Proper holomorphic self-mappings of the minimal ball

by Nabil Ourimi (Bizerte)

Abstract. The purpose of this paper is to prove that proper holomorphic self-mappings of the minimal ball are biholomorphic. The proof uses the scaling technique applied at a singular point and relies on the fact that a proper holomorphic mapping \( f : D \to \Omega \) with branch locus \( V_f \) is factored by automorphisms if and only if \( f_*(\pi_1(D \setminus f^{-1}(f(V_f)), x)) \) is a normal subgroup of \( \pi_1(\Omega \setminus f(V_f), b) \) for some \( b \in \Omega \setminus f(V_f) \) and \( x \in f^{-1}(b) \).

1. Introduction. Families of proper holomorphic mappings arise in the problem of determining which domains in \( \mathbb{C}^n \) do not possess any proper holomorphic mappings which are not biholomorphic. In this paper our aim is to study this problem in the case of a special domain in \( \mathbb{C}^n \), \( n \geq 2 \), with non-piecewise smooth boundary. This domain is the minimal ball. It is given by

\[
B_\infty = \{ z \in \mathbb{C}^n : N_\infty(z) < 1 \},
\]

where \( N_\infty(z) = (|z|^2 + |z^2|)/2 \) and \( z^2 = \sum_{1 \leq j \leq n} z_j^2 \).

The function \( \sqrt{N_\infty} \) is a norm in \( \mathbb{C}^n \) introduced by Hahn–Pflug [4] as the smallest norm in \( \mathbb{C}^n \) that extends the Euclidean norm in \( \mathbb{R}^n \) under certain restrictions. It has been studied in several recent works [7], [12], [9], [10], [11], [17]. The automorphism group of \( B_\infty \) is \( S^1 \cdot O(n, \mathbb{R}) \) (see [7]). In addition \( B_\infty \) is a non-Lu Qi-Keng domain for \( n \geq 4 \) and it is neither homogeneous nor Reinhardt. Its boundary is \( B \)-regular in the sense of Sibony [16] and Henkin–Iordan [6].

Our main result can be stated as follows:

**Theorem 1.** Every proper holomorphic self-mapping of \( B_\infty \) is biholomorphic.

2000 Mathematics Subject Classification: Primary 32H35.

Key words and phrases: proper holomorphic mappings, correspondences, branch locus, scaling, factorization, minimal ball.
The following example shows that this theorem cannot be extended to proper holomorphic self-correspondences as in the case of strongly pseudo-convex domains (see [1]).

Let \( M = \{ z \in \mathbb{C}^{n+1} : |z| < \sqrt{2} \text{ and } z^2 = 0 \} \). The group \( S^1.O(n+1, \mathbb{R}) \) is a subgroup of \( \text{Aut}(M) \). Consider the projection \( pr : \mathbb{C}^{n+1} \to \mathbb{C}^n \) defined by \( pr(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_n) \). The restriction \( F := \text{pr}|M \) is a proper holomorphic mapping with multiplicity 2 from \( M \) onto \( B_\infty \subset \mathbb{C}^n \). Let \( g \in S^1.O(n+1, \mathbb{R}) \setminus S^1.O(n, \mathbb{R}) \) (the group \( O(n, \mathbb{R}) \) can be regarded as a subgroup of \( O(n+1, \mathbb{R}) \)). Then \( h = F \circ g \circ F^{-1} \) is an irreducible proper holomorphic self-correspondence of \( B_\infty \). To prove that \( h \) is a nontrivial correspondence, assume that \( h \) is a mapping. Then \( h \) is an automorphism of \( B_\infty \) (i.e. \( h \in S^1.O(n, \mathbb{R}) \)); otherwise the multiplicity of \( h \circ F \) will be greater than the multiplicity of \( F \circ g \). This implies that \( g \in S^1.O(n, \mathbb{R}) \) and so we get a contradiction.

2. Preliminary results. In this section, we give some preliminary results useful for the proof of our theorem.

2.1. Factorization of proper holomorphic mappings. A mapping \( f : D \to \Omega \) is factored by automorphisms if there is a finite subgroup \( \Gamma \subset \text{Aut}(D) \) such that for all \( z \in D \),

\[
    f^{-1}(f(z)) = \{ \gamma(z) : \gamma \in \Gamma \}.
\]

We will denote by \( J_f(z) \) the Jacobian determinant of \( f \) and by \( V_f = \{ z \in D : J_f(z) = 0 \} \) its branch locus. A necessary and sufficient condition to factorize proper holomorphic mappings is given in the following theorem.

