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A uniqueness theorem of Cartan–Gutzmer type for holomorphic mappings in $\mathbb{C}^n$

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Abstract. We continue our previous work on a problem of Janiec connected with a uniqueness theorem, of Cartan–Gutzmer type, for holomorphic mappings in $\mathbb{C}^n$. To solve this problem we apply properties of $(\mathbf{j};k)$-symmetrical functions.

1. Introduction. The following two uniqueness theorems are well known.

Theorem (H. Cartan, [1]). *Let Ω be a bounded domain in $\mathbb{C}^n$ and $g : \Omega \to \Omega$ be a holomorphic mapping. If there exists a point $b \in \Omega$ such that $g(b) = b$, $Dg(b) = I$, then $g(z) = z$ for every $z \in \Omega$.*

Theorem (A. Gutzmer, [3]). *Let Δ be the open unit disc in the complex plane $\mathbb{C}$ and $g : \Delta \to \Delta$ be a holomorphic function of the form*

$$g(\zeta) = \sum_{\nu=0}^{\infty} a_\nu \zeta^\nu, \quad \zeta \in \Delta.$$

If there exists an integer $j \geq 0$ such that $|a_j| = 1$, then $g(\zeta) = a_j \zeta^j$, $\zeta \in \Delta$.

We consider a related problem connected with a uniqueness theorem (given by E. Janiec in [4]) for holomorphic mappings in bounded complete Reinhardt domains

$$\mathbb{B}^{\mathbf{t}}(r) = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{s=1}^{n} |z_s|^{2t_s} < r \Big\},$$

where $r \in \mathbb{R}_+ = (0, \infty)$ and $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$. We write $\mathbb{B}^{\mathbf{t}}$ for $\mathbb{B}^{\mathbf{t}}(1)$; if $\mathbf{1} = (1, \ldots, 1)$, then $\mathbb{B} = \mathbb{B}^{\mathbf{1}}$ is the euclidean open unit ball. For $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of positive integers, we set $\mathbf{t}/\mathbf{j} = (t_1/j_1, \ldots, t_n/j_n) \in \mathbb{R}_+^n$.

In [4] E. Janiec proved the following result.

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Theorem (E. Janiec, [4]). *Let* $\mathbf{t} \in \mathbb{R}_+^n$, $\mathbf{j} \in \mathbb{N}^n$ *and* $g = (g^1, \ldots, g^n)$ *be a holomorphic mapping in* $\mathbb{B}^{\mathbf{t}}$. *If each* g^s, $s = 1, \ldots, n$, *has an expansion into a series of homogeneous polynomials of the form*

$$g^s(z) = z_s^{j_s} + \sum_{\nu=j_s+1}^{\infty} P^{s,\nu}(z), \qquad z = (z_1, \ldots, z_n) \in \mathbb{B}^{\mathbf{t}}, \tag{1.1}$$

and $g : \mathbb{B}^{\mathbf{t}} \to \mathbb{B}^{\mathbf{t}/\mathbf{j}}$, *then* g *transforms* $\mathbb{B}^{\mathbf{t}}$ *onto* $\mathbb{B}^{\mathbf{t}/\mathbf{j}}$ *and*

$$g(z) = (z_1^{j_1}, \ldots, z_n^{j_n}), \qquad z = (z_1, \ldots, z_n) \in \mathbb{B}^{\mathbf{t}}.$$

It is clear that when $\mathbf{j} = (1, \ldots, 1)$, the above theorem coincides with the Cartan uniqueness theorem for $\Omega = \mathbb{B}^{\mathbf{t}}$ and $b = 0$.

