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# On fractional iterates of a homeomorphism of the plane

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**Abstract.** We find all continuous iterative roots of $n$th order of a Sperner homeomorphism of the plane onto itself.

**1. Introduction.** In the present paper we shall give all continuous solutions of the functional equation

$$g^n(x) = f(x) \quad \text{for } x \in \mathbb{R}^2, \tag{1}$$

where $n \in \mathbb{N}$, $n > 1$ and $f$ is a given *Sperner homeomorphism* of $\mathbb{R}^2$ onto itself, i.e., $f$ is a homeomorphism of $\mathbb{R}^2$ onto itself which satisfies the following condition:

(S) every Jordan domain $B$ meets at most a finite number of its images $f^n[B]$, $n \in \mathbb{Z}$,

where by a *Jordan domain* is meant the union of a Jordan curve $C$ and the *inside* of $C$ [i.e., the bounded component of $\mathbb{R}^2 \setminus C$].

MAIN RESULT. *Let $f$ be a Sperner homeomorphism of the plane onto itself. Then there exists a continuous solution $g$ of equation* (1) *if and only if one of the following three conditions holds*:

(a) *$f$ preserves orientation and $n$ is odd*;
(b) *$f$ reverses orientation and $n$ is odd*;
(c) *$f$ preserves orientation and $n$ is even.*

**2. Preliminaries.** The *index* $\mathrm{Ind}_C$ of $C$, where $C$ is a piecewise continuously differentiable closed curve in $\mathbb{R}^2$ defined on the unit interval, is the function on $\mathbb{R}^2 \setminus C$ defined by

$$\mathrm{Ind}_C(x) := \frac{1}{2\pi i}\int_0^1 \frac{C'(t)}{C(t)-x}\,dt \quad \text{for } x \in \mathbb{R}^2 \setminus C.$$

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By the *index* of a closed curve $C$ is meant the index of any piecewise continuously differentiable closed curve $C_1$ which is homotopic to $C$ (such a curve $C_1$ exists: see [3, p. 247]). On account of the Cauchy integral theorem the definition does not depend on the choice of $C_1$.

Furthermore, for every homeomorphism $f$ of $\mathbb{R}^2$ into itself there exists exactly one $d_f \in \{-1, 1\}$ such that

$$\mathrm{Ind}_C(x) = d_f \cdot \mathrm{Ind}_{f[C]}(f(x)) \tag{2}$$

for every Jordan curve $C$ and every $x \in \mathbb{R}^2 \setminus C$ (see [7, p. 197]).

The number $d_f$ is called the *degree* of $f$ and denoted by $\deg f$. We shall say that a homeomorphism $f$ of $\mathbb{R}^2$ into itself *preserves orientation* if $\deg f = 1$, and it *reverses orientation* if $\deg f = -1$.

We will study the following cases:

($\mathrm{A}_1$) there exists a homeomorphism $\varphi$ of the plane onto itself satisfying the Abel equation

$$\varphi(f(x)) = \varphi(x) + (1, 0) \quad \text{for } x \in \mathbb{R}^2; \tag{3}$$

($\mathrm{A}_2$) there exists a homeomorphism $\varphi$ of the plane onto itself satisfying the equation

$$\varphi(f(x)) = S_0(\varphi(x)) + (1, 0) \quad \text{for } x \in \mathbb{R}^2, \tag{4}$$

where

$$S_0(x_1, x_2) = (x_1, -x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2. \tag{5}$$

E. Sperner [8] proved that a homeomorphism $f$ of the plane onto itself satisfies ($\mathrm{A}_1$) if and only if it preserves orientation and condition (S) holds. Furthermore, D. Betten [2] proved that $f$ satisfies ($\mathrm{A}_2$) if and only if it is an orientation reversing Sperner homeomorphism of the plane onto itself.

A homeomorphic image of a straight line which is a closed set is called a *line*. Let us consider the following condition:

(B) there exists a line $K$ such that

$$K \cap f[K] = \emptyset, \tag{6}$$

$$U^0 \cap f[U^0] = \emptyset, \tag{7}$$

$$\bigcup_{n \in \mathbb{Z}} f^n[U^0] = \mathbb{R}^2, \tag{8}$$

where $U^0 := M^0 \cup f[K]$ and $M^0$ is the strip bounded by $K$ and $f[K]$.

Geometrically speaking, the condition is that the strips between two consecutive iterates of $K$ are pairwise disjoint and each point of the plane belongs either to one of the strips or to an iterate of $K$.

