ANNALES
POLONICI MATHEMATICI
LXXIX.2 (2002)

# Proper holomorphic self-mappings of the minimal ball

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**Abstract.** The purpose of this paper is to prove that proper holomorphic self-mappings of the minimal ball are biholomorphic. The proof uses the scaling technique applied at a singular point and relies on the fact that a proper holomorphic mapping $f : D \to \Omega$ with branch locus $V_f$ is factored by automorphisms if and only if $f_*(\pi_1(D \setminus f^{-1}(f(V_f)), x))$ is a normal subgroup of $\pi_1(\Omega \setminus f(V_f), b)$ for some $b \in \Omega \setminus f(V_f)$ and $x \in f^{-1}(b)$.

**1. Introduction.** Families of proper holomorphic mappings arise in the problem of determining which domains in $\mathbb{C}^n$ do not possess any proper holomorphic mappings which are not biholomorphic. In this paper our aim is to study this problem in the case of a special domain in $\mathbb{C}^n$, $n \geq 2$, with non-piecewise smooth boundary. This domain is the minimal ball. It is given by

$$B_\infty = \{z \in \mathbb{C}^n : N_\infty(z) < 1\},$$

where $N_\infty(z) = (|z|^2 + |z^2|)/2$ and $z^2 = \sum_{1 \leq j \leq n} z_j^2$.

The function $\sqrt{N_\infty}$ is a norm in $\mathbb{C}^n$ introduced by Hahn–Pflug [4] as the smallest norm in $\mathbb{C}^n$ that extends the Euclidean norm in $\mathbb{R}^n$ under certain restrictions. It has been studied in several recent works [7], [12], [9], [10], [11], [17]. The automorphism group of $B_\infty$ is $S^1.O(n,\mathbb{R})$ (see [7]). In addition $B_\infty$ is a non-Lu Qi-Keng domain for $n \geq 4$ and it is neither homogeneous nor Reinhardt. Its boundary is $B$-regular in the sense of Sibony [16] and Henkin–Iordan [6].

Our main result can be stated as follows:

Theorem 1. *Every proper holomorphic self-mapping of $B_\infty$ is biholomorphic.*

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2000 *Mathematics Subject Classification*: Primary 32H35.

*Key words and phrases*: proper holomorphic mappings, correspondences, branch locus, scaling, factorization, minimal ball.

The following example shows that this theorem cannot be extended to proper holomorphic self-correspondences as in the case of strongly pseudoconvex domains (see [1]).

Let $\mathbb{M} = \{z \in \mathbb{C}^{n+1} : |z| < \sqrt{2} \text{ and } z^2 = 0\}$. The group $S^1.O(n+1,\mathbb{R})$ is a subgroup of $\mathrm{Aut}(\mathbb{M})$. Consider the projection $\mathrm{pr} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ defined by $\mathrm{pr}(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_n)$. The restriction $F := \mathrm{pr}|\mathbb{M}$ is a proper holomorphic mapping with multiplicity 2 from $\mathbb{M}$ onto $B_\infty \subset \mathbb{C}^n$. Let $g \in S^1.O(n+1,\mathbb{R}) \setminus S^1.O(n,\mathbb{R})$ (the group $O(n,\mathbb{R})$ can be regarded as a subgroup of $O(n+1,\mathbb{R})$). Then $h = F \circ g \circ F^{-1}$ is an irreducible proper holomorphic self-correspondence of $B_\infty$. To prove that $h$ is a nontrivial correspondence, assume that $h$ is a mapping. Then $h$ is an automorphism of $B_\infty$ (i.e. $h \in S^1.O(n,\mathbb{R})$); otherwise the multiplicity of $h \circ F$ will be greater than the multiplicity of $F \circ g$. This implies that $g \in S^1.O(n,\mathbb{R})$ and so we get a contradiction.

**2. Preliminary results.** In this section, we give some preliminary results useful for the proof of our theorem.

**2.1.** *Factorization of proper holomorphic mappings.* A mapping $f : D \to \Omega$ is *factored by automorphisms* if there is a finite subgroup $\Gamma \subset \mathrm{Aut}(D)$ such that for all $z$ in $D$,

$$f^{-1}(f(z)) = \{\gamma(z) : \gamma \in \Gamma\}.$$

We will denote by $J_f(z)$ the Jacobian determinant of $f$ and by $V_f = \{z \in D : J_f(z) = 0\}$ its branch locus. A necessary and sufficient condition to factorize proper holomorphic mappings is given in the following theorem.