E. Janiec [4] also asked whether it is possible to omit the assumption that

$$P^{s,0} = P^{s,1} = \ldots = P^{s,j_s-1} = 0, \qquad s = 1, \ldots, n. \tag{1.2}$$

He tried to solve this problem by restricting the set of mappings considered. He obtained an affirmative answer for every $\mathbf{t} \in \mathbb{R}_+^n$ under the additional assumption that $g : \mathbb{B}^{\mathbf{t}} \to \mathbb{B}^{\mathbf{t}/\mathbf{j}}$ is holomorphic in $\overline{\mathbb{B}^{\mathbf{t}}}$. Another approach is to reduce the generality of the domains $\mathbb{B}^{\mathbf{t}}$. We have proved in [6] that assumption (1.2) is not necessary when $\mathbf{t} = \mathbf{j} = (j, \ldots, j) \in \mathbb{N}^n$. In this paper we generalize this result to the case $\mathbf{t} = \mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$.

The main theorem (Theorem 3) and its proof are given in Section 3. There we apply some properties of $(\mathbf{j}; k)$-symmetrical functions; these properties are presented in Section 2.

We now introduce some further notations. Let $\mathbb{Z}$ denote the set of all integers. For $\lambda \in \mathbb{C}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ and $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ we write $\lambda\mathbf{j} = (\lambda j_1, \ldots, \lambda j_n)$, $\lambda^{\mathbf{j}} z = (\lambda^{j_1} z_1, \ldots, \lambda^{j_n} z_n)$, $\mathbf{j} + \mathbf{l} = (j_1 + l_1, \ldots, j_n + l_n)$, $\mathbf{j}\mathbf{l} = (j_1 l_1, \ldots, j_n l_n)$ and $|\mathbf{j}| = j_1 + \ldots + j_n$.

For every fixed $k \in \mathbb{N}$, $k \geq 2$, let $\mathcal{K} = \{0, 1, \ldots, k-1\}$ and $\varepsilon = \exp(2\pi i/k)$. We will use the equality

$$\sum_{\mathbf{j} \in \mathcal{K}^n} \varepsilon^{|\mathbf{j}\mathbf{l}|} = \begin{cases} k^n & \text{if } k \mid \mathbf{l}, \\ 0 & \text{if } k \nmid \mathbf{l}, \end{cases} \qquad \mathbf{l} \in \mathbb{Z}^n, \tag{1.3}$$

where $k \mid \mathbf{l}$ means that there exists an $\mathbf{m} \in \mathbb{Z}^n$ such that $\mathbf{l} = k\mathbf{m}$. This formula follows directly from the equalities

$$\sum_{\mathbf{j} \in \mathcal{K}^n} \varepsilon^{|\mathbf{j}\mathbf{l}|} = \sum_{j_1=0,\ldots,j_n=0}^{k-1} \prod_{s=1}^{n} \varepsilon^{j_s l_s} = \prod_{s=1}^{n} \sum_{j_s=0}^{k-1} \varepsilon^{j_s l_s}$$

and from the well known fact that $\sum_{j=0}^{k-1} \varepsilon^{jl}$ is k if $k \mid l$ and zero otherwise, for $l \in \mathbb{Z}$.

2. $(\mathbf{j};k)$-symmetrical functions. Fix $k\in\mathbb{N}$, $k\geq 2$, and let $\varepsilon=\exp(2\pi i/k)$. A nonempty subset U of $\mathbb{C}^n$ will be called *separately k-symmetrical* if for any $(z_1,\ldots,z_n)\in U$ and $s=1,\ldots,n$ the point $(z_1,\ldots,z_{s-1},\varepsilon z_s,z_{s+1},\ldots,z_n)$ is also in U.

Let U be a separately k-symmetrical subset of $\mathbb{C}^n$, $\mathcal{F}(U)$ be the vector space of all functions $F:U\to\mathbb{C}$, and $\mathcal{L}_{\mathbf{j}}:\mathcal{F}(U)\to\mathcal{F}(U)$, for $\mathbf{j}\in\mathbb{Z}^n$, be a linear operator defined as follows:

$$(\mathcal{L}_{\mathbf{j}}F)(z)=F(\varepsilon^{\mathbf{j}}z),\qquad z\in U,\ F\in\mathcal{F}(U). \tag{2.1}$$

A function $F\in\mathcal{F}(U)$ will be called $(\mathbf{j};k)$-*symmetrical* if for every $\mathbf{l}\in\mathbb{Z}^n$,

$$\mathcal{L}_{\mathbf{l}}F=\varepsilon^{|\mathbf{jl}|}F. \tag{2.2}$$

It is easy to see that (2.2) is equivalent to

$$F(z_1,\ldots,z_{s-1},\varepsilon z_s,z_{s+1},\ldots,z_n)=\varepsilon^{j_s}F(z)$$

for every $z=(z_1,\ldots,z_n)\in U$ and $s=1,\ldots,n$.

