Linear differential polynomials sharing the same 1-points with weight two

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Abstract. We prove a uniqueness theorem for meromorphic functions involving differential polynomials which improves some previous results and provides a better answer to a question of C. C. Yang.

1. Introduction and definitions. Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). If for \( a \in \mathbb{C} \cup \{\infty\} \), \( f - a \) and \( g - a \) have the same set of zeros with the same multiplicities, we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities) and if we do not consider the multiplicities, \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as those are available in [2].

In [9] C. C. Yang asked: What can be said if two nonconstant entire functions \( f, g \) share the value 0 CM and their first derivatives share the value 1 CM?

A number of authors have worked on this question of Yang (e.g. [3, 6, 7, 10, 11]). To answer the question of Yang, K. Shibazaki [7] proved the following result.

Theorem A. Let \( f \) and \( g \) be two entire functions of finite order. If \( f' \) and \( g' \) share the value 1 CM with \( \delta(0; f) > 0 \) and 0 being lacunary for \( g \) then either \( f \equiv g \) or \( f'g' \equiv 1 \).

Improving Theorem A, H. X. Yi [12] obtained the following theorem.

Theorem B. Let \( f, g \) be two entire functions such that \( f^{(n)} \) and \( g^{(n)} \) share the value 1 CM. If \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)}g^{(n)} \equiv 1 \).

For meromorphic functions H. X. Yi and C. C. Yang [13] proved the following result.

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Theorem C. Let \( f \) and \( g \) be two meromorphic functions such that \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \). If \( f^{(n)} \) and \( g^{(n)} \) share the value 1 CM with \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)}g^{(n)} \equiv 1 \).

In [3] the following question was asked: What can be said if two linear differential polynomials generated by two meromorphic functions \( f \) and \( g \) share the value 1 CM?

We denote by \( \Psi(D) \) a linear differential operator with constant coefficients of the form

\[
\Psi(D) = \sum_{i=1}^{p} \alpha_i D^i,
\]

where \( D = d/dz \).

Also we denote by \( N_k(r, a; f) \) the counting function of \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( \mu \) is counted \( \mu \) times if \( \mu \leq k \) and \( k \) times if \( \mu > k \), where \( k \) is a positive integer. We put

\[
\delta_k(a; f) = \limsup_{r \to \infty} \frac{N_k(r, a; f)}{T(r, f)}.
\]

Clearly \( \delta(a; f) \leq \delta_k(a; f) \leq \delta_{k-1}(a; f) \leq \ldots \leq \delta_1(a; f) = \Theta(a; f) \).

In [3] the following two theorems were proved.

Theorem D. Let \( f \) and \( g \) be two meromorphic functions such that

(i) \( \Psi(D)f, \Psi(D)g \) are nonconstant and share 1 CM, and

(ii) \[
\frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{4(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{4(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)},
\]

where \( \sum_{a \neq \infty} \delta_p(a; f) > 0 \) and \( \sum_{a \neq \infty} \delta_p(a; g) > 0 \). Then either \( [\Psi(D)f][\Psi(D)g] \equiv 1 \) or \( f - g \equiv s \) where \( s = s(z) \) is a solution of the differential equation \( \Psi(D)w = 0 \).

Theorem E. If \( f \) and \( g \) are of finite order then Theorem D still holds if condition (ii) is replaced by the following weaker one:

\[
\frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{2(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{2(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)},
\]

where \( \sum_{a \neq \infty} \delta_p(a; f) > 0 \) and \( \sum_{a \neq \infty} \delta_p(a; g) > 0 \).

H. X. Yi [10] also answered the question of Yang and proved the following result.
Theorem F. Let \( f \) and \( g \) be two nonconstant entire functions. Assume that \( f, g \) share 0 CM and \( f^{(n)}, g^{(n)} \) share 1 CM, where \( n \) is a nonnegative integer. If \( \delta(0; f) > 1/2 \) then either \( f \equiv g \) or \( f^{(n)} g^{(n)} \equiv 1 \).

As an application of Theorem D, in [3] the following answer to the question of Yang was given.