Theorem 2. *Let $D$ be a pseudoconvex bounded domain in $\mathbb{C}^n$, $\Omega$ a domain in $\mathbb{C}^n$ and $f : D \to \Omega$ a proper holomorphic mapping with branch locus $V_f$. Denote by $F$ the restriction of $f$ to $D \setminus f^{-1}(f(V_f))$. Then the following statement are equivalent*:

(1) *there exist $b \in \Omega \setminus f(V_f)$ and $x \in f^{-1}(b)$ such that $F_*(\pi_1(D \setminus f^{-1}(f(V_f)), x))$ is a normal subgroup of $\pi_1(\Omega \setminus f(V_f), b)$,*

(2) *$f$ is factored by automorphisms.*

The existence of the group $\Gamma$ is due to W. Rudin [15] in the case of the Euclidean ball in $\mathbb{C}^n$ and to Bedford–Bell [1] in the case of strongly pseudoconvex domains in $\mathbb{C}^n$ (see also the references for related results). Theorem 2 implies that the branch locus of $f$ is given by

$$V_f = \bigcup_{\{g \in \Gamma : g \neq \mathrm{id}\}} \{z \in D : g(z) = z\}.$$

Then the factorization theorem above may be used to reduce the study of the behavior of the branch locus to the study of the group $\Gamma$. Thus far $\Gamma$ has been identified only in the case of the Euclidean ball in $\mathbb{C}^n$.

*Proof of Theorem 2.* Let $H = f(V_f)$, $E = D \setminus f^{-1}(H)$ and $B = \Omega \setminus H$. The restriction $F = f|E : E \to B$ is a connected finite covering. According to [3], $F_*(\pi_1(E,x))$ is a normal subgroup in $\pi_1(B,b)$ if and only if $F : E \to B$ is a Galois covering, i.e. $F$ is a connected covering and the group $\Gamma = \{\gamma \in \mathrm{Hom}(E) : F \circ \gamma = F\}$ acts transitively on each fiber $F^{-1}(b)$, $b \in B$. Moreover, the group $\Gamma$ is isomorphic to $\pi_1(B,b)/F_*(\pi_1(E,x))$.

Assume that $F_*(\pi_1(E,x))$ is a normal subgroup in $\pi_1(B,b)$. Then $F$ is a Galois covering. The mapping $F$ is holomorphic, so all elements of $\Gamma$ are biholomorphic and since $D$ is bounded, they extend to holomorphic mappings from $D$ onto $\overline{D}$. Suppose that there exists a point $p \in f^{-1}(H)$ such that $\gamma(p) \in \partial D$ for some $\gamma \in \Gamma$. Let $\Delta$ be an analytic disc in $D$ such that $\Delta \cap f^{-1}(H) = \{p\}$. Then $\gamma(\Delta)$ intersects $\partial D$ only at $\gamma(p)$. This contradicts the fact that $D$ is pseudoconvex. Hence for all $\gamma \in \Gamma$, $\gamma(D) \subset D$. Thus the elements of $\Gamma$ extend to automorphisms of $D$ and define a subgroup of $\mathrm{Aut}(D)$ that we denote by $\widehat{\Gamma}$. By analytic extension the equality $f \circ \widehat{\gamma} = f$ remains valid for all $\widehat{\gamma} \in \widehat{\Gamma}$.

Let now $b \in H$ and $z_1, z_2 \in f^{-1}(H)$ be such that $f(z_1) = f(z_2) = b$. Since $f$ is an open map, there exist two sequences $\{z_1^j\}_j$ and $\{z_2^j\}_j$ in $E$ that converge respectively to $z_1$ and $z_2$ and satisfy $f(z_1^j) = f(z_2^j)$ for all $j$. The mapping $F$ is a Galois covering, so for all $j$ there exists $\gamma_j \in \Gamma$ such that $z_1^j = \gamma_j(z_2^j)$. As $\Gamma$ is a finite subgroup, we may assume that $z_1^j = \gamma(z_2^j)$ for some $\gamma \in \Gamma$ and for any integer $j$. Passing to the limit, we get $z_1 = \widehat{\gamma}(z_2)$ ($\widehat{\gamma}$ is the extension of $\gamma$). This proves that the mapping $f$ is factored by $\widehat{\Gamma}$.

Conversely, assume that $f$ is factored by a subgroup $\widehat{\Gamma}$. It is clear that for all $\widehat{\gamma} \in \widehat{\Gamma}$, $\widehat{\gamma}$ maps $E$ onto itself. Then the restriction $F$ is a Galois covering. This implies that $F_*(\pi_1(E,x))$ is a normal group in $\pi_1(B,b)$. ∎

**2.2.** *Hopf's lemma for $B_\infty$.* We denote by $\varrho = \sqrt{N_\infty} - 1$ a defining function of $B_\infty$. It is easy to see that this function satisfies the following lemma.

