},\ (z,w)\in\widehat{W}^{\mathrm{o}}. \tag{6.1}$$

Now we conclude the proof in the same way as in [6, p. 23]. More precisely, taking into account (6.1), one gets

$$\begin{aligned}|\widehat{f}^N(z,w)| &\le |K[f^N](z,w)|\\ &\le \frac{C|f|_W^N}{(1-|z|)(1-|w|)(1-e^{-(1-\omega(z,w))})}, \qquad (z,w)\in\widehat{W}^{\mathrm{o}}.\end{aligned}$$

Taking the $N$th roots of both sides and letting $N$ tend to $\infty$ yields the estimate of Step 1. ∎

STEP 2: *Proof that $\widehat{f}$ is the unique function in $\mathcal{O}(\widehat{W}^{\mathrm{o}})$ which satisfies property* (1) *of Theorem A*. First we show that $\widehat{f}$ satisfies (1). Without loss of generality, it suffices to prove that there is a subset $\widetilde{B}$ of $B\cap B^*$ such that $\mathrm{mes}(\widetilde{B}) = \mathrm{mes}(B)$ and $\widehat{f}$ has angular limit $f$ at every point of $D\times\widetilde{B}$.

For any $a\in A$ put

$$B_a := \{b\in B : f(a,\cdot) \text{ has an angular limit at } b\}.$$

By hypothesis (iii), we have $\mathrm{mes}(B_a) = \mathrm{mes}(B)$, $a\in A$. Consequently, applying Fubini's theorem, we obtain

$$\int_A \mathrm{mes}(B_a)\,|da| = \mathrm{mes}(A)\,\mathrm{mes}(B) = \int_B \mathrm{mes}(\{a\in A : b\in B_a\})\,|db|.$$

Hence,

$$\mathrm{mes}(\{a\in A : b\in B_a\}) = \mathrm{mes}(A) \qquad \text{for a.e. } b\in B. \tag{6.2}$$

The same reasoning also gives

$$\mathrm{mes}(\{a\in A : f(a,b) = f_1(a,b)\}) = \mathrm{mes}(A) \qquad \text{for a.e. } b\in B. \tag{6.3}$$

Set

$$\begin{aligned}\widetilde{B} := \{b\in B\cap B^* : \mathrm{mes}(&\{a\in A : b\in B_a\}) = \mathrm{mes}(A) \text{ and}\\ &\mathrm{mes}(\{a\in A : f(a,b) = f_1(a,b)\}) = \mathrm{mes}(A)\}.\end{aligned} \tag{6.4}$$

We deduce from (6.2)–(6.4) that

$$\operatorname{mes}(\widetilde{B}) = \operatorname{mes}(B). \tag{6.5}$$

Fix $b_0 \in \widetilde{B}$ and let $(w_n)_{n=1}^{\infty}$ be a sequence in $G$ such that $\lim_{n\to\infty} w_n = b_0$ and $w_n \in \mathcal{A}_\alpha(b_0)$ for some fixed $0 < \alpha < \pi/2$. Fix $z_0 \in D$ and let $(z_n)_{n=1}^{\infty}$ be any sequence in $D$ such that $\lim_{n\to\infty} z_n = z_0$.

Clearly, we can find $0 < \delta_1 < 1$ such that

$$\sup_{n\in\mathbb{N}} \omega(z_n, A, D) < 1 - \delta_1. \tag{6.6}$$

Fix $\delta_2$ such that $0 < \delta_2 < \delta_1$. Since $b_0$ is locally regular relative to $B$ and $\lim_{n\to\infty} w_n = b_0$ and $w_n \in \mathcal{A}_\alpha(b_0)$, there is a sufficiently large number $N_0$ with

$$\omega(w_n, B, G) < \delta_2, \qquad n > N_0. \tag{6.7}$$

Let $U$ be the connected component of $D_{\delta_1}$ that contains $z_0$ (see (6.6)). Applying Theorem 5.3, we find an angular Jordan domain $\Omega := \Omega(F, h) \subset U$ such that $F \subset A \cap A^* \cap U^D$. Let $V$ be a rectifiable Jordan domain with $\Omega \subset V \subset U$, $w_0 \in V$, and $V \cap \mathcal{U} = \Omega \cap \mathcal{U}$ for some neighborhood $\mathcal{U}$ of the base $F$ of $\Omega$.

In view of (6.7) and of the fact that $V \subset U \subset D_\delta$, we obtain

$$V \times \{w_n\} \subset \widehat{W}^{\mathrm{o}}, \qquad n > N_0. \tag{6.8}$$

Consequently, Theorem 5.5 implies that for any $n > N_0$,

$$f(a, w_n) = \lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} \widehat{f}(z, w_n), \qquad 0 < \alpha < \pi/2, \tag{6.9}$$

for a.e. $a \in F$.

Next, for any $n > N_0$ let

$$F_n := \{a \in F : b_0 \in B_a \text{ and } f(a, w_n) = \lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} \widehat{f}(z, w_n)\},$$

$$F_0 := \bigcap_{n=N_0+1}^{\infty} F_n.$$

It follows from (6.4), (6.9) and the fact that $b_0 \in \widetilde{B}$ that $\operatorname{mes}(F_n) = \operatorname{mes}(F)$ for $n > N_0$. Hence

$$\operatorname{mes}(F_0) = \operatorname{mes}(F) > 0. \tag{6.10}$$

In view of (6.8), consider the following holomorphic functions on $V$:

$$h_n(t) := \widehat{f}(t, w_n) \quad \text{and} \quad h_0(t) := f(t, b_0), \qquad t \in V,\ n > N_0. \tag{6.11}$$

Since we have already shown in Step 1 that $|h_n|_V \le |f|_X < \infty$ for $n > N_0$ or $n = 0$, applying Theorem 5.8(1), we find a subset $\Delta$ of $F_0$ with $\operatorname{mes}(\Delta) = \operatorname{mes}(F_0) > 0$ such that $h_n$, $n > N_0$, (resp. $h_0$) has angular limit $f_1(t, w_n)$

(resp. $f_1(t,b_0)$) at $t\in\varDelta$. Observe that by (6.4) and the fact that $b_0\in\widetilde{B}$ we have

$$\lim_{n\to\infty} f_1(t,w_n)=f_1(t,b_0)=f(t,b_0)\quad \text{for a.e. } t\in\varDelta.$$

Using this and (6.11), we can apply the Khinchin–Ostrowski theorem (see [4, Theorem 4, p. 397]) to the sequence $(h_n)_{n=0}^{\infty}$. Consequently,

$$\lim_{n\to\infty}\widehat{f}(z_n,w_n)=f(z_0,b_0).$$

This shows that $\widehat{f}$ has angular limit $f$ at every point of $D\times\widetilde{B}$. Hence, $\widehat{f}$ satisfies property (1).

It remains to show the uniqueness of $\widehat{f}$. To do this, let $\widehat{\widehat{f}}\in\mathcal{O}(\widehat{W}^{\mathrm{o}})$ have the following property: There is a subset $\widetilde{\widetilde{A}}$ (resp. $\widetilde{\widetilde{B}}$) of $A\cap A^*$ (resp. $b\cap B^*$) such that $\mathrm{mes}(A\setminus\widetilde{\widetilde{A}})=\mathrm{mes}(B\setminus\widetilde{\widetilde{B}})=0$ and $\widehat{\widehat{f}}$ has angular limit $f$ at every point of $(\widetilde{\widetilde{A}}\times G)\cup(D\times\widetilde{\widetilde{B}})$. Fix $(z_0,w_0)\in\widehat{W}^{\mathrm{o}}$. Let $U$ be the connected component containing $z_0$ of

$$\{z\in D:\omega(z,A,D)<1-\omega(w_0,B,G)\}.$$

We deduce that both $\widehat{f}(\cdot,w_0)|_U$ and $\widehat{\widehat{f}}(\cdot,w_0)|_U$ have angular limit $f(\cdot,w_0)$ at every point of $\widetilde{A}\cap\widetilde{\widetilde{A}}\cap U^D$. Consequently, Theorem 5.4 yields $\widehat{f}(\cdot,w_0)=\widehat{\widehat{f}}(\cdot,w_0)$ on $U$. Hence, $\widehat{f}(z_0,w_0)=\widehat{\widehat{f}}(z_0,w_0)$. Since $(z_0,w_0)\in\widehat{W}^{\mathrm{o}}$ is arbitrary, the uniqueness of $\widehat{f}$ is established. ∎

STEP 3: *Proof of (2) of Theorem A.* Fix $(z_0,w_0)\in\widehat{W}^{\mathrm{o}}$. For every $b\in B$ we have

$$|f(a,b)|\le|f|_{A\times B},\quad a\in A,\quad\text{and}\quad |f(z,b)|\le|f|_W,\quad z\in D.$$

Therefore, the two-constant theorem (see Theorem 2.2 in [10]) implies that

$$|f(z,b)|\le|f|_{A\times B}^{1-\omega(z,A,D)}|f|_W^{\omega(z,A,D)},\qquad z\in D,\ b\in B. \tag{6.12}$$

Let $\delta:=\omega(z_0,A,D)$ and consider the $\delta$-level set

$$G_\delta:=\{w\in G:\omega(w,B,G)<1-\delta\}.$$

Clearly, $w_0\in G_\delta$.