In [6] we have constructed all continuous and homeomorphic solutions of the Abel equation

$$\varphi(f(x)) = \varphi(x) + a \quad \text{for } x \in \mathbb{R}^2, \tag{9}$$

where $a \neq (0,0)$ and $f$ is an orientation preserving homeomorphism of the plane onto itself satisfying (B). Moreover, it has been proved that for every homeomorphism $f$ of $\mathbb{R}^2$ onto itself which preserves orientation, conditions ($A_1$) and (B) are equivalent.

**3. Equation with reflection.** In this section we are concerned with continuous and homeomorphic solutions of the functional equation

$$\varphi(f(x)) = S_k(\varphi(x)) + a \quad \text{for } x \in \mathbb{R}^2, \tag{10}$$

where $f$ is a given orientation reversing homeomorphism of the plane onto itself such that condition (B) holds, $S_k$ is the *reflection* in a given straight line $k$ and the vector $a \in \mathbb{R}^2$ is not perpendicular to $k$.

The following statement is well known:

Proposition 1. *Let $k$ be a straight line on the plane and let $a = (a_1, a_2) \in \mathbb{R}^2$ be a vector which is not perpendicular to $k$. Then there exist a straight line $l$ and a vector $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ which is parallel to $l$ and such that*

$$S_k(x) + a = S_l(x) + b \quad \text{for } x \in \mathbb{R}^2. \tag{11}$$

According to Proposition 1 we can replace the right-hand side of (10) by $S_l(\varphi(x)) + b$, where the vector $b$ is parallel to $l$. Therefore we write (10) in the form

$$\varphi(f(x)) = S_{l,b}(\varphi(x)) \quad \text{for } x \in \mathbb{R}^2, \tag{12}$$

where $S_{l,b}$ denotes the *glide reflection* which is the composition of the reflection in $l$ and the translation by the vector $b$.

In the case where $l = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ and $b = (1,0)$ the glide reflection $S_{l,b}$ will be denoted by $S_1$. Thus

$$S_1(x) = S_0(x) + (1,0) \quad \text{for } x \in \mathbb{R}^2, \tag{13}$$

where $S_0$ is given by (5).

Now we prove

Lemma 1. *Let $l$ be a straight line on the plane. Let $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be a vector parallel to $l$. Then there exists a homeomorphism $\psi$ of the plane onto itself such that*

$$S_1 = \psi^{-1} \circ S_{l,b} \circ \psi, \tag{14}$$

*where $S_1$ is given by* (13).

*Proof.* In case $b_1 \neq 0$,

$$l = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = \frac{b_2}{b_1} x_1 + d \right\},$$

for some $d \in \mathbb{R}$, whereas in case $b_1 = 0$, $l$ has the form

$$l = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = d'\}$$

with some $d' \in \mathbb{R}$. Set

$$\psi(x_1, x_2) = \left( b_1 x_1 - \frac{b_2}{\sqrt{b_1^2 + b_2^2}} x_2, b_2 x_1 + \frac{b_1}{\sqrt{b_1^2 + b_2^2}} x_2 + d \right) \tag{15}$$

if $b_1 \neq 0$, and

$$\psi(x_1, x_2) = (x_2 + d', b_2 x_1) \tag{16}$$

if $b_1 = 0$. ∎

Now we prove

PROPOSITION 2. *Let $l$ be any straight line on the plane. Let $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be a vector parallel to $l$. Then $\varphi$ is a solution of equation* (12) *if and only if it has the form*

$$\varphi = \psi \circ \varphi_0, \tag{17}$$

*where $\varphi_0$ satisfies equation* (4) *and $\psi$ is given by* (15) *in case $b_1 \neq 0$, and by* (16) *in case $b_1 = 0$.*

*Proof.* First we show that $\varphi$ given by (17) is a solution of (12). Since $\varphi_0$ solves (4), we have, by (14),

$$\varphi_0(f(x)) = (\psi^{-1} \circ S_{l,b} \circ \psi)(\varphi_0(x)) \quad \text{for } x \in \mathbb{R}^2,$$

where $\psi$ is given by (15) or (16). Hence

$$(\psi \circ \varphi_0)(f(x)) = S_{l,b}((\psi \circ \varphi_0)(x)) \quad \text{for } x \in \mathbb{R}^2.$$

Thus $\psi \circ \varphi_0$ is a solution of (12).