Lemma 1. *For $z \in B_\infty$ one has*

$$\frac{1}{\sqrt{2}}\,\mathrm{dist}(z, \partial B_\infty) \le |\varrho(z)| \le \mathrm{dist}(z, \partial B_\infty).$$

First, we establish the uniform Hopf lemma for the unit disc $\Delta$ in $\mathbb{C}$.

Lemma 2. *Let $r$ be a subharmonic negative function on $\Delta$. Then for all $z \in \Delta$ one has*

$$|r(z)| \ge \inf_{\overline{\Delta}(0,1/2)} |r|\,\mathrm{dist}(z, \partial\Delta).$$

*Proof.* We consider the subharmonic function

$$\widehat{r}(z) = r(z) - A\,\frac{\mathrm{Ln}\,|z|}{\mathrm{Ln}\,2}, \quad \text{where} \quad A = \inf_{\overline{\Delta}(0,1/2)} |r|.$$

Since $\widehat{r}$ is negative on $\partial\Delta(0,1/2)$ and $\overline{\lim}_{z\to z_0\in\partial\Delta}\,\widehat{r}(z)\le 0$, by the maximum principle we get $\widehat{r}(z)\le 0$ on $\Delta(0,1)\setminus\Delta(0,1/2)$. Thus

$$\forall z\in\Delta,\qquad |r(z)|\ge \inf_{\overline{\Delta}(0,1/2)}|r|\min\left(1,\frac{-\operatorname{Ln}|z|}{\operatorname{Ln}2}\right)\ge \inf_{\overline{\Delta}(0,1/2)}|r|(1-|z|).\ ■$$

As an application, we get the Hopf lemma for the minimal ball.

LEMMA 3. *Let $r$ be a plurisubharmonic negative function on $B_\infty$. Then for all $z\in B_\infty$ one has*

$$|r(z)|\ge \frac{1}{\sqrt{2}}\inf_{\frac12\overline{B}_\infty}|r|\operatorname{dist}(z,\partial B_\infty),$$

*where* $\frac12\overline{B}_\infty=\{z\in\mathbb{C}^n:\sqrt{N_\infty}(z)\le 1/2\}$.

*Proof.* For $z=0$ the inequality is true. Let now $z\in B_\infty\setminus\{0\}$ and consider the subharmonic function $u$ defined on the unit disc by

$$u(\xi)=r\left(\frac{\xi z}{\sqrt{N_\infty}(z)}\right).$$

In view of Lemma 2 we have

$$|u(\xi)|\ge \inf_{\overline{\Delta}(0,1/2)}|u|\operatorname{dist}(\xi,\partial\Delta).$$

Let $\xi=\sqrt{N_\infty}(z)$. The previous inequality becomes

$$|r(z)|\ge \inf_{\overline{\Delta}(0,1/2)}|u|\cdot|1-\sqrt{N_\infty}(z)|\ge\frac{1}{\sqrt{2}}\inf_{\overline{\Delta}(0,1/2)}|u|\operatorname{dist}(z,\partial B_\infty).$$

Since

$$\inf_{w\in\overline{\Delta}(0,1/2)}\left|r\left(\frac{wz}{\sqrt{N_\infty}(z)}\right)\right|\ge\inf_{\frac12\overline{B}_\infty}|r|,$$

we have the desired inequality. ■

In the case $n=2$, the minimal ball is biholomorphic to the Reinhardt domain $\{(z,w)\in\mathbb{C}^2:|z|+|w|<1\}$. Hence according to [2], any proper holomorphic self-mapping of $B_\infty$ is biholomorphic. Now assume that $n\ge 3$. The proof of our theorem is based on the scaling technique and the notion of factorization of proper holomorphic mappings.

**3. Scaling technique.** Set $H=\{z\in\mathbb{C}^n:\sum_{1\le j\le n}z_j^2=0\}$ and $H_\infty=H\cap B_\infty$. The singular part of the boundary of $B_\infty$ is obviously the set $\partial H_\infty=H\cap\partial B_\infty$. The regular part $\partial B_\infty\setminus\partial H_\infty$ is $C^\infty$-smooth and it consists of strongly pseudoconvex points. In [10] the authors gave an explicit formula for the Bergman kernel of the minimal ball. In particular, they showed that any proper holomorphic self-mapping of the minimal ball extends holomorphically to a neighborhood of $\overline{B}_\infty$ (see Theorem 5.5 of [10]). In this section we shall prove the following:

PROPOSITION 1. *Let $f$ be a branching proper holomorphic self-mapping of $B_\infty$ with branch locus $V_f$. Then $f(V_f)$ is equal to $H_\infty$.*

*Proof.* Let $p \in \partial V_f = \overline{V}_f \setminus V_f$ and let $\{p_k\}_k$ be a sequence in $V_f$ that converges to $p$. Since $f$ is proper, the sequence $\{f(p_k)\}_k$ converges to a boundary point $q \in \partial B_\infty$. We shall prove that $q$ is a singular point (i.e. $q \in \partial H_\infty$). The proof is by contradiction. Assume that $q$ is a strong pseudoconvexity point. We will discuss two cases.