Recall from Step 2 that $\widetilde{B}\subset B\cap B^*$, $\mathrm{mes}((B\cap B^*)\setminus\widetilde{B})=0$, and

$$f(z_0,b)=\lim_{w\to b,\,w\in\mathcal{A}_\alpha(b)}\widehat{f}(z_0,w),\qquad 0<\alpha<\pi/2,\ b\in\widetilde{B}. \tag{6.13}$$

Consider the function $h:G_\delta\cup\widetilde{B}\to\mathbb{C}$ defined by

$$h(t):=\begin{cases}\widehat{f}(z_0,t), & t\in G_\delta,\\ f(z_0,t), & t\in\widetilde{B}.\end{cases} \tag{6.14}$$

Clearly, $h|_{G_\delta}\in\mathcal{O}(G_\delta)$.

On the other hand, in view of (6.14) and the result of Step 1, we have

$$|h|_{G_\delta} \le |\widehat{f}|_{\widehat{W}^\circ} \le |f|_W < \infty. \tag{6.15}$$

In addition, applying Corollary 4.11 and taking (6.13)–(6.14) into account yields

$$|h(t)| \le |h|_{\widetilde{B}}^{1-\omega_\delta(t,A,D)} |h|_{G_\delta}^{\omega_\delta(t,A,D)}, \quad t \in G_\delta,$$

where, by Theorem 4.10,

$$\omega_\delta(t,B,G) = \frac{\omega(t,B,G)}{1-\omega(z_0,A,D)}.$$

This, combined with (6.12)–(6.15), implies that

$$|\widehat{f}(z_0,w_0)| = |h(w_0)| \le |f|_{A\times B}^{1-\omega(z_0,A,D)-\omega(w_0,B,G)} |f|_W^{\omega(z_0,A,D)+\omega(w_0,B,G)}.$$

Hence (2) for the point $(z_0,w_0)$ is proved. ∎

Step 4: *Proof of (3) of Theorem A.* Let $(a_0,w_0) \in A^* \times G$ be such that the following limit exists:

$$\lambda := \lim_{(a,w)\to(a_0,w_0),\,(a,w)\in A\times G} f(a,w).$$

We now show that $\widehat{f}$ has angular limit $\lambda$ at $(a_0,w_0)$.

For any $0<\varepsilon<1/2$, we find an open neighborhood $A_{a_0}$ of $a_0$ in $A$ and a positive number $r>0$ such that $\mathbb{B}(w_0,r) \Subset G$ and

$$|f(a,w)-\lambda| < \varepsilon^2, \quad a \in A_{a_0},\ |w-w_0| \le r. \tag{6.16}$$

Put

$$\delta := \sup_{w\in\overline{\mathbb{B}(w_0,r)}} \omega(w,B,G). \tag{6.17}$$

Since $a_0 \in A^*$, it is clear that $\mathrm{mes}(A_{a_0})>0$. Next, consider the level set

$$D_\delta := \{z \in D : \omega(z,A_{a_0},D) < 1-\delta\}.$$

In view of (6.17), we can define

$$h(t,w) := \widehat{f}(t,w) - \lambda, \quad t \in D_\delta,\ w \in \overline{\mathbb{B}(w_0,r)}. \tag{6.18}$$

Clearly,

$$|h|_{D_\delta} \le 2|\widehat{f}|_{\widehat{W}^\circ} = 2|f|_W = 2. \tag{6.19}$$

By (6.18) and using the result of Step 2, we know that for every $w \in \mathbb{B}(w_0,r)$ the holomorphic function $h(\cdot,w)|_{D_\delta}$ has angular limit $f(a,w)-\lambda$ at $a$ for $a \in \widetilde{A} \cap A_{a_0}$, where $\widetilde{A}$ is given in Step 2. Consequently, applying Corollary 4.11 and taking (6.16) and (6.19) into account, we see that

$$|h(t,w)| < \varepsilon^{2(1-\omega_\delta(t,A_{a_0},D))} 2^{\omega_\delta(t,A_{a_0},D)}, \quad t \in D_\delta.$$

Let $0 < \alpha < \pi/2$. From Theorem 4.10 and the hypothesis that $a_0 \in A^*$, we deduce that $\lim_{t\to a_0,\, t\in\mathcal{A}_\alpha(a_0)} \omega_\delta(t, A_{a_0}, D) = 0$. Consequently, there is an $r_\alpha > 0$ such that

$$|f(z,w) - \lambda| = |h(z,w)| < \varepsilon, \qquad z \in \mathcal{A}_\alpha(a_0) \cap \{|z-a_0| < r_\alpha\},\ w \in \mathbb{B}(w_0, r).$$

This completes the proof of the above assertion.

Similarly, we can prove that $\widehat{f}$ has angular limit

$$\lim_{(z,b)\to(z_0,b_0),\,(z,b)\in D\times B} f(z,b)$$

at any point $(z_0, b_0)$, if the limit exists. ■

Step 5: *Proof of (4) of Theorem A.* Let $(a_0, b_0) \in A^* \times B^*$ be such that

$$\lambda := \lim_{(a,b)\to(a_0,b_0),\,(a,b)\in A\times B} f(a,b)$$

exists. We will show that $\widehat{f}$ has angular limit $\lambda$ at $(a_0, b_0)$.

Recall that $|f|_X < 1$, and fix $0 < \varepsilon < 1/2$. Since $(a_0, b_0) \in A^* \times B^*$, we find an open neighborhood $A_{a_0}$ of $a_0$ in $A$ (resp. an open neighborhood $B_{b_0}$ of $b_0$ in $B$) such that

$$|f(a,b) - \lambda| < \varepsilon^2, \qquad a \in A_{a_0},\ b \in B_{b_0}. \tag{6.20}$$

It is clear that $\mathrm{mes}(A_{a_0}) > 0$ and $\mathrm{mes}(B_{b_0}) > 0$.

Consider the function

$$h(z,w) := f(z,w) - \lambda, \qquad (z,w) \in \mathbb{X}(A_{a_0}, B_{b_0}; D, G). \tag{6.21}$$

Clearly,

$$|h(z,w)| \le 2, \qquad (z,w) \in \mathbb{X}(A_{a_0}, B_{b_0}; D, G). \tag{6.22}$$

Applying the results of Steps 1–3 to $h$, we obtain the function

$$\widehat{h} := K[h] \quad \text{on } \widehat{\mathbb{X}}^{\mathrm{o}}(A_{a_0}, B_{b_0}; D, G). \tag{6.23}$$

so that $\widehat{h}$ has angular limit $h$ on $(\widetilde{A}_{a_0} \times G) \cup (D \times \widetilde{B}_{b_0})$, where $\widetilde{A}_{a_0}$, $\widetilde{B}_{b_0}$ are given by Step 2. Clearly,

$$\widehat{\mathbb{X}}(A_{a_0}, B_{b_0}; D, G) \subset \widehat{\mathbb{X}}(A, B; D, G).$$

Consequently, arguing as in Step 1 and taking into account the above mentioned angular limit of $\widehat{h}$, we conclude that

$$\widehat{h} = \widehat{f} - \lambda \quad \text{on } \widehat{\mathbb{X}}(A_{a_0}, B_{b_0}; D, G).$$

Consequently, applying Step 3 and taking into account (6.20)–(6.23) and the inequality $|f|_X < 1$, we see that

$$\begin{aligned} |\widehat{f}(z,w) - \lambda| &= |\widehat{h}(z,w)| \\ &\le |h|_{A_{a_0}\times B_{b_0}}^{1-\omega(z,A_{a_0},D)-\omega(w,B_{b_0},G)} (2|f|_X)^{\omega(z,A_{a_0},D)+\omega(w,B_{b_0},G)} \\ &< \varepsilon^{2(1-\omega(z,A_{a_0},D)-\omega(w,B_{b_0},G))} 2^{\omega(z,A_{a_0},D)+\omega(w,B_{b_0},G)}. \end{aligned}$$