**Theorem 2.** Let \( D \) be a pseudoconvex bounded domain in \( \mathbb{C}^n \), \( \Omega \) a domain in \( \mathbb{C}^n \) and \( f : D \to \Omega \) a proper holomorphic mapping with branch locus \( V_f \). Denote by \( F \) the restriction of \( f \) to \( D \setminus f^{-1}(f(V_f)) \). Then the following statements are equivalent:

1. There exist \( b \in \Omega \setminus f(V_f) \) and \( x \in f^{-1}(b) \) such that \( F_*(\pi_1(D \setminus f^{-1}(f(V_f)), x)) \) is a normal subgroup of \( \pi_1(\Omega \setminus f(V_f), b) \),
2. \( f \) is factored by automorphisms.

The existence of the group \( \Gamma \) is due to W. Rudin [15] in the case of the Euclidean ball in \( \mathbb{C}^n \) and to Bedford–Bell [1] in the case of strongly pseudoconvex domains in \( \mathbb{C}^n \) (see also the references for related results). Theorem 2 implies that the branch locus of \( f \) is given by

\[
    V_f = \bigcup_{\{g \in \Gamma : g \neq \text{id}\}} \{ z \in D : g(z) = z \}.
\]

Then the factorization theorem above may be used to reduce the study of the behavior of the branch locus to the study of the group \( \Gamma \). Thus far \( \Gamma \) has been identified only in the case of the Euclidean ball in \( \mathbb{C}^n \).
Proof of Theorem 2. Let $H = f(V_f)$, $E = D \setminus f^{-1}(H)$ and $B = \Omega \setminus H$. The restriction $F = f|E : E \to B$ is a connected finite covering. According to [3], $F_*(\pi_1(E, x))$ is a normal subgroup in $\pi_1(B, b)$ if and only if $F : E \to B$ is a Galois covering, i.e. $F$ is a connected covering and the group $\Gamma = \{ \gamma \in \text{Hom}(E) : F \circ \gamma = F \}$ acts transitively on each fiber $F^{-1}(b)$, $b \in B$. Moreover, the group $\Gamma$ is isomorphic to $\pi_1(B, b)/F_*(\pi_1(E, x))$.

Assume that $F_*(\pi_1(E, x))$ is a normal subgroup in $\pi_1(B, b)$. Then $F$ is a Galois covering. The mapping $F$ is holomorphic, so all elements of $\Gamma$ are biholomorphic and since $D$ is bounded, they extend to holomorphic mappings from $D$ onto $\overline{D}$. Suppose that there exists a point $p \in f^{-1}(H)$ such that $\gamma(p) \in \partial D$ for some $\gamma \in \Gamma$. Let $\Delta$ be an analytic disc in $D$ such that $\Delta \cap f^{-1}(H) = \{ p \}$. Then $\gamma(\Delta)$ intersects $\partial D$ only at $\gamma(p)$. This contradicts the fact that $D$ is pseudoconvex. Hence for all $\gamma \in \Gamma$, $\gamma(D) \subset D$. Thus the elements of $\Gamma$ extend to automorphisms of $D$ and define a subgroup of $\text{Aut}(D)$ that we denote by $\hat{\Gamma}$. By analytic extension the equality $f \circ \hat{\gamma} = f$ remains valid for all $\hat{\gamma} \in \hat{\Gamma}$.

Let now $b \in H$ and $z_1, z_2 \in f^{-1}(H)$ be such that $f(z_1) = f(z_2) = b$. Since $f$ is an open map, there exist two sequences $\{ z_1^j \}_j$ and $\{ z_2^j \}_j$ in $E$ that converge respectively to $z_1$ and $z_2$ and satisfy $f(z_1^j) = f(z_2^j)$ for all $j$. The mapping $F$ is a Galois covering, so for all $j$ there exists $\gamma_j \in \Gamma$ such that $z_1^j = \gamma_j(z_2^j)$. As $\Gamma$ is a finite subgroup, we may assume that $z_1^j = \gamma(z_2^j)$ for some $\gamma \in \Gamma$ and for any integer $j$. Passing to the limit, we get $z_1 = \hat{\gamma}(z_2)$ ($\hat{\gamma}$ is the extension of $\gamma$). This proves that the mapping $f$ is factored by $\hat{\Gamma}$.

Conversely, assume that $f$ is factored by a subgroup $\hat{\Gamma}$. It is clear that for all $\hat{\gamma} \in \hat{\Gamma}$, $\hat{\gamma}$ maps $E$ onto itself. Then the restriction $F$ is a Galois covering. This implies that $F_*(\pi_1(E, x))$ is a normal group in $\pi_1(B, b)$.  