*First case*: *$p$ is a strong pseudoconvexity point*. Since the mapping $f$ extends holomorphically to a neighborhood of $\overline{B}_\infty$ ([10]), it defines a local biholomorphism near $p$ (see [14]). This contradicts the fact that $p \in \partial V_f$.

*Second case*: *$p$ is a singular point* ($p \in \partial H_\infty$). In this case we will use the scaling technique to prove that this situation is not possible.

Since $S^1.O(n,\mathbb{R})$ acts transitively on $\partial H_\infty$, we can assume without loss of generality that $p = (0,\dots,0,i,1)$. The domain $B_\infty - p$ is represented by

$$2\operatorname{Re}(z_n - iz_{n-1}) + |z_1|^2 + \dots + |z_n|^2 + |z_1^2 + \dots + z_n^2 + 2(z_n + iz_{n-1})| < 0.$$

In the new coordinates

$$\begin{aligned}\zeta_j &= z_j, \qquad j \in \{1,\dots,n-2\},\\ \zeta_{n-1} &= z_n + iz_{n-1},\\ \zeta_n &= z_n - iz_{n-1},\end{aligned}$$

the point $p$ is transformed to 0 and the domain $B_\infty - p$ corresponds to the domain $G$ defined by $\{\varphi < 0\}$ with

$$\begin{aligned}\varphi(\xi) = 2\operatorname{Re}(\zeta_n) + \frac{1}{2}(|\zeta_n|^2 + |\zeta_{n-1}|^2) + |\zeta_1|^2 + \dots + |\zeta_{n-2}^2|\\ + |\zeta_1^2 + \dots + \zeta_{n-2}^2 + 2\zeta_{n-1} + \zeta_{n-1}\zeta_n|.\end{aligned}$$

We write $z \in \mathbb{C}^n$ as $z = ('z, z_n)$ where $'z$ denotes the first $n-1$ coordinates of $z$ or $z = (''z, z_{n-1}, z_n)$ where $''z$ denotes the first $n-2$ coordinates of $z$. If $\{a_k\}_k$ and $\{b_k\}_k$ are two sequences of real positive numbers, we write $a_k \simeq b_k$ if there is a positive constant $c$ independent of $k$ such that $c^{-1}b_k \le a_k \le cb_k$.

Let $g : B_\infty \to G$ be the linear transformation mapping $B_\infty$ onto $G$ and $t_k = ('0, -\delta_k)$, $k = 1,2,\dots$, be a sequence of points in $G$ where $\{\delta_k\}_k$ is a sequence of real positive numbers converging to 0. It is clear that for large $k$, $|\varphi(t_k)| \simeq \delta_k$. Since $B_\infty$ is linearly equivalent to $G$, in view of Lemma 1 we have, for all $z \in G$,

$$\frac{1}{\sqrt{2}\,\|g\|}\operatorname{dist}(z,\partial G) \le |\varphi(z)| \le \|g^{-1}\|\operatorname{dist}(z,\partial G).$$

It follows that for large $k$,

$$\text{(1)} \qquad \operatorname{dist}(t_k, \partial G) \simeq \delta_k.$$

As the mapping $f$ is continuous, the sequence $\{q_k\}_k$, $q_k = f \circ g^{-1}(t_k)$, converges to $q = f \circ g^{-1}(0)$. Let $V$ be a neighborhood of $q$ in $\mathbb{C}^n$ which does not intersect the set of weakly pseudoconvex points of $\partial B_\infty$. For all $w \in \partial B_\infty \cap V$ we consider the change of variables $h^w$ defined by

$$z_j^* = \frac{\partial \varrho}{\partial \overline{z}_n}(w)(z_j - w_j) - \frac{\partial \varrho}{\partial \overline{z}_j}(w)(z_n - w_n), \quad 1 \le j \le n-1,$$
$$z_n^* = \sum_{1 \le j \le n} \frac{\partial \varrho}{\partial z_j}(w)(z_j - w_j).$$