Theorem G. Let \( f \) and \( g \) be two nonconstant meromorphic functions with \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \). Suppose that \( f^{(n)}, g^{(n)} (n \geq 1) \) share 1 CM and \( f, g \) share a value \( b (\neq \infty) \) IM. If \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)} g^{(n)} \equiv 1 \).

The following example shows that in Theorems D and E sharing the value 1 cannot be relaxed from CM to IM.

Example 1. Let \( f = -ie^z, g = 2^{-p} e^{2z} - 2ie^2 \) and \( \Psi(D) = D^p \). Then
\[
\Psi(D)f, \Psi(D)g \text{ share the value } 1 \text{ IM and } \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) = 3/2
\]
but neither \( f \equiv g + Q \) nor \( [\Psi(D)f][\Psi(D)g] \equiv 1 \) where \( Q \) is a polynomial of degree at most \( p - 1 \).

Now one may ask the following question: Is it possible in any way to relax the nature of sharing the value 1 in Theorems D and E?

The purpose of the paper is to study this problem. We shall not only relax the nature of sharing the value 1 but also weaken the condition on deficiencies. To this end we consider a gradation of sharing of values which measures how close a shared value is to being shared IM or being shared CM and is called weighted sharing of values as introduced in [4, 5].

Definition 1. Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{ \infty \} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \).

If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f, g \) share a value \( a \) with weight \( k \) then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) \((\leq k)\) if and only if \( z_0 \) is a zero of \( g - a \) with multiplicity \( m \) \((\leq k)\), and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) \((> k)\) if and only if \( z_0 \) is a zero of \( g - a \) with multiplicity \( n \) \((> k)\) where \( m \) is not necessarily equal to \( n \).

We write “\( f, g \) share \((a, k)\)” to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \((a, k)\) then \( f, g \) share \((a, p)\) for any integer \( p \), \( 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \((a, 0)\) or \((a, \infty)\) respectively.

Definition 2. We denote by \( N(r, a; f | = 1) \) the counting function of simple \( a \)-points of \( f \).
Definition 3. If \( s \) is a positive integer, we denote by \( \overline{N}(r, a; f \mid \geq s) \) the counting function of those \( a \)-points of \( f \) whose multiplicities are greater than or equal to \( s \), where each \( a \)-point is counted only once.

Definition 4. Let \( f, g \) share a value \( a \) IM. We denote by \( \overline{N}_s(r, a; f, g) \) the counting function of those \( a \)-points of \( f \) whose multiplicities are not equal to multiplicities of the corresponding \( a \)-points of \( g \), where each \( a \)-point is counted only once.

Clearly \( \overline{N}_s(r, a; f, g) \equiv \overline{N}_s(r, a; g, f) \).

Definition 5 (cf. [1]). For a meromorphic function \( f \) we put
\[
T_0(r, f) = \int_1^r \frac{T(t, f)}{t} \, dt, \quad N_0(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} \, dt,
\]
\[
N_k^0(r, a; f) = \int_1^r \frac{N_k(t, a; f)}{t} \, dt, \quad m_0(r, f) = \int_1^r \frac{m(t, f)}{t} \, dt,
\]
\[
S_0(r, f) = \int_1^r \frac{S(t, f)}{t} \, dt.
\]

Definition 6. If \( f \) is a meromorphic function, we put, for \( a \in \mathbb{C} \cup \{\infty\} \),
\[
\delta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a; f)}{T_0(r, f)},
\]
\[
\Theta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}_0(r, a; f)}{T_0(r, f)},
\]
\[
\delta_k^0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_k^0(r, a; f)}{T_0(r, f)}.
\]

2. Lemmas. In this section we present some lemmas which will be needed in what follows. Let \( f, g \) be two nonconstant meromorphic functions and we put
\[
h = \left( \frac{f'}{f} - \frac{2f'}{f-1} \right) - \left( \frac{g''}{g'} - \frac{2g'}{g-1} \right).
\]

Lemma 1. If \( f, g \) share \((1, 1)\) and \( h \neq 0 \) then

(i) \( N(r, 1; f \mid = 1) \leq N(r, h) + S(r, f) + S(r, g) \),

(ii) \( N(r, 1; g \mid = 1) \leq N(r, h) + S(r, f) + S(r, g) \).