The $(\mathbf{j};k)$-symmetrical functions form a complex linear subspace of $\mathcal{F}(U)$, denoted by $\mathcal{F}_{\mathbf{j}}(U)$. From now on, we write $\mathcal{F}_{\mathbf{j}}$ and $\mathcal{F}$ for $\mathcal{F}_{\mathbf{j}}(U)$ and $\mathcal{F}(U)$, respectively.

The analysis of the spaces $\mathcal{F}_{\mathbf{j}}$ can be restricted to the case when $\mathbf{j}\in\mathcal{K}^n$, because $\mathcal{L}_{\mathbf{j}+k\mathbf{l}}=\mathcal{L}_{\mathbf{j}}$ and consequently $\mathcal{F}_{\mathbf{j}+k\mathbf{l}}=\mathcal{F}_{\mathbf{j}}$ for $\mathbf{l}\in\mathbb{Z}^n$.

Observe that the notion of a $(\mathbf{j};k)$-symmetrical function $F:U\to\mathbb{C}$ on a separately k-symmetrical set $U\subset\mathbb{C}^n$ coincides for $n=1$ with the notion of a $(j;k)$-symmetrical function $F:U\to\mathbb{C}$ on a k-symmetrical set $U\subset\mathbb{C}$, given in [5].

For $\mathbf{j}\in\mathcal{K}^n$, define the following operators $\boldsymbol{\pi}_{\mathbf{j}}$ on $\mathcal{F}$:

$$\boldsymbol{\pi}_{\mathbf{j}}F=k^{-n}\sum_{\mathbf{l}\in\mathcal{K}^n}\varepsilon^{-|\mathbf{jl}|}\mathcal{L}_{\mathbf{l}}F,\qquad F\in\mathcal{F}. \tag{2.3}$$

Now we give a useful decomposition theorem.

THEOREM 1. *Let U be a separately k-symmetrical subset of $\mathbb{C}^n$ and let $F\in\mathcal{F}$. Then*

$$F=\sum_{\mathbf{j}\in\mathcal{K}^n}\boldsymbol{\pi}_{\mathbf{j}}F, \tag{2.4}$$

$$\boldsymbol{\pi}_{\mathbf{j}}F\in\mathcal{F}_{\mathbf{j}},\qquad \mathbf{j}\in\mathcal{K}^n. \tag{2.5}$$

The above decomposition is unique in the following sense: if

$$F=\sum_{\mathbf{j}\in\mathcal{K}^n}F_{\mathbf{j}} \tag{2.6}$$

where $F_{\mathbf{j}}\in\mathcal{F}_{\mathbf{j}}$ for $\mathbf{j}\in\mathcal{K}^n$, then $F_{\mathbf{j}}=\boldsymbol{\pi}_{\mathbf{j}}F$.

Proof. (2.4) follows from (2.3), (1.3) and (2.1):

$$\sum_{\mathbf{j}\in\mathcal{K}^n} \boldsymbol{\pi}_{\mathbf{j}} F = \sum_{\mathbf{j}\in\mathcal{K}^n} k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{jl}|} \mathcal{L}_{\mathbf{l}} F = \sum_{\mathbf{l}\in\mathcal{K}^n} k^{-n} \mathcal{L}_{\mathbf{l}} F \sum_{\mathbf{j}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{jl}|} = \mathcal{L}_{\mathbf{0}} F = F.$$