Therefore, for all $(z,w) \in \widehat{\mathbb{X}}(A_{a_0}, B_{b_0}; D, G)$ satisfying

$$\omega(z, A_{a_0}, D) + \omega(w, B_{b_0}, G) < 1/3, \tag{6.24}$$

we deduce from the last estimate that

$$|\widehat{f}(z,w) - \lambda| < \varepsilon. \tag{6.25}$$

Since $a_0$ (resp. $b_0$) is locally regular relative to $A_{a_0}$ (resp. $B_{b_0}$), there is an $r_\alpha > 0$ such that (6.24) holds for

$$(z,w) \in (\mathcal{A}_\alpha(a_0) \cap \{|z - a_0| < r_\alpha\}) \times (\mathcal{A}_\alpha(b_0) \cap \{|w - b_0| < r_\alpha\}).$$

This, combined with (6.25), completes the proof. ■

Step 6: *Proof of (5) of Theorem A.* By Step 5, we only need to show that $\widehat{f}$ has angular limit $f$ on $(A^* \times G) \cup (D \times B^*)$. To do this let $(a_0, w_0) \in A^* \times G$ and choose $0 < \varepsilon < 1$. Fix a compact subset $K$ of $B \cap B^*$ such that $\mathrm{mes}(K) > 0$ and a sufficiently large $N$ such that

$$\varepsilon^{N(1-\omega(w_0,K,G))}(2|f|_X)^{\omega(w_0,K,G)} < \varepsilon/2. \tag{6.26}$$

Using the hypothesis that $f$ can be extended to a continuous function on $A^* \times B^*$, we find an open neighborhood $A_{a_0}$ of $a_0$ in $A^*$ such that

$$|f(a,b) - f(a_0,b)| \le \varepsilon^N, \quad a \in A_{a_0} \cap A^*_{a_0},\ b \in K. \tag{6.27}$$

On the other hand,

$$|f(a,w) - f(a_0,w_0)| \le 2|f|_X < 2, \quad a \in A_{a_0} \cap A^*_{a_0},\ w \in G. \tag{6.28}$$

For $a \in A_{a_0} \cap A^*_{a_0}$, applying the two-constant theorem to the function $f(a,\cdot) - f(a_0,\cdot) \in \mathcal{O}(G)$ and taking (6.26)–(6.28) into account, we deduce that

$$|f(a,w_0) - f(a_0,w_0)| \le \varepsilon^{N(1-\omega(w_0,K,G))}(2|f|_X)^{\omega(w_0,K,G)} < \varepsilon/2. \tag{6.29}$$

Since $f(a,\cdot)|_G$ is a bounded holomorphic function for $a \in A$, there is an open neighborhood $V$ of $w_0$ such that

$$|f(a,w) - f(a,w_0)| < \varepsilon/2, \quad a \in A,\ w \in V.$$

This, combined with (6.29), implies that

$$\begin{aligned}|f(a,w) - f(a_0,w_0)| &\le |f(a,w_0) - f(a_0,w_0)| + |f(a,w) - f(a,w_0)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad a \in A_{a_0},\ w \in V.\end{aligned}$$

Therefore, $f$ is continuous at $(a_0,w_0)$. Consequently, by Step 4, $\widehat{f}$ has angular limit $f(a_0,w_0)$ at $(a_0,w_0)$. Similarly, we can also show that $\widehat{f}$ has angular limit $f(z_0,b_0)$ at every point $(z_0,b_0) \in D \times B^*$. ■

**7. Preparatory results.** We first develop some auxiliary results. This will enable us to extend the results of Section 6 to the general case of Theorem A.

DEFINITION 7.1. Let $\Omega$ be a complex manifold of dimension 1 and $A \subset \Omega$. Define

$$\omega(\cdot, A, \Omega) := \sup\{u : u \in \mathcal{SH}(\Omega),\ u \le 1 \text{ on } \Omega,\ u \le 0 \text{ on } A\}.$$

The function $\omega(\cdot, A, \Omega)$ is called the *harmonic measure of $A$ relative to $\Omega$*. A point $\zeta \in \overline{A} \cap \Omega$ is said to be a *locally regular point relative to $A$* if

$$\lim_{z\to\zeta} \omega(z, A \cap U, \Omega \cap U) = 0$$

for any open neighborhood $U$ of $\zeta$. If, moreover, $\zeta \in A$, then $\zeta$ is said to be a *locally regular point of $A$*. The set of all locally regular points relative to $A$ is denoted by $A^*$. $A$ is said to be *locally regular* if $A = A^*$.

PROPOSITION 7.2. *Let $X$ be a complex manifold of dimension* 1, $D \subset X$ *an open set and $A \subset \partial D$ a Jordan measurable subset of positive length. Let $\{a_j\}_{j\in J}$ be a finite or countable subset of $A$ with the following properties*:

- (i) *for any $j \in J$, there is an open neighborhood $U_j$ of $a_j$ such that $D\cap U_j$ is either a Jordan domain or the disjoint union of two Jordan domains*;
- (ii) $A \subset \bigcup_{j\in J} U_j$.

*For any* $0 < \delta < 1/2$, *define*

$$\begin{aligned} U_{j,\delta} &:= \{z \in D \cap U_j : \omega(z, A^* \cap U_j, D \cap U_j) < \delta\}, \qquad j \in J,\\ A_\delta &:= \bigcup_{j\in J} U_{j,\delta},\\ D_\delta &:= \{z \in D : \omega(z, A^*, D) < 1 - \delta\}. \end{aligned}$$

*Then*:

(1) $A \cap A^* \subset A^D_\delta$ *and* $A_\delta \subset D_{1-\delta} \subset D_\delta$;
(2) $\omega(z, A^*, D) - \delta \le \omega(z, A_\delta, D) \le \omega(z, A^*, D)$, $z \in D$.

*Proof.* To prove (1), let $a \in A\cap A^*$ and fix $j \in J$ such that $a \in U_j$. Then

$$\lim_{z\to a,\, z\in\mathcal{A}_\alpha(a)} \omega(z, A^* \cap U_j, D \cap U_j) = 0, \qquad 0 < \alpha < \pi/2.$$

Consequently, for every $0 < \alpha < \pi/2$, there is an open neighborhood $V_\alpha \subset U_j$ of $a$ such that

$$\omega(z, A^* \cap U_j, D \cap U_j) < \delta, \qquad z \in \mathcal{A}_\alpha(a) \cap V_\alpha.$$

This proves $A \cap A^* \subset A^D_\delta$.

To prove the second assertion of (1), one applies the subordination principle to obtain, for $z \in U_{j,\delta}$,

$$\omega(z, A^*, D) \le \omega(z, A^* \cap U_j, D \cap U_j) < \delta < 1 - \delta. \tag{7.1}$$

Hence, $z \in D_{1-\delta}$. This implies that $A_\delta \subset D_{1-\delta}$. In addition, since $0 < \delta < 1/2$, it follows that $D_{1-\delta} \subset D_\delta$. Hence, (1) is proved.

We turn to (2). Since $A_\delta$ is an open set and, by (1), $A \cap A^* \subset A_\delta^D$, it follows from Definitions 4.8 and 7.1 that

$$\omega(z, A_\delta, D) \le \omega(z, A \cap A^*, D), \qquad z \in D.$$

Hence, Theorem 4.6 shows that

$$\omega(z, A_\delta, D) \le \omega(z, A^*, D), \qquad z \in D,$$

which proves the second estimate of (2).

To complete the proof of (2), let $z \in A_\delta$. Choose $j \in J$ such that $z \in U_{j,\delta}$. We deduce from (7.1) that $\omega(z, A^*, D) - \delta \le 0$. Hence,

$$\omega(z, A^*, D) - \delta \le 0, \qquad z \in A_\delta.$$

On the other hand, $\omega(z, A^*, D) - \delta < 1$ for all $z \in D$. Consequently, the first estimate of (2) follows. ■

The main ingredient in the proof of Theorem A is the following mixed cross theorem.

Theorem 7.3. *Let $X$ and $Y$ be complex manifolds of dimension* 1, *$D \subset X$ and $\Omega \subset Y$ open subsets, and $A \subset D$ and $B \subset \partial\Omega$. Assume that $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k$ locally regular compact subsets of $D$, $A_k \subset A_{k+1}$, $k \ge 1$. In addition, assume that $B \subset \partial\Omega$ is a Jordan measurable subset of positive length. For $0 \le \delta < 1$ put $G := \{w \in \Omega : \omega(w, B, \Omega) < 1 - \delta\}$. Let $W := \mathbb{X}(A, B; D, G)$, $W^{\mathrm{o}} := \mathbb{X}^{\mathrm{o}}(A, B; D, G)$, and* (*using the notation $\omega_\delta(\cdot, B, \Omega)$ of Theorem* 4.10)

$$\widehat{W}^{\mathrm{o}} = \widehat{\mathbb{X}}^{\mathrm{o}}(A, B; D, G) := \{(z, w) \in D \times G : \omega(z, A^*, D) + \omega_\delta(w, B, \Omega) < 1\}.$$

*Let $f : W \to \mathbb{C}$ be such that*:

(i) $f \in \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$;
(ii) *$f$ is Jordan measurable and locally bounded on $W$*;
(iii) *for any $z \in A$,*

$$\lim_{w \to \eta,\, w \in \mathcal{A}_\alpha(\eta)} f(z, w) = f(z, \eta), \qquad \eta \in B,\ 0 < \alpha < \pi/2.$$

*Then there is a unique function $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ such that $\widehat{f} = f$ on $A \times G$ and*

$$\lim_{z \to z_0,\, w \to \eta_0,\, w \in \mathcal{A}_\alpha(\eta_0)} \widehat{f}(z, w) = f(z_0, \eta_0), \qquad 0 < \alpha < \pi/2,$$

*for every $z_0 \in D$ and $\eta_0 \in B \cap B^*$. Moreover, $|\widehat{f}|_{\widehat{W}^{\mathrm{o}}} \le |f|_W$.*

*Proof.* First we prove the existence and uniqueness of $\widehat{f}$. Fix $f : W \to \mathbb{C}$ which satisfies (i)–(iii) above.