The mapping $h^w$ maps $w$ to 0 and the real normal to $\partial B_\infty$ at $w$ onto the line $\{'z = 0,\ y_n = 0\}$. Let $w_k$ be the projection of $q_k$ on the boundary of $B_\infty$. For simplicity we denote $h^{w_k}$ (the mapping as above) by $h^k$. Set $D^k = h^k(B_\infty)$, $\varrho^k = \varrho \circ {h^k}^{-1}$ and $\gamma_k = \operatorname{dist}(h^k(q_k), \partial D^k)$. We have $h^k(q_k) = ('0, -\gamma_k)$. The sequence of proper holomorphic mappings defined by $f^k = h^k \circ f \circ g^{-1} : G \to D^k$ satisfies $f^k('0, -\delta_k) = ('0, -\gamma_k)$ for all $k$. Now we introduce the inhomogeneous dilatation of coordinates as follows:

$$\alpha^k('z, z_n) = \left(\frac{'z}{\sqrt{\gamma_k}}, \frac{z_n}{\gamma_k}\right), \quad \beta^k(''z, z_{n-1}, z_n) = \left(\frac{''z}{\sqrt{\delta_k}}, \frac{z_{n-1}}{\delta_k}, \frac{z_n}{\delta_k}\right).$$

The idea is to follow the argument of Pinchuk [13] and to consider the mapping $\widehat{f^k} = \alpha^k \circ f^k \circ (\beta^k)^{-1}$. In the new coordinates, $G$ and $D^k$ correspond to the domains $\widehat{G}^k$ and $\widehat{D}^k$ with defining functions

$$\widehat{\varphi}^k(z) = \frac{1}{\delta_k}\, \varphi \circ {\beta^k}^{-1}(z), \quad \widehat{\varrho}^k(z) = \frac{1}{\gamma_k}\, \varrho^k \circ (\alpha^k)^{-1}(z)$$

respectively. Thus, $\widehat{f^k}$ is a proper holomorphic mapping from $\widehat{G}^k$ onto $\widehat{D}^k$ and satisfies $\widehat{f^k}('0, -1) = ('0, -1)$. Let $\widehat{G} = \{\widehat{\varphi} < 0\}$ and $\Sigma = \{\widehat{\varrho} < 0\}$ where $\widehat{\varphi}(z) = 2\operatorname{Re}(z_n) + |''z|^2 + |''z^2 + 2z_{n-1}|$, $\widehat{\varrho}(z) = 2\operatorname{Re}(z_n) + |'z|^2$ and $''z^2 = \sum_{1 \le j \le n-2} z_j^2$. The sequence $\{\widehat{\varphi}^k\}_k$ (resp. $\{\widehat{\varrho}^k\}_k$) converges uniformly to the function $\widehat{\varphi}$ on compact subsets of $\widehat{G}$ (resp. to the function $\widehat{\varrho}$ on compact subsets of $\Sigma$). Consequently, for all compact $K \subset \widehat{G}$, the mappings $\widehat{f^k}$ are well defined on $K$, starting from some $k^0 = k^0(K)$. By exhausting $\widehat{G}$ with an increasing sequence of compact sets and by passing to the limit, we conclude that we may assume that $\{\widehat{f^k}\}_k$ converges to a holomorphic function $\widehat{f} : \widehat{G} \to \overline{\Sigma}$. Since $\widehat{\varrho}$ is plurisubharmonic and $\widehat{f}('0, -1) = ('0, -1) \in \Sigma$, the maximum principle implies that $\widehat{f}(\widehat{G}) \subset \Sigma$.

We shall prove that $\widehat{f}$ is proper. For this we need some estimates on the distance.

LEMMA 4. *There exists a constant $c > 0$ such that for all $z \in B_\infty$,*

$$\frac{1}{c}\operatorname{dist}(z, \partial B_\infty) \le \operatorname{dist}(f(z), \partial B_\infty) \le c \operatorname{dist}(z, \partial B_\infty).$$

*Proof.* Recall that $\varrho$ denotes a defining function of $B_\infty$. Since $\varrho \circ f$ is plurisubharmonic and negative on $B_\infty$, in view of Lemmas 1 and 3 there exists a constant $c > 0$ such that for all $z \in B_\infty$, $\operatorname{dist}(f(z), \partial B_\infty) \ge c^{-1}\operatorname{dist}(z, \partial B_\infty)$. To prove the right-hand inequality, we consider the function $r(w) = \max\{\varrho(z) : z \in f^{-1}(w)\}$, which is well defined and plurisubharmonic on $B_\infty \setminus f(V_f)$ and also bounded there. Since $f$ is proper, $f(V_f)$ is an analytic subvariety, and so $r$ extends as a plurisubharmonic function on $B_\infty$. Now we apply Lemmas 1 and 3 again. ■