Proof. Since \( f, g \) share \((1, 1)\), it follows that a simple 1-point of \( f \) is a simple 1-point of \( g \) and conversely. Let \( z_0 \) be a simple 1-point of \( f \) and \( g \). Then by a simple calculation we see that in some neighbourhood of \( z_0 \),
\[
h = (z - z_0)\phi(z),
\]
where \( \phi \) is analytic at \( z_0 \).
Hence by the first fundamental theorem and the Milloux theorem [2, p. 47] we get

\[ N(r, 1; f | =1) \leq N(r, 0; h) \leq N(r, h) + S(r, f) + S(r, g), \]

which is (i).

Now (ii) follows from (i) because \( N(r, 1; f | =1) \equiv N(r, 1; g | =1). \) This proves the lemma.

**Lemma 2.** Let \( f, g \) share \((1,0)\) and \( h \neq 0. \) Then for any number \( b \) \((\neq 0, 1, \infty),\)

\[
N(r, h) \leq \overline{N}(r, \infty; f | \geq 2) + \overline{N}(r, 0; f | \geq 2) + \overline{N}(r, b; f | \geq 2) \\
+ \overline{N}(r, \infty; g | \geq 2) + \overline{N}(r, 0; g | \geq 2) + \overline{N}_\ast(r, 1; f, g) \\
+ \overline{N}_\odot(r, 0; f') + \overline{N}_\odot(r, 0; g'),
\]

where \( \overline{N}_\odot(r, 0; f') \) is the reduced counting function of those zeros of \( f' \) which are not zeros of \( f(f - 1)(f - b), \) and \( \overline{N}_\odot(r, 0; g') \) is the reduced counting function of those zeros of \( g' \) which are not zeros of \( g(g - 1). \)

**Proof.** We can easily verify that possible poles of \( h \) occur at (i) multiple zeros of \( f, g; \) (ii) multiple poles of \( f, g; \) (iii) zeros of \( f - 1, g - 1; \) (iv) multiple zeros of \( f - b; \) (v) zeros of \( f' \) which are not zeros of \( f(f - 1)(f - b); \) (vi) zeros of \( g' \) which are not zeros of \( g(g - 1). \)

Let \( z_0 \) be a zero of \( f - 1 \) with multiplicity \( m \) \((\geq 1)\) and of \( g - 1 \) with multiplicity \( n \) \((\geq 1). \) Then in some neighbourhood of \( z_0 \) we get

\[ h = \frac{(n - m)\psi}{z - z_0} + \phi, \]

where \( \phi, \psi \) are analytic at \( z_0 \) and \( \psi(z_0) \neq 0. \)

This shows that if \( m = n \) then \( z_0 \) is not a pole of \( h \) and if \( m \neq n \) then \( z_0 \) is a simple pole of \( h. \) Since all the poles of \( h \) are simple, the lemma is proved.

**Lemma 3.** If \( f, g \) share \((1,2)\) then

\[
N_\odot(r, 0; g') + \overline{N}(r, 1; g | \geq 2) + \overline{N}_\ast(r, 1; f, g) \\
\leq \overline{N}(r, \infty; g) + \overline{N}(r, 0; g) + S(r, g),
\]

where \( N_\odot(r, 0; g') \) is the counting function of those zeros of \( g' \) which are not zeros of \( g(g - 1). \)

**Proof.** Since \( f, g \) share \((1,2)\), it follows that \( \overline{N}_\ast(r, 1; f, g) \leq \overline{N}(r, 1; g | \geq 3). \) So remembering the definition of \( N_\odot(r, 0; g') \) we get
(1) \[ N_\otimes(r, 0; g') + N(r, 1; g \geq 2) + N_\ast(r, 1; f, g) + N(r, 0; g) = N(r, 0; g) \leq N_\otimes(r, 0; g') + N(r, 1; g \geq 2) + N(r, 1; g \geq 3) + N(r, 0; g) - N(r, 0; g) \leq N(r, 0; g'). \]

By the first fundamental theorem and the Milloux theorem [2, p. 55] we get

(2) \[ N(r, 0; g') \leq N(r, 0; g' / g) + N(r, 0; g) - N(r, 0; g) \leq N(r, 0; g') + N(r, 0; g) - N(r, 0; g) + S(r, g) = N(r, \infty; g) + N(r, 0; g) + N(r, 0; g) - N(r, 0; g) + S(r, g) = N(r, \infty; g) + N(r, 0; g) + S(r, g). \]

Now the lemma follows from (1) and (2).