Now let $\mathbf{m} \in \mathcal{K}^n$. Since (2.3) and (2.1) give

$$\begin{aligned}
\mathcal{L}_{\mathbf{m}}(\boldsymbol{\pi}_{\mathbf{j}} F) &= \mathcal{L}_{\mathbf{m}}\Big(k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{jl}|} \mathcal{L}_{\mathbf{l}} F\Big) = k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{jl}|} \mathcal{L}_{\mathbf{m}} \mathcal{L}_{\mathbf{l}} F \\
&= \varepsilon^{|\mathbf{jm}|} k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{j}(\mathbf{l}+\mathbf{m})|} \mathcal{L}_{\mathbf{l}+\mathbf{m}} F = \varepsilon^{|\mathbf{jm}|} k^{-n} \sum_{\mathbf{s}=\mathbf{m}}^{\mathbf{m}+(k-1,\ldots,k-1)} \varepsilon^{-|\mathbf{js}|} \mathcal{L}_{\mathbf{s}} F \\
&= \varepsilon^{|\mathbf{jm}|} k^{-n} \sum_{\mathbf{s}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{js}|} \mathcal{L}_{\mathbf{s}} F = \varepsilon^{|\mathbf{jm}|} \pi_{\mathbf{j}} F,
\end{aligned}$$

we see that $\boldsymbol{\pi}_{\mathbf{j}} F$ satisfies condition (2.2), and consequently (2.5) holds.

To prove the uniqueness, suppose that (2.6) holds and $F_{\mathbf{j}} \in \mathcal{F}_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{K}^n$. Then, in view of (2.3), (2.5), (2.2) and (1.3), for every $\mathbf{m} \in \mathcal{K}^n$ we obtain

$$\begin{aligned}
\boldsymbol{\pi}_{\mathbf{m}} F &= \boldsymbol{\pi}_{\mathbf{m}}\Big(\sum_{\mathbf{j}\in\mathcal{K}^n} F_{\mathbf{j}}\Big) = \sum_{\mathbf{j}\in\mathcal{K}^n} \boldsymbol{\pi}_{\mathbf{m}} F_{\mathbf{j}} = \sum_{\mathbf{j}\in\mathcal{K}^n} k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{lm}|} \mathcal{L}_{\mathbf{l}} F_{\mathbf{j}} \\
&= \sum_{\mathbf{j}\in\mathcal{K}^n} k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{lm}|} \varepsilon^{|\mathbf{jl}|} F_{\mathbf{j}} = \sum_{\mathbf{j}\in\mathcal{K}^n} k^{-n} F_{\mathbf{j}} \sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{|\mathbf{l}(\mathbf{j}-\mathbf{m})|} = F_{\mathbf{m}}. \ \blacksquare
\end{aligned}$$

The functions $F_{\mathbf{j}} = \pi_{\mathbf{j}} F$ will be called the $(\mathbf{j};k)$-*symmetrical parts* of F.

Theorem 1 can also be proved by the methods of the representation theory of finite groups (see for instance [2]). Another proof of Theorem 1, in the case of $n = 2$, has been given in [7] by a reduction to the case $n = 1$, considered in [5].

Now we give two corollaries of Theorem 1.

COROLLARY 1. *Let $U \subset \mathbb{C}^n$ be separately k-symmetrical, $F \in \mathcal{F}$ and $F_{\mathbf{j}}$ be its $(\mathbf{j};k)$-symmetrical parts, $\mathbf{j} = (j_1,\ldots,j_n) \in \mathcal{K}^n$. Then*

$$\sum_{\mathbf{l}\in\mathcal{K}^n} |F(\varepsilon^{\mathbf{l}} z)|^2 = k^n \sum_{\mathbf{j}\in\mathcal{K}^n} |F_{\mathbf{j}}(z)|^2, \qquad z \in U.$$

Proof. Since $F_{\mathbf{j}} = \boldsymbol{\pi}_{\mathbf{j}} F$, we have

$$k^{2n} \sum_{\mathbf{j}\in\mathcal{K}^n} |F_{\mathbf{j}}(z)|^2 = \sum_{\mathbf{j}\in\mathcal{K}^n} \Big(\sum_{\mathbf{l}\in\mathcal{K}^n} \varepsilon^{-|\mathbf{jl}|} F(\varepsilon^{\mathbf{l}} z)\Big)\Big(\sum_{\mathbf{m}\in\mathcal{K}^n} \varepsilon^{|\mathbf{jm}|} \overline{F(\varepsilon^{\mathbf{m}} z)}\Big).$$