Step I: *Reduction to the case where $D \Subset X$ is an open hyperconvex set* ([3]), *$A$ is a locally regular compact subset of $D$ and $|f|_W < \infty$.* Since $X$ is countable at infinity, we find an exhaustion sequence $(D_k)_{k=1}^{\infty}$ of relatively compact, hyperconvex open subsets $D_k$ of $D$ with $A_k \subset D_k \nearrow D$ (for example, we can choose open subsets $D_k$ of $D$ with smooth boundary which contain $A_k$). Similarly, since $Y$ is countable at infinity, we find a sequence $(\Omega_k)_{k=1}^{\infty}$ of open subsets of $\Omega$ and a sequence $(B_k)_{k=1}^{\infty}$ of Jordan measurable subsets of $B$ which satisfy the hypothesis of Proposition 4.7. Let $G_k := \{w \in \Omega_k : \omega(w, B_k, \Omega_k) < 1 - \delta\}$. Using a compactness argument, we see that $|f|_{\mathbb{X}(A_k,B_k;D_k,G_k)} < \infty$.

By assumption, for each $k$ there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{\mathbb{X}}^{\mathrm{o}}(A_k, B_k; D_k, G_k))$ such that $\widehat{f}_k$ has angular limit $f|_{\mathbb{X}(A_k,B_k\cap B_k^*;D_k,G_k)}$ on $\mathbb{X}(A_k, B_k \cap B_k^*; D_k, G_k)$.

We claim that $\widehat{f}_{k+1} = \widehat{f}_k$ on $\widehat{\mathbb{X}}^{\mathrm{o}}(A_k, B_k; D_k, G_k)$. Indeed, fix $k_0 \geq 1$ and $(z_0, w_0) \in \widehat{\mathbb{X}}^{\mathrm{o}}(A_{k_0}, B_{k_0}; D_{k_0}, G_{k_0})$. Let $k \in \mathbb{N}$ be such that $k \geq k_0$. Let $\mathcal{D}$ be the connected component containing $z_0$ of the open set

$$\{z \in D : \omega(z, A_{k_0}, D_{k_0}) < 1 - \omega_\delta(w_0, B_k, \Omega_k)\}.$$

Observe that both $\widehat{f}_{k_0}(\cdot, w_0)|_{\mathcal{D}}$ and $\widehat{f}_k(\cdot, w_0)|_{\mathcal{D}}$ are holomorphic and

$$\widehat{f}_k(z, w_0) = f_k(z, w_0) = \widehat{f}_{k_0}(z, w_0), \qquad z \in A_k \cap \mathcal{D}.$$

Since $A_k \cap \mathcal{D}$ is nonpolar, we deduce that $\widehat{f}_{k_0}(\cdot, w_0)|_{\mathcal{D}} = \widehat{f}_k(\cdot, w_0)|_{\mathcal{D}}$. Hence, $\widehat{f}_{k_0}(z_0, w_0) = \widehat{f}_k(z_0, w_0)$, which proves the above assertion.

On the other hand, by Proposition 4.7 one gets $\widehat{\mathbb{X}}^{\mathrm{o}}(A_k, B_k; D_k, G_k) \nearrow \widehat{W}^{\mathrm{o}}$ as $k \nearrow \infty$. Therefore, we can glue the $\widehat{f}_k$ together to obtain $\widehat{f} \in \mathcal{O}(\widehat{X}^{\mathrm{o}})$ with angular limit $f$ on $W$ and $\widehat{f} = f$ on $A \times G$. The uniqueness of $\widehat{f}$ can be proved as in the previous paragraph.

Step II: *The case where $D \Subset X$ is an open hyperconvex set, $A$ is a locally regular compact subset of $D$, and $|f|_W < \infty$.* Suppose without loss of generality that $|f|_W < 1$. We will apply Théorème 3.3 of [15] to the pair of condensers $(A, D)$. In the following, we use the notation from that work.

Let $\mu_0 := \mu_{A,D}$ and $\mu_1$ a $B$-admissible Lebesgue measure of $D$. Let $H_1 := L^2_{\mathrm{h}}(D, \mu_1)$, $H_0 :=$ the closure of $H_1|_A$ in $L^2(A, \mu_0)$, and let $(b_j)_{j=1}^{\infty} \subset H_1$ be a system of doubly orthogonal bases in $H_1$ and $H_0$. Recall that $\|b_j\|_{H_0} = 1$.

([3]) An open set $D \subset X$ is said to be *hyperconvex* if it admits an exhaustion function which is bounded subharmonic.

Putting $\gamma_j := \|b_j\|_{H_1}$, $j \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} \gamma_j^{-\varepsilon} < \infty, \qquad \varepsilon > 0. \tag{7.2}$$

For any $w \in B$, we have $f(\cdot, w) \in H_1$ and $f(\cdot, w)|_A \in H_0$. Hence

$$f(\cdot, w) = \sum_{j=1}^{\infty} c_j(w) b_j, \tag{7.3}$$

where

$$c_j(w) = \frac{1}{\gamma_j^2} \int_D f(z, w)\, \overline{b_j(z)}\, d\mu_1(z) = \int_A f(z, w)\, \overline{b_j(z)}\, d\mu_0(z), \qquad j \in \mathbb{N}. \tag{7.4}$$

Taking the hypotheses (i)–(iii) into account and applying Lebesgue's dominated convergence theorem, we see that the formula

$$\widehat{c}_j(w) := \int_A f(z, w)\, \overline{b_j(z)}\, d\mu_0(z), \qquad w \in G \cup B,\ j \in \mathbb{N}; \tag{7.5}$$

defines a bounded function which is holomorphic in $G$. Moreover, by (iii) and (7.4)–(7.5) it follows that

$$\lim_{w \to \eta,\, w \in \mathcal{A}_\alpha(\eta)} \widehat{c}_j(w) = \widehat{c}_j(\eta) = c_j(\eta), \qquad \eta \in B,\ 0 < \alpha < \pi/2. \tag{7.6}$$

Using (7.4)–(7.6), we obtain the estimates

$$\frac{\log |\widehat{c}_j(w)|}{\log \gamma_j} \le \frac{\log \sqrt{\mu_0(A)}}{\log \gamma_j}, \qquad w \in G,\ j \in \mathbb{N},$$

$$\limsup_{w \to \eta,\, w \in \mathcal{A}_\alpha(\eta)} \frac{\log |\widehat{c}_j(\eta)|}{\log \gamma_j} \le \frac{\log \sqrt{\mu_1(D)}}{\log \gamma_j} - 1, \qquad \eta \in B,\ 0 < \alpha < \pi/2,\ j \in \mathbb{N}.$$

This shows that for any $\varepsilon > 0$, there is an $N$ such that for all $j \ge N$,

$$\frac{\log |\widehat{c}_j|}{\log \gamma_j} \le \omega_\delta(\cdot, B, \Omega) + \varepsilon - 1 \quad \text{on } G. \tag{7.7}$$

Take a compact set $K \Subset D$ and let $1 > \alpha = \alpha(K) > \max_K \omega(\cdot, A, D)$. Choose an $\varepsilon = \varepsilon(K) > 0$ so small that $\alpha + 2\varepsilon < 1$. Consider the open set

$$G_K := \{w \in G : \omega_\delta(\cdot, B, \Omega) < 1 - \alpha - 2\varepsilon\}.$$

By (7.7) there is a constant $C'(K)$ such that

$$|\widehat{c}_j|_{G_j} \le C'(K) \gamma_j^{\omega_\delta(\cdot, B, \Omega) + \varepsilon - 1} \le C'(K) \gamma_j^{-\alpha - \varepsilon}, \qquad j \ge 1. \tag{7.8}$$

Now we show that

$$\sum_{j=1}^{\infty} \widehat{c}_j(w) b_j(z) \tag{7.9}$$

converges locally uniformly in $\widehat{W}^{\mathrm{o}}$. Indeed, by (7.2) and (7.8),

$$\sum_{j=1}^{\infty} |\widehat{c}_j|_{G_K} |b_j|_K \le \sum_{j=1}^{\infty} C'(K)\gamma_j^{-\alpha-\varepsilon} C(K,\alpha)\gamma_j^{\alpha} \le C'(K)C(K,\alpha)\sum_{j=1}^{\infty}\gamma_j^{-\varepsilon} < \infty, \tag{7.10}$$

which gives the normal convergence on $K \times G_K$. Since $K$ and $\varepsilon > 0$ are arbitrary, the series in (7.9) converges uniformly on compact subsets of $\widehat{W}^{\mathrm{o}}$; call the limit function $\widehat{f}$.