Since the coordinates $h^{w_k}$ depend continuously on $w_k$ and the domain $G$ is linearly equivalent to $B_\infty$, in view of Lemma 4 the following estimates hold:

$$c_1 \operatorname{dist}(z, \partial G) \le \operatorname{dist}(f^k(z), \partial D^k) \le c_2 \operatorname{dist}(z, \partial G), \tag{2}$$

with $c_1, c_2 > 0$ do not depend on $k$. In addition, in $G$ and $D^k$ we have the estimates

$$\begin{aligned} c_3|\varphi(z)| &\le \operatorname{dist}(z, \partial G) \le c_4|\varphi(z)|, \\ c_3|\varrho^k(w)| &\le \operatorname{dist}(w, \partial D^k) \le c_4|\varrho^k(w)|, \end{aligned} \tag{3}$$

where $c_3, c_4 > 0$ do not depend on $k$ (the estimates (3) follow from Lemmas 1 and 3). According to (1) and (2), there exist positive constants $c_5$ and $c_6$ independent of $k$ such that for all $k$,

$$c_5 < \gamma_k/\delta_k < c_6. \tag{4}$$

Let $K$ be a compact set in $\widehat{G}$ and $z \in K$. Set $w^k = \widehat{f}^k(z)$. In view of (2)–(4),

$$\begin{aligned} \widehat{\varrho}^k(w^k) &= {\gamma_k}^{-1}\varrho^k(\sqrt{\gamma_k}'w^k, \gamma_k w_n^k) \le (\gamma_k c_3)^{-1}\operatorname{dist}((\sqrt{\gamma_k}'w^k, \gamma_k w_n^k), \partial D^k) \\ &\le c_2(\gamma_k c_3)^{-1}\operatorname{dist}((\sqrt{\delta_k}''z, \delta_k z_{n-1}, \delta_k z_n), \partial G) \\ &\le c_2 c_4(\gamma_k c_3)^{-1}\varphi(\sqrt{\delta_k}''z, \delta_k z_{n-1}, \delta_k z_n) \\ &= c_2 c_4 \delta_k (c_3\gamma_k)^{-1}\widehat{\varphi}^k(z) \le c_2 c_4 (c_5 c_3)^{-1}\widehat{\varphi}^k(z). \end{aligned}$$

Passing to a convergent subsequence and to the limit, we get

$$\widehat{\varrho}(\widehat{f}(z)) \le c_7\widehat{\varphi}(z) \tag{5}$$

for $z \in K$ and for some positive constant $c_7$ independent of $z$. Since $K$ is an arbitrary compact set in $\widehat{G}$, the estimate (5) holds for all $z \in \widehat{G}$.

LEMMA 5. *The sequence $\{\widehat{f}^k\}_k$ admits a subsequence converging uniformly on compact subsets of $\widehat{G}$ to a proper holomorphic mapping $\widehat{f} : \widehat{G} \to \Sigma$.*

*Proof.* The proof is based on certain ideas of S. Pinchuk [13]. For the convenience of the reader and for the sake of completeness we include a proof. We consider the function $v(z) = e^{z_n}$. It satisfies $|v(z)| < 1$ on $\overline{G} \setminus \{0\}$ and $v(0) = 1$. For $\delta_k < 1$ the functions

$$v^k(z) = \frac{v(z) - (1 - \delta_k)}{1 - v(z)(1 - \delta_k)}$$

are holomorphic in a neighborhood of $\overline{G}$ and $|v^k(z)| < 1$ for all $z \in G$. For large $k$ we consider the function $u^k(z) = v^k(\sqrt{\delta_k}''z, \delta_k z_{n-1}, \delta_k z_n)$, which is holomorphic in a neighborhood of $\widehat{G}^k$ and has the form

$$u^k(z) = \frac{1 + z_n + \delta_k\left(\sum_{p \geq 2} \frac{\delta_k^{p-2}}{p!} z_n^p\right)}{1 - z_n(1 - \delta_k) - \delta_k\left(\sum_{p \geq 2} \frac{\delta_k^{p-2}}{p!} z_n^p\right)(1 - \delta_k)}.$$