2.2. Hopf’s lemma for $B_\infty$. We denote by $\varrho = \sqrt{N_\infty} - 1$ a defining function of $B_\infty$. It is easy to see that this function satisfies the following lemma.

**Lemma 1.** For $z \in B_\infty$ one has

\[
\frac{1}{\sqrt{2}} \text{dist}(z, \partial B_\infty) \leq |\varrho(z)| \leq \text{dist}(z, \partial B_\infty).
\]

First, we establish the uniform Hopf lemma for the unit disc $\Delta$ in $\mathbb{C}$.

**Lemma 2.** Let $r$ be a subharmonic negative function on $\Delta$. Then for all $z \in \Delta$ one has

\[
|r(z)| \geq \inf_{\Delta(0,1/2)} |r| \text{dist}(z, \partial \Delta).
\]

**Proof.** We consider the subharmonic function

\[
\hat{r}(z) = r(z) - A \frac{\ln |z|}{\ln 2}, \quad \text{where} \quad A = \inf_{\Delta(0,1/2)} |r|.
\]
Since $\hat{r}$ is negative on $\partial \Delta(0, 1/2)$ and $\lim_{z \to z_0 \in \partial \Delta} \hat{r}(z) \leq 0$, by the maximum principle we get $\hat{r}(z) \leq 0$ on $\Delta(0, 1) \setminus \Delta(0, 1/2)$. Thus
\[
\forall z \in \Delta, \quad |r(z)| \geq \inf_{\Delta(0, 1/2)} |r| \min \left( 1, \frac{-\ln |z|}{\ln 2} \right) \geq \inf_{\Delta(0, 1/2)} |r|(1 - |z|). \quad \blacksquare
\]

As an application, we get the Hopf lemma for the minimal ball.

**Lemma 3.** Let $r$ be a plurisubharmonic negative function on $B_\infty$. Then for all $z \in B_\infty$ one has
\[
|r(z)| \geq \frac{1}{\sqrt{2}} \inf_{\frac{1}{2}B_\infty} |r| \operatorname{dist}(z, \partial B_\infty),
\]
where $\frac{1}{2}B_\infty = \{ z \in \mathbb{C}^n : \sqrt{N_\infty}(z) \leq 1/2 \}$.

**Proof.** For $z = 0$ the inequality is true. Let now $z \in B_\infty \setminus \{0\}$ and consider the subharmonic function $u$ defined on the unit disc by
\[
u(\xi) = r\left( \frac{\xi z}{\sqrt{N_\infty}(z)} \right).
\]In view of Lemma 2 we have
\[
|u(\xi)| \geq \inf_{\Delta(0, 1/2)} |u| \operatorname{dist}(\xi, \partial \Delta).
\]Let $\xi = \sqrt{N_\infty}(z)$. The previous inequality becomes
\[
|r(z)| \geq \inf_{\Delta(0, 1/2)} |u| \cdot |1 - \sqrt{N_\infty}(z)| \geq \frac{1}{\sqrt{2}} \inf_{\Delta(0, 1/2)} |u| \operatorname{dist}(z, \partial B_\infty).
\]Since
\[
\inf_{w \in \Delta(0, 1/2)} \left| r\left( \frac{w z}{\sqrt{N_\infty}(z)} \right) \right| \geq \inf_{\frac{1}{2}B_\infty} |r|,
\]we have the desired inequality. $\blacksquare$

In the case $n = 2$, the minimal ball is biholomorphic to the Reinhardt domain $\{(z, w) \in \mathbb{C}^2 : |z| + |w| < 1\}$. Hence according to [2], any proper holomorphic self-mapping of $B_\infty$ is biholomorphic. Now assume that $n \geq 3$. The proof of our theorem is based on the scaling technique and the notion of factorization of proper holomorphic mappings.

**3. Scaling technique.** Set $H = \{ z \in \mathbb{C}^n : \sum_{1 \leq j \leq n} z_j^2 = 0 \}$ and $H_\infty = H \cap B_\infty$. The singular part of the boundary of $B_\infty$ is obviously the set $\partial H_\infty = H \cap \partial B_\infty$. The regular part $\partial B_\infty \setminus \partial H_\infty$ is $C^\infty$-smooth and it consists of strongly pseudoconvex points. In [10] the authors gave an explicit formula for the Bergman kernel of the minimal ball. In particular, they showed that any proper holomorphic self-mapping of the minimal ball
extends holomorphically to a neighborhood of $\overline{B}_\infty$ (see Theorem 5.5 of [10]). In this section we shall prove the following:

**PROPOSITION 1.** Let $f$ be a branching proper holomorphic self-mapping of $B_\infty$ with branch locus $V_f$. Then $f(V_f)$ is equal to $H_\infty$.