Let now $\varphi$ be any solution of (12). Set $\varphi_0 := \psi^{-1} \circ \varphi$. By (12) and (14) we have

$$\varphi(f(x)) = (\psi \circ S_1 \circ \psi^{-1})(\varphi(x)) \quad \text{for } x \in \mathbb{R}^2.$$

Hence

$$(\psi^{-1} \circ \varphi)(f(x)) = S_1((\psi^{-1} \circ \varphi)(x)) \quad \text{for } x \in \mathbb{R}^2.$$

Thus $\varphi_0$ is a solution of (4) such that (17) holds. ∎

**4. Equations with reflection in the $x$-axis.** By Propositions 1 and 2 in order to find all solutions of (10) it suffices to know all solutions of (4).

LEMMA 2. *If a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies* ($A_2$), *then it is an orientation reversing homeomorphism of the plane onto itself such that condition* (B) *holds.*

*Proof.* Let $\varphi$ be a homeomorphism of the plane onto itself which satisfies (4). Then

$$f = \varphi^{-1} \circ S_1 \circ \varphi,$$

where $S_1$ is given by (13). Hence $f$ is a homeomorphism of the plane onto itself which reverses orientation.

Putting $K := \varphi^{-1}[L]$, where $L := \{0\} \times \mathbb{R}$, we get condition (B). This completes the proof. ∎

All continuous and homeomorphic solutions of (4) are found in

THEOREM 1. *Let $f$ be a homeomorphism of the plane onto itself which reverses orientation. Assume that condition* (B) *is satisfied. Let $\varphi_0 : U^0 \cup K \to \mathbb{R}^2$ be continuous and suppose that*

$$\varphi_0(f(x)) = S_0(\varphi_0(x)) + (1,0) \quad \text{for } x \in K,$$

*where $S_0$ is given by* (5). *Then*:

(a) *There exists a unique solution $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of equation* (4) *such that*

$$\varphi(x) = \varphi_0(x) \quad \text{for } x \in U^0 \cup K.$$

*The function $\varphi$ is continuous.*

(b) *If $\varphi_0$ is one-to-one and $\varphi_0[U^0] \cap (S_0^n[\varphi_0[U^0]] + (n,0)) = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $\varphi$ is a homeomorphism.*

(c) *If $\varphi_0$ is one-to-one, $\varphi_0[K]$ is a line and $\varphi_0[K] \cap D_\gamma \neq \emptyset$ for all $\gamma \in \mathbb{R}$, where $D_\gamma = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = \gamma\}$, then $\varphi$ is a homeomorphism.*

(d) *If $\varphi_0$ is as in* (c) *and $\varphi_0[M^0] = N^0$, where $N^0$ is the strip bounded by $\varphi_0[K]$ and $S_0[\varphi_0[K]] + (1,0)$, then $\varphi$ is a homeomorphism of $\mathbb{R}^2$ onto itself.*

*Proof.* By (6) we have

$$f^n[K] \cap f^{n+1}[K] = \emptyset \quad \text{for } n \in \mathbb{Z}.$$

Furthermore, $f^n[K]$ is a line for $n \in \mathbb{Z}$, as so is $K$. For each $n \in \mathbb{Z}$, denote by $M^n$ the strip bounded by $f^n[K]$ and $f^{n+1}[K]$. Let $U^n := M^n \cup f^{n+1}[K]$ for $n \in \mathbb{Z}$. Then $f^n[U^0] = U^n$ for $n \in \mathbb{Z}$.

Define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$\varphi(x) = S_0^n(\varphi_0(f^{-n}(x))) + (n,0) \quad \text{for } x \in U^n,\ n \in \mathbb{Z}.$$

By (7) and (8), $\varphi$ is a function defined on $\mathbb{R}^2$. The rest of the proof is similar to that of the theorem describing the construction of solutions of (9). ∎

From Theorem 1(d), by the Schönflies Theorem (see [1]), we get

COROLLARY 1. *Let $f$ be an orientation reversing homeomorphism of $\mathbb{R}^2$ onto itself. Then* (B) *implies* ($A_2$).