Fix $z_0 \in D$ and $\eta_0 \in B \cap B^*$. We choose a compact $K_0 \Subset D$ so that $K_0$ is a neighborhood of $z_0$. Let $\varepsilon_0 > 0$.

By (7.10), there is an $N_0$ such that

$$\sum_{j=N_0+1}^{\infty} |\widehat{c}_j|_{G_{K_0}} |b_j|_{K_0} < \varepsilon_0/2. \tag{7.11}$$

On the other hand, by (7.3)–(7.6), we can find, for any $0 < \alpha < \pi/2$, an open neighborhood $V_\alpha$ of $\eta_0$ such that

$$\Big| \sum_{j=1}^{N_0} \widehat{c}_j(w) b_j(z) - \sum_{j=1}^{N_0} c_j(\eta_0) b_j(z) \Big| < \varepsilon_0/2, \qquad z \in K_0,\ w \in \mathcal{A}_\alpha(\eta_0) \cap V_\alpha.$$

This, combined with (7.9) and (7.11), implies that

$$\limsup_{z\to z_0,\, w\to\eta_0,\, w\in\mathcal{A}_\alpha(\eta_0)} |\widehat{f}(z,w) - f(z_0,\eta_0)| < \varepsilon_0, \qquad 0 < \alpha < \pi/2.$$

We conclude that

$$\lim_{z\to z_0,\, w\to\eta_0,\, w\in\mathcal{A}_\alpha(\eta_0)} \widehat{f}(z,w) = f(z_0,\eta_0),\ (z_0,\eta_0) \in D \times (B \cap B^*),\ 0 < \alpha < \pi/2.$$

To complete the proof of Step II, it remains to show that $\widehat{f} = f$ on $A \times G$. To do this, fix $(z_0, w_0) \in A \times G$. Let $\mathcal{G}$ be the connected component of $G$ containing $w_0$. Recall that $G = \{w \in \Omega : \omega(w, B, \Omega) < 1 - \delta\}$. Observe that both $\widehat{f}(z_0,\cdot)|_{\mathcal{G}}$ and $f(z_0,\cdot)|_{\mathcal{G}}$ have the same angular limit $f$ on $B \cap \mathcal{G}^{\Omega}$. Consequently, Theorem 5.4 shows that $\widehat{f}(z_0,\cdot)|_{\mathcal{G}} = f(z_0,\cdot)|_{\mathcal{G}}$. Hence, $\widehat{f}(z_0, w_0) = f(z_0, w_0)$, which proves the above assertion.

It remains to prove the estimate $|\widehat{f}|_{\widehat{W}^{\mathrm{o}}} \le |f|_W$. Assume for contradiction that $|\widehat{f}(z^0)| > |f|_W$ for some $z^0 \in \widehat{W}^{\mathrm{o}}$. Put $\alpha := \widehat{f}(z^0)$ and consider the function

$$g(z) := \frac{1}{f(z) - \alpha}, \qquad z \in W. \tag{7.12}$$

It can be checked that $g$ satisfies hypotheses (i)–(iii) of Theorem 7.3. Hence the first assertion of the theorem shows that there is exactly one $\widehat{g} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ with $\widehat{g} = g$ on $A \times G$. Therefore, by (7.12) we have $g(f - \alpha) \equiv 1$ on $A \times G$. Thus $\widehat{g}(\widehat{f} - \alpha) \equiv 1$ on $\widehat{W}^{\mathrm{o}}$. In particular,

$$0 = \widehat{g}(z^0)(\widehat{f}(z^0) - \alpha) = 1;$$

a contradiction. ∎

We conclude this section with two uniqueness results.

PROPOSITION 7.4. *Let $X$, $Y$ be two complex manifolds of dimension* 1, *$D \subset X$, $G \subset Y$ two open sets and $A \subset \partial D$, $B \subset \partial G$ two Jordan measurable subsets of positive length. Let $\widetilde{D} \subset X$ be an open set with $D \cap \widetilde{D} \neq \emptyset$, and let $\widetilde{A} \subset \partial\widetilde{D}$ be a Jordan measurable subset of positive measure. Put*

$$\widehat{W}^{\mathrm{o}} := \widehat{\mathbb{X}}^{\mathrm{o}}(A, B; D, G), \qquad \widehat{\widetilde{W}}^{\mathrm{o}} := \widehat{\mathbb{X}}^{\mathrm{o}}(\widetilde{A}, B; \widetilde{D}, G).$$

*Let $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$, $\widehat{\widetilde{f}} \in \mathcal{O}(\widehat{\widetilde{W}}^{\mathrm{o}})$, and $z_0 \in D \cap \widetilde{D}$ be such that both $\widehat{f}$ and $\widehat{\widetilde{f}}$ have the same angular limit at $(z_0, b)$ for a.e. $b \in B$. Then $\widehat{f}(z, w) = \widehat{\widetilde{f}}(z, w)$ for every $(z, w) \in \widehat{W}^{\mathrm{o}} \cap \widehat{\widetilde{W}}^{\mathrm{o}}$.*

*Proof.* Fix $w_0 \in G$ such that $(z_0, w_0) \in \widehat{W}^{\mathrm{o}} \cap \widehat{\widetilde{W}}^{\mathrm{o}}$. Choose $0 < \varepsilon < 1$ so that

$$(z_0, w_0) \in D_{1-\varepsilon} \times G_\varepsilon \cap \widetilde{D}_{1-\varepsilon} \times G_\varepsilon,$$

where we have used the notation of level sets introduced in Section 4. Applying Theorem 5.4 to $\widehat{f}(z_0, \cdot)|_{G_\varepsilon}$ and $\widehat{\widetilde{f}}(z_0, \cdot)|_{G_\varepsilon}$ shows that $\widehat{\widetilde{f}}(z_0, w_0) = \widehat{f}(z_0, w_0)$. ∎

Now we are able to prove the uniqueness stated in Theorem A.

COROLLARY 7.5. *Under the hypotheses and the notation of Theorem A, there is at most one $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$ which satisfies property* (1) *of Theorem A.*

*Proof.* This follows immediately from Proposition 7.4. ∎

**8. Proof of Theorem A.** Recall that by Corollary 7.5, the function $\widehat{f}$ satisfying (1) of Theorem A is uniquely determined (if it exists). We only give the proof of (1). We then conclude the proof of (2)–(5) of Theorem A in exactly the same way as we did in Section 6 starting from Step 2 of that section. The proof is divided into two steps.

STEP 1: *Proof of Theorem A for the case where $G$ is a Jordan domain.* By Proposition 7.2, let $\{a_j\}_{j\in J}$ be a finite or countable subset of $A$ with the following properties:

- for any $j \in J$, there is an open neighborhood $U_j$ of $a_j$ such that $D \cap U_j$ is either a Jordan domain or the disjoint union of two Jordan domains (according to the type of $a_j$);
- $A \subset \bigcup_{j\in J} U_j$.