Now change the order of summation and use (1.3). $\blacksquare$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for $\mathbf{m} = (m_1,\ldots,m_n) \in \mathbb{N}_0^n$, $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, let $a_{\mathbf{m}} z^{\mathbf{m}} = a_{\mathbf{m}} z_1^{m_1} \cdot \ldots \cdot z_n^{m_n}$.

COROLLARY 2. *Let $U \subset \mathbb{C}^n$ be a bounded complete Reinhardt domain with centre at the origin, and let $F \in \mathcal{F}$ be a holomorphic function of the* *form*

$$F(z)=\sum_{\mathbf{m}\in\mathbb{N}_0^n} a_{\mathbf{m}} z^{\mathbf{m}}, \qquad z=(z_1,\dots,z_n)\in U.$$

Then, for any fixed $k\in\mathbb{N}$, $k\ge 2$, *and* $\mathbf{j}\in\mathcal{K}^n$,

$$F_{\mathbf{j}}(z)=\sum_{\mathbf{d}\in\mathbb{N}_0^n} a_{\mathbf{j}+k\mathbf{d}} z^{\mathbf{j}+k\mathbf{d}}, \qquad z\in U. \tag{2.7}$$

Proof. As U is a separately k-symmetrical domain for every $k\ge 2$ and F is the sum of an absolutely convergent power series, we can change the order of summation after using formula (2.3). Now apply (1.3). ■

For $s\in\{1,\dots,n\}$ and $\mathbf{j}=(j_1,\dots,j_n)\in\mathbb{N}^n$ let $\mathbf{j}^s=(0,\dots,0,j_s,0,\dots,0)$. Now we prove a uniqueness theorem for $(\mathbf{j};k)$-symmetrical functions.

THEOREM 2. *Let* $\mathbf{j}=(j_1,\dots,j_n)\in\mathbb{N}^n\cap\mathcal{K}^n$. *If* $f=(f^1,\dots,f^n):\mathbb{B}^{\mathbf{j}}\to\mathbb{B}$ *is holomorphic and the* $(\mathbf{j}^s;k)$*-symmetrical part* $f^s_{\mathbf{j}^s}$, $s=1,\dots,n$, *of* f^s *has the form*

$$f^s_{\mathbf{j}^s}(z)=z_s^{j_s}, \qquad z=(z_1,\dots,z_n)\in\mathbb{B}^{\mathbf{j}}, \tag{2.8}$$

then

$$f(z)=(z_1^{j_1},\dots,z_n^{j_n}), \qquad z=(z_1,\dots,z_n)\in\mathbb{B}^{\mathbf{j}}.$$

Proof. In view of the uniqueness of the decomposition (2.6) it is sufficient to show that the $(\mathbf{l};k)$-symmetrical part $f^q_{\mathbf{l}}$ of f^q, $q=1,\dots,n$, vanishes in $\mathbb{B}^{\mathbf{j}}$ if $\mathbf{l}\in\mathcal{K}^n$ and $\mathbf{l}\ne\mathbf{j}^q$. Observe that applying Corollary 1 to f^s, $s=1,\dots,n$, we obtain

$$k^{-n}\sum_{s=1}^{n}\sum_{\mathbf{m}\in\mathcal{K}^n}|f^s(\varepsilon^{\mathbf{m}}z)|^2=\sum_{s=1}^{n}|f^s_{\mathbf{j}^s}(z)|^2+\sum_{s=1}^{n}\sum_{\mathbf{j}^s\ne\mathbf{l}\in\mathcal{K}^n}|f^s_{\mathbf{l}}(z)|^2. \tag{2.9}$$