Lemma 4 (see [1]). \( \lim_{r \to \infty} S_0(r, f) / T_0(r, f) = 0 \) through all values of \( r \).

Lemma 5 (see [3]). For \( a \in \mathbb{C} \cup \{ \infty \} \), \( \delta(a; f) \leq \delta_0(a; f) \), \( \Theta(a; f) \leq \Theta_0(a; f) \) and \( \delta_k(a; f) \leq \delta_k^0(a; f) \).

Lemma 6 (see [3]).

(i) \( \lim_{r \to \infty} \inf \frac{T_0(r, \Psi(D) f)}{T_0(r, f)} \geq \sum_{a \neq \infty} \delta_0^0(a; f) \),

(ii) \( \delta_0(0; \Psi(D) f) \geq \frac{\sum_{a \neq \infty} \delta_0(a; f)}{1 + p(1 - \Theta_0(\infty; f))} \).

Lemma 7 (see [3]). If \( \sum_{a \neq \infty} \delta_0^0(a; f) > 0 \) then

\( \Theta_0(\infty; \Psi(D) f) \geq 1 - \frac{1 - \Theta_0(\infty; f)}{\sum_{a \neq \infty} \delta_0^0(a; f)} \).

Lemma 8 (see [8]). If \( f \) is transcendental then \( \lim_{r \to \infty} T_0(r, f) / (\log r)^2 = \infty \) through all values of \( r \).

3. The main result. In this section we discuss the main result of the paper.

Theorem 1. Let \( f, g \) be two meromorphic functions such that

(i) \( \Psi(D) f, \Psi(D) g \) are transcendental and share \( (1, 2) \) and
We put

\[(\text{ii}) \quad \frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} + \min\{\delta_2(b; \Psi(D)f), \delta_2(b; \Psi(D)g)\} > 1 + \frac{2(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{2(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)} \]

for some \(b \neq 0, 1, \infty, 1/2, 2, -\omega, -\omega^2\), with \(\sum_{a \neq \infty} \delta_p(a; f) > 0\), \(\sum_{a \neq \infty} \delta_p(a; g) > 0\) and \(\omega\) being the imaginary cube root of unity.

Then either \([\Psi(D)f][\Psi(D)g] \equiv 1\) or \(f - g \equiv s\), where \(s = s(z)\) is a solution of the differential equation \(\Psi(D)w = 0\).

The following example shows that Theorem 1 is sharp.

**Example 2.** Let \(f = \frac{1}{2}e^z(e^z - 1)\), \(g = \frac{1}{2}e^{-z}(\frac{1}{2} - \frac{1}{5}e^{-z})\) and \(\Psi(D) = D^2 - 3D\). Then \(\Psi(D)f = e^z(1 - e^z)\), \(\Psi(D)g = e^{-z}(1 - e^{-z})\), \(\sum_{a \neq \infty} \delta(a; f) = \sum_{a \neq \infty} \delta(a; g) = 1/2\), \(\Theta(\infty; f) = \Theta(\infty; g) = 1\), \(\delta_2(b; \Psi(D)f) = \delta_2(b; \Psi(D)g) = 0\) for \(b \neq 0, \infty\) and \(\Psi(D)f, \Psi(D)g\) share \((1, 2)\). It is easily seen that neither \([\Psi(D)f][\Psi(D)g] \equiv 1\) nor \(f - g \equiv c_1 - c_2e^{3z}\) for any constants \(c_1\) and \(c_2\).