For any $0 < \delta < 1/2$, define

$$\begin{aligned} U_{j,\delta} &:= \{z \in D \cap U_j : \omega(z, A^* \cap U_j, D \cap U_j) < \delta\}, \qquad j \in J, \\ A_\delta &:= \bigcup_{j\in J} U_{j,\delta}, \\ G_\delta &:= \{w \in G : \omega(w, B, G) < 1 - \delta\}. \end{aligned}$$

Moreover, for every $j \in J$ let

$$\begin{aligned} W_j &:= \mathbb{X}(\partial(D \cap U_j) \cap A, B; D \cap U_j, G), \\ \widehat{W}_j^{\mathrm{o}} &:= \widehat{\mathbb{X}}^{\mathrm{o}}(\partial(D \cap U_j) \cap A, B; D \cap U_j, G), \\ \widetilde{f}_j &:= f|_{W_j}. \end{aligned} \tag{8.1}$$

Using the hypotheses on $f$, we conclude that $\widetilde{f}_j$, $j \in J$, satisfies (i)–(iii) of Theorem A. Moreover, since $G$ is a Jordan domain and $D \cap U_j$, $j \in J$, is either a Jordan domain or the disjoint union of two Jordan domains, we can apply the result of Section 6 to $\widetilde{f}_j$. Consequently, for each $j \in J$, we obtain a unique function $\widehat{f}_j \in \mathcal{O}(\widehat{W}_j^{\mathrm{o}})$, a subset $A_j$ of $\partial(D \cap U) \cap A$, and a subset $B_j$ of $B$ such that

$$\begin{aligned} &A_j \subset A_j^*, \\ &(\partial(D \cap U) \cap A) \setminus A_j \text{ and } B \setminus B_j \text{ are of zero length}, \\ &\widehat{f}_j \text{ has angular limit } f \text{ on } ((\partial(D \cap U_j) \cap A_j) \times G) \cup (D \times B_j). \end{aligned} \tag{8.2}$$

Put

$$\begin{aligned} &\widetilde{A} := \bigcap_{j\in J} A_j, \qquad \widetilde{B} := \bigcap_{j\in J} B_j, \\ &W_\delta := \mathbb{X}(A_\delta, \widetilde{B}; D, G_\delta), \qquad \widehat{W}_\delta^{\mathrm{o}} := \widehat{\mathbb{X}}^{\mathrm{o}}(A_\delta, \widetilde{B}; D, G_\delta). \end{aligned} \tag{8.3}$$

By Proposition 7.4, the family $(\widehat{f}_j|_{U_{j,\delta}\times G_\delta})_{j\in J}$ yields a function $\widetilde{\widetilde{f}}_\delta \in \mathcal{O}(A_\delta \times G_\delta)$.

Next, consider the function $\widetilde{f}_\delta : W_\delta \to \mathbb{C}$ given by

$$\widetilde{f}_\delta := \begin{cases} \widetilde{\widetilde{f}}_\delta & \text{on } A_\delta \times G_\delta, \\ f & \text{on } D \times (\widetilde{B} \cap \widetilde{B}^*). \end{cases} \tag{8.4}$$

From (8.1)–(8.4), we deduce that

$$A \setminus \widetilde{A} \text{ and } B \setminus \widetilde{B} \text{ are of zero length}, \tag{8.5}$$

and

$$(8.6)\quad \begin{aligned} &\lim_{z\to z_0,\, w\to b_0,\, w\in\mathcal{A}_\alpha(b_0)} \widetilde{f_\delta}(z,w) = f(z_0,b_0),\\ &\qquad\qquad 0<\alpha<\pi/2,\ z_0\in D,\ b_0\in \widetilde{B}\cap\widetilde{B}^*,\\ &\lim_{z\to a_0,\, z\in\mathcal{A}_\alpha(a_0),\, w\to w_0} \widetilde{f_\delta}(z,w) = f(a_0,w_0),\\ &\qquad\qquad 0<\alpha<\pi/2,\ a_0\in \widetilde{A},\ w_0\in G_\delta. \end{aligned}$$

By (8.4)–(8.6), $\widetilde{f_\delta}$ satisfies hypotheses (i)–(iii) of Theorem 7.3. Applying this theorem to $\widetilde{f_\delta}$, we obtain, for every $0<\delta<1/2$, a function $\widehat{f_\delta}\in\mathcal{O}(\widehat{W}^{\mathrm{o}}_\delta)$. By (8.6), we see that

$$(8.7)\quad \begin{aligned} &\widehat{f_\delta} = \widetilde{f_\delta} \quad \text{on } A_\delta\times G_\delta,\\ &\lim_{z\to z_0,\, w\to b_0,\, w\in\mathcal{A}_\alpha(b_0)} \widehat{f_\delta}(z,w) = f(z_0,b_0),\\ &\qquad\qquad 0<\alpha<\pi/2,\ z_0\in D,\ b_0\in \widetilde{B}\cap\widetilde{B}^*,\\ &\lim_{z\to a_0,\, z\in\mathcal{A}_\alpha(a_0),\, w\to b_0} \widehat{f_\delta}(z,w) = f(a_0,w_0),\\ &\qquad\qquad 0<\alpha<\pi/2,\ a_0\in \widetilde{A},\ w_0\in G_\delta. \end{aligned}$$

We are now in a position to define the desired extension $\widehat{f}$. Indeed, one glues $(\widehat{f_\delta})_{0<\delta<1/2}$ together to obtain $\widehat{f}$ in the following way:

$$(8.8)\qquad \widehat{f} := \lim_{\delta\to 0}\widehat{f_\delta} \quad \text{on } \widehat{W}^{\mathrm{o}} = \widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D,G).$$

One has to check that the limit (8.8) exists and has all the required properties. This will be an immediate consequence of the following

Lemma 8.1. *For any $(z,w)\in\widehat{W}^{\mathrm{o}}$ put*

$$(8.9)\qquad \delta_{(z,w)} := \frac{1-\omega(z,A^*,D)-\omega(w,B^*,G)}{2}.$$

*Then $\widehat{f}(z,w)=\widehat{f_\delta}(z,w)$ for all $0<\delta\le\delta_{(z,w)}$.*

*Proof.* Fix $(z_0,w_0)\in\widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D,G)$ and let $\delta_0:=\delta_{(z_0,w_0)}$. Let $0<\delta\le\delta_0$. Then $\omega(w_0,B^*,G)<1-\delta_0$ and

$$\begin{aligned} \omega(z_0,A_\delta,D)+\omega_{\delta_0}(w_0,B,G) &\le \omega(z_0,A^*,D)+\frac{\omega(w_0,B^*,G)}{1-\delta_0}\\ &\le \frac{\omega(z_0,A^*,D)+\omega(w_0,B^*,G)}{1-\delta_0}<1, \end{aligned}$$

where the last estimate follows from (8.9). Consequently,

$$(8.10)\qquad (z_0,w_0)\in\widehat{\mathbb{X}}^{\mathrm{o}}(A_\delta,B;D,G_{\delta_0}).$$

On the other hand, by Proposition 7.2(1), it is clear that

$$\widehat{\mathbb{X}}^{\mathrm{o}}(A_\delta, B; D, G_{\delta_0}) \subset \widehat{\mathbb{X}}^{\mathrm{o}}(A_\delta, B; D, G_\delta) \cap \widehat{\mathbb{X}}^{\mathrm{o}}(A_{\delta_0}, B; D, G_{\delta_0}). \tag{8.11}$$

Moreover, in view of (8.4) and (8.7), we have

$$\widehat{f}_\delta = \widehat{\widetilde{\widetilde{f}}}_\delta = \widehat{f}_{\delta_0} \quad \text{on } A_\delta \times G_{\delta_0}. \tag{8.12}$$

Next, let $\mathcal{D}$ be the connected component containing $z_0$ of the open set

$$\{z \in D : \omega(z, A_\delta, D) < 1 - \omega_{\delta_0}(w_0, B, G)\}.$$

By (8.10)–(8.11), both $\widehat{f}_\delta|_{\mathcal{D}}$ and $\widehat{f}_{\delta_0}|_{\mathcal{D}}$ are holomorphic and $\mathcal{D} \cap A_\delta$ is a nonempty open set. Therefore, (8.12) yields $\widehat{f}_\delta = \widehat{f}_{\delta_0}$ on $\mathcal{D}$. Hence, $\widehat{f}_\delta(z_0, w_0) = \widehat{f}_{\delta_0}(z_0, w_0)$. ■

We now complete the proof of (1) as follows. An immediate consequence of Lemma 8.1 is that $\widehat{f} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$. Next, applying Lemma 8.1, (8.4)–(8.9) and the fact that $\widehat{W}^{\mathrm{o}}_\delta \to \widehat{W}^{\mathrm{o}}$ as $\delta \searrow 0$ we conclude that $\widehat{f}$ satisfies the conclusion of (1). ■

STEP 2: *Proof of Theorem A for the general case.* We proceed using Step 1 in exactly the same way as we proved Step 1 using the result of Section 6. ■

We conclude this section with the following remark. Using the above proof, one can also derive Gonchar's theorem (Theorem 1) from Drużkowski's theorem (Theorem 3). Indeed, in Step 1 above, let $\{a_j\}_{j\in J}$ be a finite or countable subset of $A$ with the following properties:

- for any $j \in J$, there is an open neighborhood $U_j$ of $a_j$ such that $D \cap U_j$ is a Jordan domain and $A \cap U_j$ is an open arc;
- $A \subset \bigcup_{j\in J} U_j$.

Then we repeat Step 1 ($B$ is only one open arc) and Step 2 (the general case) above using Drużkowski's theorem, and Gonchar's theorem follows. ■

**9. Proof of Theorem B.** We will give the proof for the case when $D$ and $G$ are the unit disc $E$. The general case can be proved using the scheme of Sections 6 and 8. The proof is divided into two steps.