Set $u^0 = \lim_{k \to \infty} u^k$. The limit $u^0$ is holomorphic on $\widehat{G}$ and it is defined by

$$u^0(z) = \frac{1 + z_n}{1 - z_n}.$$

As $|z| \to \infty$ and $z \in \widehat{G}$, clearly $|z_n| \to \infty$. Thus $u^0(z) \to -1$ as $|z| \to \infty$. Since the mappings $\widehat{f}^k : \widehat{G}^k \to \widehat{D}^k$ are ramified analytic coverings (see [5]), for each $k$ there exists a polynomial

$$P^k(t, w) = t^m + S_1^k(w)t^{m-1} + \ldots + S_m^k(w)$$

(where $S_\alpha^k$ are holomorphic functions on $\widehat{D}^k$ for $\alpha \in \{1, \ldots, m\}$ and $m$ is the multiplicity of $\widehat{f}^k$) such that $P^k(u^k(z), \widehat{f}^k(z)) \equiv 0$ on $\widehat{G}^k$ and for $w \in \widehat{D}^k$ one has $P^k(t, w) = 0$ if and only if $t \in u^k \circ (\widehat{f}^k)^{-1}(w)$. As $|u^k(z)| < 1$ in $\widehat{G}^k$, it follows that all the roots of $P^k(\cdot, w)$ are of modulus less than one for all $w \in \widehat{D}^k$. Then there exists a constant $c > 0$ such that $|S_\alpha^k(w)| < c$ in $\widehat{D}^k$. Consequently, we can assume (after passing to a subsequence) that for all $\alpha \in \{1, \ldots, m\}$ the sequence $\{S_\alpha^k\}_k$ converges to a function $S_\alpha$, defined and holomorphic in $\Sigma$. Let $P(t, w) = t^m + S_1(w)t^{m-1} + \ldots + S_m(w)$. We have $P(u^o(z), \widehat{f}(z)) \equiv 0$ and the roots of $P$ lie in the closed unit disc for all $w \in \Sigma$. We write $P(t, w)$ in the form $(t + 1)^l Q(t, w)$, where $Q(-1, w) \not\equiv 0$.

Assume that $\widehat{f}$ is not proper. In view of (5) there exists a sequence $\{z_\mu\}_\mu$ of $\widehat{G}$ such that $z_\mu \to \infty$ and $\widehat{f}(z_\mu) \to w^0 \in \Sigma$ as $\mu \to \infty$. By the Weierstrass preparation theorem, $Q(t, w)$ can be written as $Q_1(t, w) \cdot Q_2(t, w)$ in a neighborhood of $(-1, w^o)$ where $Q_1(-1, w^o) \neq 0$ and

$$Q_2(t, w) = (w_n - w_n^o)^r + a_1(t, 'w)(w_n - w_n^o)^{r-1} + \ldots + a_r(t, 'w).$$

If $t^0$ is close to $-1$ and $|t^o| > 1$, the equation $Q_2(t, w) = 0$ has a root $w^1$ in $\Sigma$ close to $w^o$, and so $P(t^o, w^1) = 0$. This contradicts the fact that the roots of $P$ lie in the closure of the unit disc and proves that $\widehat{f}$ is proper. ■

We need the following proposition.

PROPOSITION 2 ([13]). *Let at least one of the domains $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be strongly pseudoconvex, and assume that there exists a proper holomorphic mapping from $\Omega_1$ onto $\Omega_2$ which is not biholomorphic. Then there is no proper holomorphic mapping from $\Omega_2$ onto $\Omega_1$. In particular $\Omega_1$ and $\Omega_2$ are biholomorphically inequivalent.*

*Conclusion of the proof of Proposition 1.* It is clear that $\widehat{G}$ is biholomorphic to the domain $E_1 = \{(''z, z_{n-1}, z_n) \in \mathbb{C}^n : 2\operatorname{Re}(z_n) + |''z|^2 + |z_{n-1}| < 0\}$. The fractional transformation

$$(''z, z_{n-1}, z_n) \mapsto \left( \frac{\sqrt{2}\,''z}{z_n - 1}, \frac{2z_{n-1}}{(z_n - 1)^2}, \frac{z_n + 1}{z_n - 1} \right)$$

maps $E_1$ biholomorphically onto the domain $E_2 = \{(''z, z_{n-1}, z_n) \in \mathbb{C}^n : |''z|^2 + |z_{n-1}| + |z_n|^2 < 1\}$ and $\Sigma$ is biholomorphic to the unit ball $\mathbb{B}$ by means of Cayley's transformation. The mapping $(''z, z_{n-1}, z_n) \mapsto (''z, z_{n-1}^2, z_n)$ is proper from $\mathbb{B}$ onto $E_2$. Thus there exists a proper holomorphic mapping from $\Sigma$ onto $\widehat{G}$ which is not biholomorphic. So Proposition 2 implies that there is no proper holomorphic mapping from $\widehat{G}$ onto $\Sigma$. This contradicts the fact that $\widehat{f}$ is proper. Therefore $q$ is a singular point of $\partial B_\infty$. Finally, by using the maximum principle and the irreducibility of $H_\infty$ $(n \ge 3)$, we get $f(V_f) = H_\infty$. This completes the proof of Proposition 1. ■

**4. Factorization by automorphisms and proof of Theorem 1.** In this section we give the proof of Theorem 1. First of all, we need the following lemma to prove that a proper holomorphic self-mapping of $B_\infty$ is factored by automorphisms.