**Proof.** Let $p \in \partial V_f = \overline{V_f \setminus V_f}$ and let $\{p_k\}_k$ be a sequence in $V_f$ that converges to $p$. Since $f$ is proper, the sequence $\{f(p_k)\}_k$ converges to a boundary point $q \in \partial B_\infty$. We shall prove that $q$ is a singular point (i.e. $q \in \partial H_\infty$). The proof is by contradiction. Assume that $q$ is a strong pseudoconvexity point. We will discuss two cases.

*First case: $p$ is a strong pseudoconvexity point.* Since the mapping $f$ extends holomorphically to a neighborhood of $\overline{B}_\infty$ ([10]), it defines a local biholomorphism near $p$ (see [14]). This contradicts the fact that $p \in \partial V_f$.

*Second case: $p$ is a singular point ($p \in \partial H_\infty$).* In this case we will use the scaling technique to prove that this situation is not possible.

Since $S^1O(n, \mathbb{R})$ acts transitively on $\partial H_\infty$, we can assume without loss of generality that $p = (0, \ldots, 0, i, 1)$. The domain $B_\infty - p$ is represented by $2 \text{Re}(z_n - iz_{n-1}) + |z_1|^2 + \ldots + |z_n|^2 + |z_1^2 + \ldots + z_n^2 + 2(z_n + iz_{n-1})| < 0$.

In the new coordinates

$$
\zeta_j = z_j, \quad j \in \{1, \ldots, n-2\},
\zeta_{n-1} = z_n + iz_{n-1},
\zeta_n = z_n - iz_{n-1},
$$

the point $p$ is transformed to 0 and the domain $B_\infty - p$ corresponds to the domain $G$ defined by $\{ \varphi < 0 \}$ with

$$
\varphi(\xi) = 2 \text{Re}(\zeta_n) + \frac{1}{2}(|\zeta_n|^2 + |\zeta_{n-1}|^2) + |\zeta_1|^2 + \ldots + |\zeta_{n-2}|^2 + |\zeta_1^2 + \ldots + \zeta_{n-2}^2 + 2\zeta_{n-1} + \zeta_{n-1}\zeta_n|.
$$

We write $z \in \mathbb{C}^n$ as $z = (\zeta, z_n)$ where $\zeta$ denotes the first $n-1$ coordinates of $z$ or $z = (\zeta', z_{n-1}, z_n)$ where $\zeta'$ denotes the first $n-2$ coordinates of $z$. If $\{a_k\}_k$ and $\{b_k\}_k$ are two sequences of real positive numbers, we write $a_k \simeq b_k$ if there is a positive constant $c$ independent of $k$ such that $c^{-1}b_k \leq a_k \leq cb_k$.

Let $g : B_\infty \to G$ be the linear transformation mapping $B_\infty$ onto $G$ and $t_k = (0, -\delta_k), k = 1, 2, \ldots$, be a sequence of points in $G$ where $\{\delta_k\}_k$ is a sequence of real positive numbers converging to 0. It is clear that for large $k$, $|\varphi(t_k)| \simeq \delta_k$. Since $B_\infty$ is linearly equivalent to $G$, in view of Lemma 1 we have, for all $z \in G$,

$$
\frac{1}{\sqrt{2} \|g\|} \text{dist}(z, \partial G) \leq |\varphi(z)| \leq \|g^{-1}\| \text{dist}(z, \partial G).
$$
It follows that for large $k$,

\begin{equation}
\text{dist}(t_k, \partial G) \simeq \delta_k.
\end{equation}

As the mapping $f$ is continuous, the sequence $\{q_k\}_k$, $q_k = f \circ g^{-1}(t_k)$, converges to $q = f \circ g^{-1}(0)$. Let $V$ be a neighborhood of $q$ in $\mathbb{C}^n$ which does not intersect the set of weakly pseudoconvex points of $\partial B_\infty$. For all $w \in \partial B_\infty \cap V$ we consider the change of variables $h^w$ defined by

$$z_j^* = \frac{\partial}{\partial z_j}(w)(z_j - w_j) - \frac{\partial}{\partial z_j}(w)(z_n - w_n), \quad 1 \leq j \leq n - 1,$$