STEP 1: *Proof of Theorem B when $f(a,\cdot)|_G$ and $f(\cdot,b)|_D$ are bounded for every $a \in A$ and $b \in B$.* For any $N \in \mathbb{N}$ let

$$A_N := \{a \in A : |f(a,\cdot)|_G \le N\}, \qquad B_N := \{b \in B : |f(\cdot,b)|_D \le N\}. \tag{9.1}$$

Then

$$A_N \nearrow A \text{ and } B_N \nearrow B \quad \text{as } N \nearrow \infty. \tag{9.2}$$

Now we show that for every $N \in \mathbb{N}$,

$$\begin{aligned} &A_N \text{ is a closed subset of } A \text{ and } f|_{A_N\times G} \in \mathcal{C}(A_N\times G),\\ &B_N \text{ is a closed subset of } B \text{ and } f|_{D\times B_N} \in \mathcal{C}(D\times B_N). \end{aligned} \tag{9.3}$$

To do this fix $N \in \mathbb{N}$ and let $(a_n)_{n=1}^{\infty} \subset A_N$ with $\lim_{n\to\infty} a_n = a_0 \in A_N$. Then, by hypothesis (i),

$$\lim_{n\to\infty} f(a_n,t) = f(a_0,t), \qquad t \in B. \tag{9.4}$$

On the other hand, by the hypothesis of Step 1,

$$|f(a_n,\cdot)|_G \le N \quad \text{and} \quad |f(a_0,\cdot)|_G < \infty.$$

Now the Khinchin–Ostrowski theorem (see [4, Theorem 4, p. 397]) shows that the sequence $(f(a_n,\cdot)|_G)_{n=1}^{\infty} \subset \mathcal{O}(G)$ converges uniformly on compact subsets of $G$ to $f(a_0,\cdot)$. This completes the proof of (9.3).

On the other hand, by hypothesis (ii), the holomorphic function $f(a,\cdot)$ has angular limit $f(a,b)$ at $b \in B$. It follows that $f|_{A_N\times B_N}$ is measurable. Moreover, by (9.1), $|f|_{\mathbb{X}(A_N,B_N;D,G)} \le N$ for every $N \in \mathbb{N}$. In addition, in view of (9.2), there exists an $N_0$ such that $\operatorname{mes}(A_N) > 0$ and $\operatorname{mes}(B_N) > 0$ for $N \ge N_0$. Consequently, Theorem A applied to the restruction of $f$ to the cross $\mathbb{X}(A_N,B_N;D,G)$ for $N \ge N_0$ yields a function $\widehat{f}_N \in \mathcal{O}(\widehat{\mathbb{X}}^{\mathrm{o}}(A_N,B_N;D,G))$ and a subset $\widetilde{A}_N$ (resp. $\widetilde{B}_N$) of $A_N$ (resp. $B_N$), for $N \ge N_0$, such that

$$\begin{aligned} &\operatorname{mes}(A_N \setminus \widetilde{A}_N) = \operatorname{mes}(B_N \setminus \widetilde{B}_N) = 0,\\ &\widehat{f}_N \text{ has angular limit } f \text{ on } (\widetilde{A}_N \times G) \cup (D \times \widetilde{B}_N). \end{aligned} \tag{9.5}$$

Put

$$\widetilde{A} := \bigcup_{N=N_0}^{\infty} \widetilde{A}_N, \qquad \widetilde{B} := \bigcup_{N=N_0}^{\infty} \widetilde{B}_N. \tag{9.6}$$

Applying (9.2), (9.5), and Corollary 7.5, we obtain

$$\widehat{f}_N = \widehat{f}_{N+1} \quad \text{on } \widehat{\mathbb{X}}^{\mathrm{o}}(\widetilde{A}_N,\widetilde{B}_N;D,G),\ N \ge N_0. \tag{9.7}$$

Therefore, the $\widehat{f}_N$ glue together to yield the desired extension as

$$\widehat{f} = \lim_{N\to\infty} \widehat{f}_N \quad \text{on } \widehat{W}^{\mathrm{o}} := \widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D,G). \tag{9.8}$$

Moreover, by (9.5)–(9.8), we infer that

$$\begin{aligned} &\operatorname{mes}(A \setminus \widetilde{A}) = \operatorname{mes}(B \setminus \widetilde{B}) = 0,\\ &\widehat{f} \text{ has angular limit } f \text{ on } (\widetilde{A} \times G) \cup (D \times \widetilde{B}). \end{aligned} \tag{9.9}$$

Next, for every $N \ge N_0$, in view of (9.2)–(9.3) and (9.5), one can find a sequence $(F_{N,n})_{n=1}^{\infty}$ (resp. $(H_{N,n})_{n=1}^{\infty}$) of compact subsets of $\partial D$ (resp. $\partial G$)

such that

$$\begin{aligned}
&F_{N,n} \subset F_{N,n+1} \subset A, \quad H_{N,n} \subset H_{N,n+1} \subset B,\\
(9.10)\qquad &\mathrm{mes}(F_{N,n}) > 0, \quad \mathrm{mes}(H_{N,n}) > 0,\\
&\mathrm{mes}\Big(\widetilde{A}_N \setminus \bigcup_{n=1}^{\infty} F_{N,n}\Big) = 0, \quad \mathrm{mes}\Big(\widetilde{B}_N \setminus \bigcup_{n=1}^{\infty} H_{N,n}\Big) = 0.
\end{aligned}$$

Moreover, for any $k \in \mathbb{N}$, $k \geq 1$, and for any $m \in \mathbb{N}$, put

$$\begin{aligned}
&A_{Nnmk} := \{a \in A_N : |f(a,\zeta) - f(a,\eta)| \leq 1/2k^2,\\
&\qquad\qquad\qquad \forall \zeta, \eta \in H_{N,n} : |\zeta - \eta| < 1/m\},\\
(9.11)\qquad &B_{Nnmk} := \{b \in B_N : |f(\zeta,b) - f(\eta,b)| \leq 1/2k^2,\\
&\qquad\qquad\qquad \forall \zeta, \eta \in F_{N,n} : |\zeta - \eta| < 1/m\}.
\end{aligned}$$

Since, by hypothesis (i), $f \in \mathcal{C}_{\mathrm{s}}(A \times B)$, we deduce from (9.10) and (9.11) that $A_{Nnmk}$ (resp. $B_{Nnmk}$) is a closed subset of $A_N$ (resp. $B_N$) and

$$(9.12)\qquad A_{Nnmk} \nearrow A_N \text{ and } B_{Nnmk} \nearrow B_N \qquad \text{as } m \nearrow \infty,\ k \geq 1.$$

Consequently, there is an $m_0 := m_0(N,n,k)$ such that $\mathrm{mes}(A_{Nnmk} \cap F_{N,n}) > 0$ and $\mathrm{mes}(B_{Nnmk} \cap H_{N,n}) > 0$ for any $m > m_0$. Now we apply Theorem A to the restriction of $f$ to $\mathbb{X}(A_{Nnmk} \cap F_{N,n}, B_{Nnmk} \cap H_{N,n}; D, G)$. Using (9.7)–(9.9) and Corollary 7.5, we then obtain exactly the restriction of $\widehat{f}$ to $\widehat{\mathbb{X}}^{\mathrm{o}}(A_{Nnmk} \cap F_{N,n}, B_{Nnmk} \cap H_{N,n}; D, G)$. Let ([4])

$$\begin{aligned}
(9.13)\qquad &\widetilde{A}_{Nnmk} := (A_{Nnmk} \cap F_{M,n}) \cap (A_{Nnmk} \cap F_{N,n})^*,\\
&\widetilde{B}_{Nnmk} := (B_{Nnmk} \cap H_{M,n}) \cap (B_{Nnmk} \cap H_{N,n})^*,
\end{aligned}$$

Taking (9.11)–(9.13) into account and arguing as in Step 5 of Section 6, we can show that

$$\begin{aligned}
&\mathrm{mes}(\widetilde{A}_{Nnmk} \setminus F_{N,n}) = 0, \quad \mathrm{mes}(\widetilde{B}_{Nnmk} \setminus H_{N,n}) = 0,\\
(9.14)\qquad &\limsup_{(z,w)\to(a,b),\, z\in\mathcal{A}_\alpha(a),\, w\in\mathcal{A}_\alpha(b),\, (z,w)\in\widehat{X}^{\mathrm{o}}} |\widehat{f}(z,w) - f(a,b)| < 1/k,
\end{aligned}$$

for any $0 < \alpha < \pi/2$, and $(a,b) \in \widetilde{A}_{Nnmk} \times \widetilde{B}_{Nnmk}$. Now it suffices to put

$$\widetilde{A} := \bigcap_{k=1}^{\infty} \bigcup_{N=N_0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=m_0(N,n,k)}^{\infty} \widetilde{A}_{Nnmk}, \qquad \widetilde{B} := \bigcap_{k=1}^{\infty} \bigcup_{N=N_0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=m_0(N,n,k)}^{\infty} \widetilde{B}_{Nnmk}.$$

Combining this and (9.14), (9.12), (9.9) and (9.2), we can check that all the conclusions of Theorem B are satisfied. ■

([4]) Recall from Subsection 2.2 that for a boundary subset $T$, $T^*$ denotes the set of locally regular points relative to $T$.