**Proof of Theorem 1.** Let \(F = \Psi(D)f\) and \(G = \Psi(D)g\). Then in view of Lemmas 5–7 condition (ii) implies

\[(3) \quad \delta_0(0; F) + \delta_0(0; G) + 2\Theta_0(\infty; F) + 2\Theta_0(\infty; G) + \min\{\delta_2^0(b; F), \delta_2^0(b; G)\} > 5.\]

We put

\[H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).\]

Suppose \(H \not\equiv 0\). Then by Lemmas 1–3 we get

\[(4) \quad N(r, 1; F = 1) \leq \bar{N}(r, \infty; F = 2) + \bar{N}(r, 0; F = 2) + \bar{N}(r, b; F = 2) + \bar{N}(r, \infty; G = 2) + \bar{N}(r, 0; G = 2) + \bar{N}(r, 0; F') + \bar{N}(r, \infty; G) + \bar{N}(r, 0; G) - \bar{N}(r, 1; G = 2) + S(r, F) + S(r, G).\]

By the second fundamental theorem we get

\[(5) \quad 2T(r, F) \leq \bar{N}(r, \infty; F) + \bar{N}(r, 1; F) + \bar{N}(r, b; F') + \bar{N}(r, 0; F) - N_\oplus(r, 0; F') + S(r, F),\]

where \(N_\oplus(r, 0; F')\) is the counting function of those zeros of \(F'\) which are not zeros of \(F(F - 1)(F - b)\).

Since \(F, G\) share \((1, 2)\), we see that

\[(6) \quad \bar{N}(r, 1; F) = \bar{N}(r, 1; F = 1) + \bar{N}(r, 1; F = 2) = \bar{N}(r, 1; F = 1) + \bar{N}(r, 1; G = 2).\]

**Linear differential polynomials**
Since $N_2(r, \infty; F) \leq 2\overline{N}(r, \infty; F)$ and $N_2(r, \infty; G) \leq 2\overline{N}(r, \infty; G)$, we get from (4)–(6) on integration
\begin{align*}
(7) \quad 2T_0(r, F) &\leq N_2^0(r, 0; F) + N_2^0(r, b; F) + N_2^0(r, 0; G) + 2\overline{N}_0(r, \infty; F) \\
&\quad + 2\overline{N}_0(r, \infty; G) + S_0(r, F) + S_0(r, G).
\end{align*}

Similarly we obtain
\begin{align*}
(8) \quad 2T_0(r, G) &\leq N_2^0(r, 0; F) + N_2^0(r, b; G) + N_2^0(r, 0; G) + 2\overline{N}_0(r, \infty; F) \\
&\quad + 2\overline{N}_0(r, \infty; G) + S_0(r, F) + S_0(r, G).
\end{align*}

From (7) and (8) we get
\begin{align*}
(9) \quad 2T_0(r) &\leq N_2^0(r, 0; F) + N_2^0(r, 0; G) + 2\overline{N}_0(r, \infty; F) \\
&\quad + 2\overline{N}_0(r, \infty; G) + S_0(r, F) + S_0(r, G),
\end{align*}

where $T_0(r) = \max\{T_0(r, F), T_0(r, G)\}$ and $N_2^0(r, b) = \max\{N_2^0(r, b; F), N_2^0(r, b; G)\}$.

Since (9) contradicts (3), it follows that $H \equiv 0$. Then
\begin{align*}
(10) \quad F = \frac{AB + C}{CG + D},
\end{align*}

where $A, B, C, D$ are complex numbers such that $AD - BC \neq 0$.

In view of (10) we get
\begin{align*}
(11) \quad T_0(r, F) = T_0(r, G) + O(\log r).
\end{align*}

Now we consider the following cases.

**Case 1:** $AC \neq 0$. Then
\begin{align*}
(12) \quad F - \frac{A}{C} = \frac{B - \frac{AD}{C}}{CG + D}.
\end{align*}

**Subcase 1.1:** $A/C \neq b$. Then by the second fundamental theorem we get on integration
\begin{align*}
2T_0(r, F) &\leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, A/C; F) + \overline{N}_0(r, b; F) + S_0(r, F) \\
&\quad = \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + \overline{N}_0(r, \infty; G) + S_0(r, F),
\end{align*}

which implies (9) in view of (11) and Lemma 8 and finally contradicts (3).