$$z_n^* = \sum_{1 \leq j \leq n} \frac{\partial}{\partial z_j}(w)(z_j - w_j).$$

The mapping $h^w$ maps $w$ to 0 and the real normal to $\partial B_\infty$ at $w$ onto the line \{z = 0, y_n = 0\}. Let $w_k$ be the projection of $q_k$ on the boundary of $B_\infty$. For simplicity we denote $h^{w_k}$ (the mapping as above) by $h^k$. Set $D^k = h^k(B_\infty)$, $g^k = \rho \circ h^{k-1}$ and $\gamma_k = \text{dist}(h^k(q_k), \partial D^k)$. We have $h^k(q_k) = (0, -\gamma_k)$. The sequence of proper holomorphic mappings defined by $f^k = h^k \circ f \circ g^{-1} : G \to D^k$ satisfies $f^k(0, -\delta_k) = (0, -\gamma_k)$ for all $k$. Now we introduce the inhomogeneous dilatation of coordinates as follows:

$$\alpha^k(z, z_n) = \left(\frac{z}{\sqrt{\gamma_k}}, \frac{z_n}{\gamma_k}\right), \quad \beta^k(z, z_{n-1}, z_n) = \left(\frac{\gamma z}{\sqrt{\gamma_k}}, \frac{z_{n-1}}{\delta_k}, \frac{z_n}{\delta_k}\right).$$

The idea is to follow the argument of Pinchuk [13] and to consider the mapping $\hat{f}^k = \alpha^k \circ f^k \circ (\beta^k)^{-1}$. In the new coordinates, $G$ and $D^k$ correspond to the domains $\hat{G}^k$ and $\hat{D}^k$ with defining functions

$$\varphi^k(z) = \frac{1}{\delta_k} \varphi \circ \beta^{k-1}(z), \quad \hat{\varphi}^k(z) = \frac{1}{\gamma_k} \varphi \circ (\alpha^k)^{-1}(z)$$

respectively. Thus, $\hat{f}^k$ is a proper holomorphic mapping from $\hat{G}^k$ onto $\hat{D}^k$ and satisfies $\hat{f}^k(0, -1) = (0, -1)$. Let $\hat{G} = \{\hat{\varphi} < 0\}$ and $\Sigma = \{\hat{\varphi} < 0\}$ where $\hat{\varphi}(z) = 2 \Re(z_n) + |z|^2 + |z|^2 + 2z_{n-1}|, \hat{\varphi}(z) = 2 \Re(z_n) + |z|^2$ and $n^2 = \sum_{1 \leq j \leq n-2} z_j^2$. The sequence $\{\hat{\varphi}^k\}_k$ (resp. $\{\hat{\varphi}^k\}_k$) converges uniformly to the function $\hat{\varphi}$ on compact subsets of $\hat{G}$ (resp. to the function $\hat{\varphi}$ on compact subsets of $\Sigma$). Consequently, for all compact $K \subset \hat{G}$, the mappings $\hat{f}^k$ are well defined on $K$, starting from some $k^0 = k^0(K)$. By exhausting $\hat{G}$ with an increasing sequence of compact sets and by passing to the limit, we conclude that we may assume that $\{\hat{f}^k\}_k$ converges to a holomorphic function $\hat{f} : \hat{G} \to \Sigma$. Since $\hat{\varphi}$ is plurisubharmonic and $\hat{f}(0, -1) = (0, -1) \in \Sigma$, the maximum principle implies that $\hat{f}(\hat{G}) \subset \Sigma$.

We shall prove that $\hat{f}$ is proper. For this we need some estimates on the distance.
Lemma 4. There exists a constant $c > 0$ such that for all $z \in B_\infty$,
\[
\frac{1}{c} \dist(z, \partial B_\infty) \leq \dist(f(z), \partial B_\infty) \leq c \dist(z, \partial B_\infty).
\]

Proof. Recall that $\varphi$ denotes a defining function of $B_\infty$. Since $\varphi \circ f$ is plurisubharmonic and negative on $B_\infty$, in view of Lemmas 1 and 3 there exists a constant $c > 0$ such that for all $z \in B_\infty$, $\dist(f(z), \partial B_\infty) \geq c^{-1} \dist(z, \partial B_\infty)$. To prove the right-hand inequality, we consider the function $r(w) = \max\{\varphi(z) : z \in f^{-1}(w)\}$, which is well defined and plurisubharmonic on $B_\infty \setminus f(V_f)$ and also bounded there. Since $f$ is proper, $f(V_f)$ is an analytic subvariety, and so $r$ extends as a plurisubharmonic function on $B_\infty$. Now we apply Lemmas 1 and 3 again. \]