Moreover, on account of Lemma 2 and Corollary 1 we have

Corollary 2. *Let $f$ be an orientation reversing homeomorphism of $\mathbb{R}^2$ onto itself. Then conditions* (B) *and* ($A_2$) *are equivalent.*

As a consequence of the results of Sperner [8] and Betten [2], the result [6] described in Section 1, and Corollary 2 we have

Theorem 2. *Let $f$ be a homeomorphism of $\mathbb{R}^2$ onto itself. Then conditions* (B) *and* (S) *are equivalent.*

*Proof.* Let $f$ satisfy (B). Then either $\deg f = 1$, or $\deg f = -1$. If $\deg f = 1$, then $f$ satisfies ($A_1$) (see [6]). Hence condition (S) holds (see [8]). Likewise, if $\deg f = -1$, then by Corollary 1, $f$ satisfies ($A_2$), and consequently condition (S) holds (see [2]). In a similar manner we can show that (S) implies (B). ∎

**5. Roots of order preserving homeomorphisms.** In this section we shall find all continuous solutions of equation (1). Let us start with

Remark 1. *If $f$ is a homeomorphism of $\mathbb{R}^2$ onto itself, $g$ is continuous and $g^n = f$ for some $n \in \mathbb{N}$, then $g$ is also a homeomorphism of $\mathbb{R}^2$ onto itself.*

*Proof.* Since $f$ is a one-to-one map of $\mathbb{R}^2$ onto itself, so is $g$ (see e.g. [5, p. 422]). Thus $g$, being a continuous one-to-one mapping of the plane onto itself, is a homeomorphism (see e.g. [4, p. 186]). ∎

Now we prove

Proposition 3. *Let $f$ be a homeomorphism of $\mathbb{R}^2$ onto itself and $g$ be a continuous function such that $g^n = f$ for some $n \in \mathbb{N}$. If $f$ satisfies condition* (S), *then $g$ is a homeomorphism of $\mathbb{R}^2$ onto itself satisfying* (S).

*Proof.* Let $B$ be a Jordan domain. Then there exists a Jordan domain $D$ such that

$$g^r[B] \subset D \quad \text{for } r \in \{0, 1, \ldots, n-1\},$$

since $B$ is a compact set and $g$ is continuous. Hence

$$g^{mn+r}[B] \subset f^m[D] \quad \text{for } m \in \mathbb{Z},\ r \in \{0, 1, \ldots, n-1\}. \tag{18}$$

By (S) there exists $m_0 \in \mathbb{N}$ such that

$$f^m[D] \cap D = \emptyset \quad \text{for } |m| \ge m_0.$$

Hence, by (18),

$$g^{mn+r}[B] \cap B = \emptyset \quad \text{for } |m| \ge m_0 \text{ and } r = 0, 1, \ldots, n-1.$$

This means that

$$g^k[B] \cap B = \emptyset \quad \text{for } |k| \ge m_0 n. \ ∎$$

From Proposition 3 and Theorem 2 we get

COROLLARY 3. *Let $f$ be a homeomorphism of $\mathbb{R}^2$ onto itself and $g$ be a continuous function such that $g^n = f$ for some $n \in \mathbb{N}$. If $f$ satisfies condition* (B), *then $g$ is a homeomorphism of $\mathbb{R}^2$ onto itself satisfying* (B).

Now we shall give all continuous solutions of equation (1) in the case where $f$ preserves orientation. First note that from Remark 1 we can get

REMARK 2. *Let $f$ be a homeomorphism of $\mathbb{R}^2$ onto itself which preserves orientation. Let $g$ be a continuous function defined on $\mathbb{R}^2$ such that $g^n = f$ for some odd $n \in \mathbb{N}$. Then $g$ is a homeomorphism of $\mathbb{R}^2$ onto itself which preserves orientation.*

*Proof.* By the definition of the degree of a homeomorphism of the plane we have

$$(\deg g)^n = \deg f. \tag{19}$$

Hence $\deg g = 1$, since $\deg f = 1$ and $n$ is odd. ∎

Let

$$T_{1/n}(x_1, x_2) := (x_1 + 1/n, x_2) \qquad \text{for } (x_1, x_2) \in \mathbb{R}^2, \tag{20}$$

$$S_{1/n}(x_1, x_2) := (x_1 + 1/n, -x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2. \tag{21}$$

Now we can state

THEOREM 3. *Let $f$ be an orientation preserving Sperner homeomorphism of $\mathbb{R}^2$ onto itself. Then*

(a) *for every even $n \in \mathbb{N}$ a function $g$ is a continuous solution of equation* (1) *if and only if it can be expressed in either of the forms*

$$g = \varphi^{-1} \circ T_{1/n} \circ \varphi \tag{22}$$

*and*

$$g = \varphi^{-1} \circ S_{1/n} \circ \varphi, \tag{23}$$

*where $\varphi$ is a homeomorphic solution of* (3) *and $T_{1/n}$, $S_{1/n}$ are given by* (20) *and* (21), *respectively*;

(b) *for every odd $n \in \mathbb{N}$, $n > 1$, a function $g$ is a continuous solution of equation* (1) *if and only if it has the form* (22), *where $\varphi$ is a homeomorphic solution of* (3).