STEP 2: *The general case.* We begin with the following

DEFINITION 9.1. For a closed subset $F$ of $\partial E$ and $n \in \mathbb{N}$ with $n > 1$, define the following open set:

$$\Delta = \Delta(F,n) := \bigcup_{\zeta\in F}\{z \in \mathcal{A}_{\pi/4}(\zeta) : |z| \ge 1 - 1/n\} \cup \mathbb{B}(0, 1 - 1/n).$$

The reader should compare this definition with Definition 5.1. Below we list some properties of such open sets.

PROPOSITION 9.2. *Let $F$ be a closed subset of $\partial E$.*

(1) *$\Delta(F,n)$ is a rectifiable Jordan domain and $F \subset \partial\Delta(F,n)$.*
(2) *$\Delta(F,n) \nearrow E$ as $n \nearrow \infty$.*
(3) *If $f : E \cup F \to \mathbb{C}$ is locally bounded, then $|f|_{\Delta(F,n)} < \infty$ for every $n > 1$.*
(4) *$\omega(z,F,E) = \lim_{n\to\infty} \omega(z,F,\Delta(F,n))$ for all $z \in E$.*

*Proof.* (1) is shown as in the proof of Proposition 5.2; (2) is an immediate consequence of Definition 9.1; (3) follows immediately from the compactness of $F$; and the proof of Proposition 4.7 with the obvious changes yields (4). ∎

Now we are in a position to complete Step 2. Indeed, first suppose that both $A$ and $B$ are closed, and consider the sequences $(D_n)_{n=2}^{\infty}$ and $(G_n)_{n=2}^{\infty}$ of rectifiable Jordan domains given by

$$D_n := \Delta(A,n), \qquad G_n := \Delta(B,n), \qquad n \ge 2.$$

For $n \ge 2$ let $f_n := f|_{\mathbb{X}(A,B;D_n,G_n)}$. By Proposition 9.2, we can apply the result of Step 1 to $f_n$ to obtain a function $\widehat{f}_n \in \widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D_n,G_n)$. We then glue the $\widehat{f}_n$ together to obtain the desired extension

$$\widehat{f} = \lim_{n\to\infty} \widehat{f}_n \quad \text{ on } \widehat{W}^{\mathrm{o}} = \widehat{\mathbb{X}}^{\mathrm{o}}(A,B;D,G).$$

Proposition 9.2 shows that $\widehat{f}$ satisfies all the assertions of Theorem B.

The case when $A$ and $B$ are only measurable is similar. It suffices to find a sequence $(A_m)_{m=1}^{\infty}$ of subsets of $A$ such that $A_m$ is compact and $\mathrm{mes}(A \setminus \bigcup_{m=1}^{\infty} A_m) = 0$, and a similar sequence $(B_m)_{m=1}^{\infty}$ for $B$. Then we apply the previous discussion to $f|_{\mathbb{X}(A_m,B_m;D,G)}$ to obtain $\widehat{f}_m \in \widehat{\mathbb{X}}^{\mathrm{o}}(A_m,B_m;D,G)$, and define the desired extension by $\widehat{f} := \lim_{m\to\infty} \widehat{f}_m$ on $\widehat{W}^{\mathrm{o}}$. ∎

**10. Examples and concluding remarks.** The following examples of Drużkowski [2] show the optimality of Theorems A and B. Let $D = G = E$, $A = B = \{t \in \partial E : \mathrm{Re}\, t > 0\}$, $W := \mathbb{X}(A,B;D,G)$, and $T := (D \cup A) \times (G \cup B)$.

EXAMPLE 1. Define $h : T \to \mathbb{C}$ by

$$h(z,w) := \begin{cases} \exp\left(-[\operatorname{Log}(1-z) + \operatorname{Log}(1-w)] \operatorname{Log}\dfrac{2+zw}{3}\right), & \\ & z \neq 1,\ w \neq 1, \\ 0, & z = 1 \text{ or } w = 1. \end{cases}$$

where Log is the principal branch of the logarithm. Put $f := h|_W$. As in [2] observe that $f$ is measurable, $f \in \mathcal{C}_{\mathrm{s}}(W) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$, $|f|_W < \infty$, but $f|_{A\times B}$ is not continuous at $(1,1)$. Since $h|_{\widehat{W}^{\mathrm{o}}} \in \mathcal{O}(\widehat{W}^{\mathrm{o}})$, using the uniqueness established in Theorem A, we conclude that the solution $\widehat{f}$ provided by Theorems A and B satisfies $\widehat{f} = h|_{\widehat{W}^{\mathrm{o}}}$. In addition, for $0 < \alpha < \pi/2$, the angular limit of $\widehat{f}$ at $(1,1)$ does not exist. Thus the condition in assertion (3) of Theorem A is necessary. Moreover, the sets $\widetilde{A}$, $\widetilde{B}$ given by Theorem B do depend on $f$.

EXAMPLE 2. Define $h : T \to \mathbb{C}$ by

$$h(z,w) := \begin{cases} \exp\left(-(z-\lambda)\operatorname{Log}^2 \dfrac{3+w}{1-w}\right), & w \neq 1, \\ 0, & w = 1 \end{cases}$$

where $(z,w) \in T$, $0 < \lambda \le \sqrt{2}/2$. Define $f := h|_W$. Then $\widehat{f} = h|_{\widehat{W}^{\mathrm{o}}}$. As in [2] observe that $f|_{A\times B}$ is continuous, $f \in \mathcal{C}_{\mathrm{s}}(W) \cap \mathcal{O}_{\mathrm{s}}(W^{\mathrm{o}})$, but $f$ is not locally bounded on $W$.

In addition, for $\pi/3 < \alpha < \pi/2$, consider the functions $z_{\alpha,\lambda}, w_\alpha : [0,1] \to \mathbb{C}$ given by

$$\begin{aligned} w_\alpha(t) &:= 1 + te^{i(\pi - 9\alpha/10)}, \\ z_{\alpha,\lambda}(t) &:= \lambda + \left(\operatorname{Re}\operatorname{Log}^2 \frac{3+w_\alpha(t)}{1-w_\alpha(t)}\right)^{-1} + i\lambda, \qquad t \in [0,1]. \end{aligned}$$

We can prove that there is a $t_{\alpha,\lambda} > 0$ and a neighborhood $U_{\alpha,\lambda}$ of $\lambda + i\lambda$ in $\mathbb{C}$ such that

$$(z_{\alpha,\lambda}(t), w_\alpha(t)) \in \begin{cases} ((\mathcal{A}_\alpha(\lambda + i\lambda) \cap U_{\alpha,\lambda}) \times \mathcal{A}_\alpha(1)) \cap \widehat{W}^{\mathrm{o}}, & \\ & 0 < t < t_{\alpha,\lambda},\ \lambda = \sqrt{2}/2, \\ (U_{\alpha,\lambda} \times \mathcal{A}_\alpha(1)) \cap \widehat{W}^{\mathrm{o}}, & 0 < t < t_{\alpha,\lambda},\ 0 < \lambda < \sqrt{2}/2. \end{cases}$$

In addition, it can be checked that

$$\lim_{t\to 0}(z_{\alpha,\lambda}(t), w_\alpha(t)) = (\lambda + i\lambda, 1), \qquad \lim_{t\to 0} |\widehat{f}(z_{\alpha,\lambda}(t), w_\alpha(t))| = \infty.$$

This shows that the assumption of the local boundedness of $f$ is necessary in Theorem A.

We conclude the article with some remarks and open questions.

1. It can be proved that $\widehat{W}^{\text{o}}$ provided by Theorem A is the maximal domain of holomorphic extension of $f$ (see [12]).

2. Does Theorem A still hold if we omit the assumption (ii) that $f|_{A\times B}$ is Jordan-measurable?

3. Does Theorem B still hold if we omit the assumption that $f|_{A\times B} \in \mathcal{C}_{\mathrm{s}}(A \times B)$?

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Fachbereich Mathematik
Carl von Ossietzky Universität Oldenburg
Postfach 2503
D-26111 Oldenburg, Germany
E-mail: pflug@mathematik.uni-oldenburg.de

Mathematics Section
The Abdus Salam International Centre
for Theoretical Physics
Strada Costiera, 11
34014 Trieste, Italy
E-mail: vnguyen0@ictp.trieste.it

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