Since the coordinates $h^{w_k}$ depend continuously on $w_k$ and the domain $G$ is linearly equivalent to $B_\infty$, in view of Lemma 4 the following estimates hold:
\[
(2) \quad c_1 \dist(z, \partial G) \leq \dist(f^k(z), \partial D^k) \leq c_2 \dist(z, \partial G),
\]
with $c_1, c_2 > 0$ do not depend on $k$. In addition, in $G$ and $D^k$ we have the estimates
\[
(3) \quad c_3 |\varphi(z)| \leq \dist(z, \partial G) \leq c_4 |\varphi(z)|,
\]
\[
\quad c_3 |\varphi^k(w)| \leq \dist(w, \partial D^k) \leq c_4 |\varphi^k(w)|,
\]
where $c_3, c_4 > 0$ do not depend on $k$ (the estimates (3) follow from Lemmas 1 and 3). According to (1) and (2), there exist positive constants $c_5$ and $c_6$ independent of $k$ such that for all $k$,
\[
(4) \quad c_5 < \gamma_k/\delta_k < c_6.
\]

Let $K$ be a compact set in $\hat{G}$ and $z \in K$. Set $w^k = \hat{f}^k(z)$. In view of (2)–(4),
\[
\hat{\varphi}^k(w^k) = \gamma_k^{-1} \varphi^k(\sqrt{\gamma_k} w^k, \gamma_k w_n^k) \leq (\gamma_k c_3)^{-1} \dist((\sqrt{\gamma_k} w^k, \gamma_k w_n^k), \partial D^k)
\]
\[
\leq c_2 (\gamma_k c_3)^{-1} \varphi(\sqrt{\delta_k} z, \delta_k z_n, \delta_k z_n^{-1}, \delta_k z_n^{-1}) \leq c_2 c_4 (\gamma_k c_3)^{-1} \varphi^k(z) \leq c_2 c_4 (c_5 c_3)^{-1} \varphi^k(z).
\]
Passing to a convergent subsequence and to the limit, we get
\[
(5) \quad \hat{\varphi}(\hat{f}(z)) \leq c_7 \varphi(z)
\]
for $z \in K$ and for some positive constant $c_7$ independent of $z$. Since $K$ is an arbitrary compact set in $\hat{G}$, the estimate (5) holds for all $z \in \hat{G}$.

Lemma 5. The sequence $\{\hat{f}^k\}_k$ admits a subsequence converging uniformly on compact subsets of $\hat{G}$ to a proper holomorphic mapping $\hat{f} : \hat{G} \to \Sigma$. 


Proof. The proof is based on certain ideas of S. Pinchuk [13]. For the convenience of the reader and for the sake of completeness we include a proof. We consider the function \( v(z) = e^{zn} \). It satisfies \( |v(z)| < 1 \) on \( \overline{G} \setminus \{0\} \) and \( v(0) = 1 \). For \( \delta_k < 1 \) the functions

\[
v^k(z) = \frac{v(z) - (1 - \delta_k)}{1 - v(z)(1 - \delta_k)}
\]

are holomorphic in a neighborhood of \( \overline{G} \) and \( |v^k(z)| < 1 \) for all \( z \in G \). For large \( k \) we consider the function \( u^k(z) = v^k(\sqrt[2]{\delta_k}z, \delta_kz_n, \delta_kz_n) \), which is holomorphic in a neighborhood of \( \tilde{G}^k \) and has the form

\[
u^k(z) = \frac{1 + z_n + \delta_k \left( \sum_{p \geq 2} \frac{\delta_k^{p-2}}{p!} z_n^p \right)}{1 - z_n(1 - \delta_k) - \delta_k \left( \sum_{p \geq 2} \frac{\delta_k^{p-2}}{p!} z_n^p \right)(1 - \delta_k)}.
\]

Set \( u^0 = \lim_{k \to \infty} u^k \). The limit \( u^0 \) is holomorphic on \( \tilde{G} \) and it is defined by

\[
u^0(z) = \frac{1 + z_n}{1 - z_n}.
\]

As \( |z| \to \infty \) and \( z \in \tilde{G} \), clearly \( |z_n| \to \infty \). Thus \( u^0(z) \to -1 \) as \( |z| \to \infty \).