*Proof.* Let

$$T_1(x_1, x_2) := (x_1 + 1, x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

Take any homeomorphic solution $\varphi$ of (3). Fix $n \in \mathbb{N}$, $n > 1$. Let $g$ be given by (22). We shall show that $g$ is a solution of (1).

Since $T_{1/n}^n = T_1$, we have $g^n = \varphi^{-1} \circ T_1 \circ \varphi$. On the other hand $f = \varphi^{-1} \circ T_1 \circ \varphi$, since $\varphi$ satisfies (3). Thus $g^n = f$.

If $n$ is even, then $g$ given by (23) is also a solution of equation (1), since $S^n_{1/n} = T_1$ for every even $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$, $n > 1$. Let $g$ be any continuous solution of (1). Then, by Proposition 3 and Theorem 2, $g$ is a homeomorphism of $\mathbb{R}^2$ onto itself which satisfies (B). Moreover, if $n$ is odd, then by Remark 2, $g$ preserves orientation.

Conversely, assume that $g$ preserves orientation. Then, by the Schönflies theorem and the theorem describing the solutions of (9) given in [6], there exists a homeomorphism $\varphi$ satisfying

$$\varphi(g(x)) = \varphi(x) + (1/n, 0) \quad \text{for } x \in \mathbb{R}^2. \tag{24}$$

Hence

$$\varphi(g^n(x)) = \varphi(x) + (1, 0) \quad \text{for } x \in \mathbb{R}^2. \tag{25}$$

Since $g^n = f$, $\varphi$ is a solution of (3). By (24) the function $g$ has the form (22).

Now assume that $n$ is even and $g$ reverses orientation. Then, by Corollary 1 and Proposition 2, there exists a homeomorphism $\varphi$ satisfying

$$\varphi(g(x)) = S_0(\varphi(x)) + (1/n, 0) \quad \text{for } x \in \mathbb{R}^2. \tag{26}$$

Hence $\varphi$ is a solution of (25), since $S^n_0 = \mathrm{id}$ for every even $n \in \mathbb{Z}$. Thus $\varphi$ satisfies (3) and clearly (23) holds. ∎

From Theorem 3(a) we obtain the following

COROLLARY 4. *Let $f$ be an orientation preserving Sperner homeomorphism of $\mathbb{R}^2$ onto itself. Then for every even positive integer $n$ there exist solutions of equation* (1) *which preserve orientation and ones which reverse orientation.*

**6. Roots of order reversing homeomorphisms.** Now we find all continuous solutions of equation (1) in the case where $f$ reverses orientation. Immediately from Remark 1 and relation (19) we obtain

REMARK 3. *Let $f$ be an orientation preserving homeomorphism of the plane onto itself. Then*

(a) *if $g$ is a continuous function such that $g^n = f$ for some odd $n \in \mathbb{N}$, then $g$ is a homeomorphism of $\mathbb{R}^2$ onto itself which reverses orientation*;

(b) *if $n$ is even*, *then there exist no solutions of equation* (1).

Now we prove

THEOREM 4. *Let $f$ be an orientation reversing Sperner homeomorphism of $\mathbb{R}^2$ onto itself. Let $n$ be an odd integer greater than* 1. *Then a function $g$ is a continuous solution of equation* (1) *if and only if it has the form* (23), *where $\varphi$ is a homeomorphic solution of equation* (4) *and $S_{1/n}$ is given by* (21).

*Proof.* Let $\varphi$ be any homeomorphism of the plane onto itself satisfying (4). Assume that $g$ is given by (23). Then $g^n = \varphi^{-1} \circ S_1 \circ \varphi$, since $S_{1/n}^n = S_1$ for odd $n$, where $S_1$ is given by (13). Hence $g^n = f$, since $\varphi$ satisfies (4).

Let $g$ be a continuous solution of (1). Then, by Theorem 2 and Corollary 3, $g$ is a homeomorphism of the plane onto itself satisfying (B). Moreover, Remark 3(a) shows that $g$ reverses orientation. By Corollary 1 and Proposition 2, there exists a homeomorphism $\varphi$ satisfying (26). Hence

$$\varphi(g^n(x)) = S_0(\varphi(x)) + (1,0) \quad \text{for } x \in \mathbb{R}^2,$$

since $n$ is odd. Thus $\varphi$ is a solution of (4). ∎

REMARK 4. Theorems 3 and 4 and Remark 3(b) yield our Main Result.

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