LEMMA 6. *For $n \ge 3$, $\pi_1(B_\infty \setminus H_\infty) = \mathbb{Z}$.*

*Proof.* The function $z \mapsto z/(1 + N_\infty(z))$ maps homeomorphically $\mathbb{C}^n$ onto $B_\infty$ and $H$ onto $H_\infty$. Thus $\mathbb{C}^n \setminus H$ is homeomorphic to $B_\infty \setminus H_\infty$. Since $\mathbb{C}^n \setminus H$ retracts by deformation onto $S^{2n-1} \setminus (H \cap S^{2n-1})$, we have $\pi_1(\mathbb{C}^n \setminus H) = \pi_1(S^{2n-1} \setminus K)$ with $K = H \cap S^{2n-1}$. The mapping

$$S^{2n-1} \setminus K \to S^1, \qquad x \mapsto \frac{g(x)}{|g(x)|},$$

$(g(x) = \sum_{1 \le j \le n} x_j^2)$ is a fibration. Its fiber $F$ is $(n-2-s)$-connected, (i.e. $\pi_i(F) = \{0\}$ for all $0 \le i \le n-2-s$) where $s = \dim H_{\text{sing}}$ (see [8]). Since $s = 0$ ($H$ has only one singularity at 0), for $n \ge 3$ we obtain

$\pi_1(F) = \pi_0(F) = \{0\}$. The homotopy sequence

$$\pi_1(F) \to \pi_1(S^{2n-1} \setminus K) \to \pi_1(S^1) \to \pi_0(F)$$

is exact; consequently, $\pi_1(S^{2n-1} \setminus K) = \pi_1(S^1) = \mathbb{Z}$. ∎

*Proof of Theorem 1.* Let $f$ be a proper holomorphic self-mapping of $B_\infty$ with branch locus $V_f$. We denote by $f^2$ the mapping $f \circ f$ and by $V_{f^2}$ its branch locus. Assume that $V_f$ is not empty. In view of Proposition 1, Lemma 6 and Theorem 2 the mapping $f$ is factored by a finite subgroup $\Gamma$. In particular we have $f^{-1} \circ f(V_f) = V_f$. Then Proposition 1 implies that

$$f^{-1}(H_\infty) = V_f. \tag{6}$$

Since $V_{f^2} = V_f \cup f^{-1}(V_f)$ and again using Proposition 1 (applied to $f$ and $f^2$) one has $H_\infty = f(H_\infty) \cup H_\infty$. It follows from the irreducibility of $H_\infty$ that

$$f(H_\infty) = H_\infty. \tag{7}$$

As the automorphism group of $B_\infty$ is $S^1.O(n, \mathbb{R})$, the elements of $\Gamma$ stabilize $H_\infty$. So in view of (7) and the factorization of $f$ we have

$$f^{-1}(H_\infty) = H_\infty. \tag{8}$$

From (6) and (8) we conclude that $V_f = H_\infty$. But the factorization theorem implies that there exists $\gamma \in \Gamma$ such that $\{\gamma(z) = z\} = H_\infty$. This is impossible, since $H_\infty$ has a singularity at 0. This contradiction shows that $V_f$ is empty. As the domain $B_\infty$ is simply connected, we conclude that $f$ is a biholomorphism. ∎

*Remarks.* We can repeat the same argument used in the proof of Proposition 1 (second case) to show that there is no proper holomorphic mapping from the minimal ball onto a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with $C^2$ boundary. The problem of existence of proper holomorphic mappings from a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ ($n \geq 3$) with $C^2$ boundary onto the minimal ball was answered in the negative in [11]. These results solve a question raised by Hahn and Pflug regarding the existence of proper holomorphic mappings between the Euclidean ball and the minimal ball, in a more general context. Note that this question was solved earlier by Oeljeklaus and Youssfi [9] in the case of the Euclidean ball.

The author is grateful to the referee for his encouragement and for his useful remarks on this material.

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*Reçu par la Rédaction le 14.7.2000*
*Révisé le 9.6.2001 et 9.2.2002* (1186)