Since the mappings \( \tilde{f}^k : \tilde{G}^k \to \tilde{D}^k \) are ramified analytic coverings (see [5]), for each \( k \) there exists a polynomial

\[
P^k(t, w) = t^m + S_1^k(w)t^{m-1} + \ldots + S_m^k(w)
\]

(where \( S_{\alpha}^k \) are holomorphic functions on \( \tilde{D}^k \) for \( \alpha \in \{1, \ldots, m\} \) and \( m \) is the multiplicity of \( \tilde{f}^k \) such that \( P^k(u^k(z), \tilde{f}^k(z)) \equiv 0 \) on \( \tilde{G}^k \) and for \( w \in \tilde{D}^k \) one has \( P^k(t, w) = 0 \) if and only if \( t \in u^k \circ (\tilde{f}^k)^{-1}(w) \). As \( |u^k(z)| < 1 \) in \( \tilde{G}^k \), it follows that all the roots of \( P^k(\cdot, w) \) are of modulus less than one for all \( w \in \tilde{D}^k \). Then there exists a constant \( c > 0 \) such that \( |S_{\alpha}^k(w)| < c \) in \( \tilde{D}^k \).

Consequently, we can assume (after passing to a subsequence) that for all \( \alpha \in \{1, \ldots, m\} \) the sequence \( \{S_{\alpha}^k\}_k \) converges to a function \( S_{\alpha} \), defined and holomorphic in \( \Sigma \). Let \( P(t, w) = t^m + S_1(w)t^{m-1} + \ldots + S_m(w) \). We have \( P(u^0(z), \tilde{f}(z)) \equiv 0 \) and the roots of \( P \) lie in the closed unit disc for all \( w \in \Sigma \). We write \( P(t, w) \) in the form \( (t + 1)Q(t, w) \), where \( Q(-1, w) \neq 0 \).

Assume that \( \tilde{f} \) is not proper. In view of (5) there exists a sequence \( \{z_\mu\}_\mu \) of \( \tilde{G} \) such that \( z_\mu \to \infty \) and \( \tilde{f}(z_\mu) \to w^0 \in \Sigma \) as \( \mu \to \infty \). By the Weierstrass preparation theorem, \( Q(t, w) \) can be written as \( Q_1(t, w) \cdot Q_2(t, w) \) in a neighborhood of \( (-1, w^0) \) where \( Q_1(-1, w^0) \neq 0 \) and

\[
Q_2(t, w) = (w_n - w_n^0)^r + a_1(t, w)(w_n - w_n^0)^{r-1} + \ldots + a_r(t, w).
\]
If \( t^0 \) is close to \(-1\) and \( |t^0| > 1 \), the equation \( Q_2(t, w) = 0 \) has a root \( w^1 \) in \( \Sigma \) close to \( w^o \), and so \( P(t^o, w^1) = 0 \). This contradicts the fact that the roots of \( P \) lie in the closure of the unit disc and proves that \( \hat{f} \) is proper. ■

We need the following proposition.

PROPOSITION 2 ([13]). Let at least one of the domains \( \Omega_1, \Omega_2 \subset \subset \mathbb{C}^n \) be strongly pseudoconvex, and assume that there exists a proper holomorphic mapping from \( \Omega_1 \) onto \( \Omega_2 \) which is not biholomorphic. Then there is no proper holomorphic mapping from \( \Omega_2 \) onto \( \Omega_1 \). In particular \( \Omega_1 \) and \( \Omega_2 \) are biholomorphically inequivalent.

Conclusion of the proof of Proposition 1. It is clear that \( \hat{G} \) is biholomorphic to the domain \( E_1 = \{(''z, z_{n-1}, z_n) \in \mathbb{C}^n : 2 \text{Re}(z_n) + |''z|^2 + |z_{n-1}| < 0\} \). The fractional transformation

\[
('''z, z_{n-1}, z_n) \mapsto \left( \frac{2z_{n-1}}{z_n-1}, \frac{z_n+1}{(z_n-1)^2} \right)
\]

maps \( E_1 \) biholomorphically onto the domain \( E_2 = \{(''z, z_{n-1}, z_n) \in \mathbb{C}^n : |''z|^2 + |z_{n-1}| + |z_n|^2 < 1\} \) and \( \Sigma \) is biholomorphic to the unit ball \( B \) by means of Cayley’s transformation. The mapping \( (''z, z_{n-1}, z_n) \mapsto (''z, z_{n-1}, z_n) \) is proper from \( B \) onto \( E_2 \). Thus there exists a proper holomorphic mapping from \( \Sigma \) onto \( \hat{G} \) which is not biholomorphic. So Proposition 2 implies that there is no proper holomorphic mapping from \( \hat{G} \) onto \( \Sigma \). This contradicts the fact that \( \hat{f} \) is proper. Therefore \( q \) is a singular point of \( \partial B_\infty \). Finally, by using the maximum principle and the irreducibility of \( H_\infty \) \((n \geq 3)\), we get \( f(V_f) = H_\infty \). This completes the proof of Proposition 1. ■

4. Factorization by automorphisms and proof of Theorem 1. In this section we give the proof of Theorem 1. First of all, we need the following lemma to prove that a proper holomorphic self-mapping of \( B_\infty \) is factored by automorphisms.