Now change the order of summation on the left-hand side of this equality and use the assumption

$$\sum_{s=1}^{n}|f^s(\varepsilon^{\mathbf{m}}z)|^2<1, \qquad z\in\mathbb{B}^{\mathbf{j}},\ \mathbf{m}\in\mathcal{K}^n.$$

Simultaneously, apply (2.8) to the first sum on the right-hand side of (2.9). Then, for every component $|f^q_{\mathbf{l}}(z)|^2$, $q=1,\dots,n$ and $\mathbf{l}\ne\mathbf{j}^q$, of the multiple sum on the right-hand side of (2.9), we obtain

$$|f^q_{\mathbf{l}}(z)|^2<1-\sum_{s=1}^{n}|z_s|^{2j_s}, \qquad z=(z_1,\dots,z_n)\in\mathbb{B}^{\mathbf{j}}.$$

Thus for $q=1,\dots,n$ and $\mathbf{l}\ne\mathbf{j}^q$,

$$\max_{\partial B^{\mathbf{j}}(r)}|f^q_{\mathbf{l}}(z)|^2\le 1-r, \qquad r\in(0,1).$$

Hence, by the maximum principle,

$$0 \le \max_{B^{\mathbf{j}}(r)} |f^q_{\mathbf{l}}(z)| \le \sqrt{1-r}, \qquad r \in (0,1).$$

Now, observe that the maximum above is a nondecreasing function of $r \in (0,1)$, while the right-hand side decreases in $(0,1)$ and $\lim_{r\to 1^-} \sqrt{1-r} = 0$. Therefore, $\max_{\overline{B^{\mathbf{j}}(r)}} |f^q_{\mathbf{l}}(z)| = 0$, for $r \in (0,1)$ and $\mathbf{l} \ne \mathbf{j}^q$. Consequently, $f^q_{\mathbf{l}}(z) = 0$ for $\mathbf{l} \ne \mathbf{j}^q$, $q = 1,\dots,n$ and $z \in \mathbb{B}^{\mathbf{j}}$, because $\mathbb{B}^{\mathbf{j}} = \bigcup_{r\in(0,1)} \overline{B^{\mathbf{j}}(r)}$. ■

3. Main result

THEOREM 3. *Let* $\mathbf{j} = (j_1,\dots,j_n) \in \mathbb{N}^n$ *and assume that* $f = (f^1,\dots,f^n) : \mathbb{B}^{\mathbf{j}} \to \mathbb{B}$ *is holomorphic. If, for every* $s \in \{1,\dots,n\}$,

$$f^s(z) = \sum_{\nu=0}^{\infty} P^{s,\nu}(z), \qquad z \in \mathbb{B}^{\mathbf{j}},$$

is an expansion of f^s *into a series of homogeneous polynomials and*

$$P^{s,j_s}(z) = z_s^{j_s}, \qquad z = (z_1,\dots,z_n) \in \mathbb{B}^{\mathbf{j}}, \tag{3.1}$$

then

$$f(z) = (z_1^{j_1},\dots,z_n^{j_n}), \qquad z = (z_1,\dots,z_n) \in \mathbb{B}^{\mathbf{j}}.$$

Proof. Set $k = 1+\max(j_1,\dots,j_n)$ and let $g = (g^1,\dots,g^n)$ where $g^s = f^s_{\mathbf{j}^s}$ for $\mathbf{j}^s = (0,\dots,0,j_s,0,\dots,0)$ and $s = 1,\dots,n$. Since f is holomorphic in $\mathbb{B}^{\mathbf{j}}$, g is holomorphic in $\mathbb{B}^{\mathbf{j}}$. We will show that g satisfies the assumptions of the Janiec theorem with $\mathbf{t} = \mathbf{j}$.

Observe first that g maps $\mathbb{B}^{\mathbf{j}}$ into $\mathbb{B}$. Indeed, from the definition of g^s, in view of Corollary 1 and by the assumption that $f(\mathbb{B}^{\mathbf{j}}) \subset \mathbb{B}$, we have, for $z \in \mathbb{B}^{\mathbf{j}}$,

$$\begin{aligned}\sum_{s=1}^{n} |g^s(z)|^2 &= \sum_{s=1}^{n} |f^s_{\mathbf{j}^s}(z)|^2 \le k^{-n} \sum_{s=1}^{n} \sum_{\mathbf{l}\in\mathcal{K}^n} |f^s(\varepsilon^{\mathbf{l}} z)|^2 \\ &= k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} \sum_{s=1}^{n} |f^s(\varepsilon^{\mathbf{l}} z)|^2 < k^{-n} \sum_{\mathbf{l}\in\mathcal{K}^n} 1 = 1.\end{aligned}$$

Now we show that

$$g^s(z) = z_s^{j_s} + \sum_{\nu>j_s}^{\infty} \widetilde{P^{s,\nu}}(z), \qquad z = (z_1,\dots,z_n) \in \mathbb{B}^{\mathbf{j}}, \tag{3.2}$$

where $\widetilde{P^{s,\nu}}$ are homogeneous polynomials of order ν.

Indeed, as f^s is holomorphic, from (3.1) we have

$$f^s(z) = \sum_{\mathbf{m}\in\mathbb{N}_0^n} a_{\mathbf{m}} z^{\mathbf{m}}, \qquad z = (z_1,\dots,z_n) \in \mathbb{B}^{\mathbf{j}},$$

where $a_{\mathbf{m}} = 1$ for $\mathbf{m} = \mathbf{j}^s$ and $a_{\mathbf{m}} = 0$ for the remaining $\mathbf{m}$ such that $|\mathbf{m}| = j_s$. Therefore, by Corollary 2,

$$g^s(z) = f^s_{\mathbf{j}^s}(z) = z_s^{j_s} + \sum_{\mathbf{0} \neq \mathbf{d} \in \mathbb{N}_0^n} a_{\mathbf{j}^s + k\mathbf{d}} z^{\mathbf{j}^s + k\mathbf{d}}.$$

This implies (3.2).

From the Janiec uniqueness theorem we infer that, for $s = 1, \ldots, n$,

$$g^s(z) = z_s^{j_s}, \qquad z = (z_1, \ldots, z_n) \in \mathbb{B}^{\mathbf{j}}.$$

Since $g^s = f^s_{\mathbf{j}^s}$, this means that for every $s = 1, \ldots, n$ the $(\mathbf{j}^s; k)$-symmetrical part $f^s_{\mathbf{j}^s}$ of f^s satisfies (2.8). Theorem 2 also shows that

$$f(z) = (z_1^{j_1}, \ldots, z_n^{j_n}), \ z = (z_1, \ldots, z_n) \in \mathbb{B}^{\mathbf{j}}. \ \blacksquare$$

Remark 1. Putting $\mathbf{j} = \mathbf{1} = (1, \ldots, 1)$ in Theorem 3, we see that the assumption $g(0) = 0$ is not necessary in the Cartan theorem with $\Omega = \mathbb{B}$ and $b = 0$.

Remark 2. If we put $n = 1$ in Theorem 3, we obtain Gutzmer's uniqueness theorem.

References

[1] H. Cartan, *Les fonctions de deux variables complexes et la problème de la représentation analytique*, J. Math. Pures Appl. 96 (1931), 1–114.

[2] W. Fulton and J. Harris, *Representation Theory*, Grad. Texts in Math. 129, Springer, 1991.

[3] A. Gutzmer, *Ein Satz über Potenzreihen*, Math. Ann. 32 (1888), 596–600.

[4] E. Janiec, *Some uniqueness theorems concerning holomorphic mappings*, Demonstratio Math. 23 (1990), 879–892.

[5] P. Liczberski and J. Połubiński, *On (j, k)-symmetrical functions*, Math. Bohem. 120 (1995), 13–28.

[6] —, —, *Some remarks on Janiec uniqueness theorem for holomorphic mappings in C^n*, Ann. Univ. Mariae Curie-Skłodowska 53 (1999), 115–119.

[7] J. Połubiński, *Symmetrical functions of several variables and their applications to solving some functional equations*, Math. Bohem. (2002), to appear.

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