**Lemma 6.** For \( n \geq 3 \), \( \pi_1(B_\infty \setminus H_\infty) = \mathbb{Z} \).

*Proof.* The function \( z \mapsto z/(1 + N_\infty(z)) \) maps homeomorphically \( \mathbb{C}^n \) onto \( B_\infty \) and \( H \) onto \( H_\infty \). Thus \( \mathbb{C}^n \setminus H \) is homeomorphic to \( B_\infty \setminus H_\infty \). Since \( \mathbb{C}^n \setminus H \) retracts by deformation onto \( S^{2n-1} \setminus (H \cap S^{2n-1}) \), we have \( \pi_1(\mathbb{C}^n \setminus H) = \pi_1(S^{2n-1} \setminus K) \) with \( K = H \cap S^{2n-1} \). The mapping

\[
S^{2n-1} \setminus K \to S^1, \quad x \mapsto \frac{g(x)}{|g(x)|},
\]

\((g(x) = \sum_{1 \leq j \leq n} x_j^2)\) is a fibration. Its fiber \( F \) is \((n - 2 - s)\)-connected, (i.e. \( \pi_i(F) = \{0\} \) for all \( 0 \leq i \leq n - 2 - s \)) where \( s = \dim H_{\text{sing}} \) (see [8]). Since \( s = 0 \) (\( H \) has only one singularity at 0), for \( n \geq 3 \) we obtain
\[ \pi_1(F) = \pi_0(F) = \{0\}. \] The homotopy sequence

\[ \pi_1(F) \to \pi_1(S^{2n-1} \setminus K) \to \pi_1(S^1) \to \pi_0(F) \]

is exact; consequently, \( \pi_1(S^{2n-1} \setminus K) = \pi_1(S^1) = \mathbb{Z}. \)

**Proof of Theorem 1.** Let \( f \) be a proper holomorphic self-mapping of \( B_\infty \) with branch locus \( V_f \). We denote by \( f^2 \) the mapping \( f \circ f \) and by \( V_{f^2} \) its branch locus. Assume that \( V_f \) is not empty. In view of Proposition 1, Lemma 6 and Theorem 2 the mapping \( f \) is factored by a finite subgroup \( \Gamma \). In particular we have \( f^{-1} \circ f(V_f) = V_f \). Then Proposition 1 implies that

\[ f^{-1}(H_\infty) = V_f. \]

Since \( V_{f^2} = V_f \cup f^{-1}(V_f) \) and again using Proposition 1 (applied to \( f \) and \( f^2 \)) one has \( H_\infty = f(H_\infty) \cup H_\infty \). It follows from the irreducibility of \( H_\infty \) that

\[ f(H_\infty) = H_\infty. \]

As the automorphism group of \( B_\infty \) is \( S^1 \cdot O(n, \mathbb{R}) \), the elements of \( \Gamma \) stabilize \( H_\infty \). So in view of (7) and the factorization of \( f \) we have

\[ f^{-1}(H_\infty) = H_\infty. \]

From (6) and (8) we conclude that \( V_f = H_\infty \). But the factorization theorem implies that there exists \( \gamma \in \Gamma \) such that \( \{ \gamma(z) = z \} = H_\infty \). This is impossible, since \( H_\infty \) has a singularity at 0. This contradiction shows that \( V_f \) is empty. As the domain \( B_\infty \) is simply connected, we conclude that \( f \) is a biholomorphism. \( \blacksquare \)

**Remarks.** We can repeat the same argument used in the proof of Proposition 1 (second case) to show that there is no proper holomorphic mapping from the minimal ball onto a strongly pseudoconvex bounded domain in \( \mathbb{C}^n \) with \( C^2 \) boundary. The problem of existence of proper holomorphic mappings from a strongly pseudoconvex bounded domain in \( \mathbb{C}^n \) with \( C^2 \) boundary onto the minimal ball was answered in the negative in [11]. These results solve a question raised by Hahn and Pflug regarding the existence of proper holomorphic mappings between the Euclidean ball and the minimal ball, in a more general context. Note that this question was solved earlier by Oeljeklaus and Youssfi [9] in the case of the Euclidean ball.

The author is grateful to the referee for his encouragement and for his useful remarks on this material.

**References**


ICTP
Math. Section
Strada Costiera, II
34014 Trieste, Italy

Current address:
Faculté des sciences de Bizerte
7021 Jarzouna, Tunisie
E-mail: ourimin@yahoo.com

Reçu par la Rédaction le 14.7.2000