**Subcase 1.2:** $A/C = b$. Also we suppose that $BD \neq 0$. Then $B/D \neq b$ because $AD - BC \neq 0$. So by the second fundamental theorem we get on integration
\begin{align*}
2T_0(r, F) &\leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + \overline{N}_0(r, B/D; F) + S_0(r, F) \\
&\quad = \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + \overline{N}_0(r, 0; G) + S_0(r, F),
\end{align*}

which by (11) and Lemma 8 implies (9) and so contradicts (3).
Let $B = 0$. Then $D \neq 0$ because $F$ is nonconstant. Now from (12) we get
\begin{equation}
F - b = \frac{-b}{\alpha G + 1},
\end{equation}
where $\alpha = C/D$.

Let $1$ be a Picard exceptional value (e.v.P.) of $F$ and so of $G$. Then by the second fundamental theorem we get on integration
\[
2T_0(r, F) \leq N_0(r, \infty; F) + N_0(r, 0; F) + N_0(r, b; F) + S_0(r, F),
\]
which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

Let $1$ be not an e.v.P. of $F$ and $G$. Then from (13) we get $\alpha = \frac{1}{b-1}$ so that
\[
F = \frac{bG}{(b-1) + G}.
\]
Since $b \neq 1/2$, by the second fundamental theorem we get on integration
\[
2T_0(r, G) \leq N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, 1-b; G) + S_0(r, G)
\]
\[
= N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, \infty; F) + S_0(r, G),
\]
which by (11) and Lemma 8 implies (9) and so contradicts (3).

Suppose $1$ is not an e.v.P. of $F$ and $G$. Then from (14) we obtain
\begin{equation}
F = b + \frac{\beta}{G},
\end{equation}
where $\beta = B/C$.

If $1$ is an e.v.P. of $F$ and so of $G$, by the second fundamental theorem we get on integration
\[
2T_0(r, F) \leq N_0(r, \infty; F) + N_0(r, 0; F) + N_0(r, b; F) + S_0(r, F),
\]
which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

Suppose $1$ is not an e.v.P. of $F$ and $G$. Then from (14) we get $\beta = 1 - b$ so that
\[
F = b + \frac{1-b}{G}.
\]
Since $b \neq -\omega, -\omega^2$, by the second fundamental theorem we get on integration
\[
2T_0(r, G) \leq N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, 1-1/b; G) + S_0(r, G)
\]
\[
= N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, 0; F) + S_0(r, G),
\]
which implies (9) in view of (11) and Lemma 8 and so contradicts (3).
CASE 2: $AC = 0$. Since $F$ is nonconstant, it follows that $A$ and $C$ are not simultaneously zero.

SUBCASE 2.1: $A = 0$ and $C \neq 0$. Then $B \neq 0$ and from (10) we get

(15) \[ \frac{1}{F} = \alpha G + \beta, \]

where $\alpha = C/B$ and $\beta = D/B$.

If $1$ is an e.v.P. of $F$ and $G$, by the second fundamental theorem we get on integration

$$2T_0(r, F) \leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + S_0(r, F),$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

Suppose $1$ is not an e.v.P. of $F$ and $G$. Then from (15) we get $1 = \frac{1}{F} + \frac{1}{1} = \frac{1}{1}$, i.e. $[\Psi(D)f][\Psi(D)g] \equiv 1$.

If $\alpha = 1$ then $FG \equiv 1$, i.e. $[\Psi(D)f][\Psi(D)g] \equiv 1$.

If $\alpha = 1 - 1/b$ then $F = \frac{b}{1 + (b-1)G}$.

Since $b \neq -\omega, -\omega^2$, by the second fundamental theorem we get on integration

$$2T_0(r, G) \leq \overline{N}_0(r, \infty; G) + \overline{N}_0(r, 0; G) + \overline{N}_0(r, b; G) + \overline{N}_0(r, 1/(1 - b); G) + S_0(r, G)$$

$$= \overline{N}_0(r, \infty; G) + \overline{N}_0(r, 0; G) + \overline{N}_0(r, b; G) + \overline{N}_0(r, \infty; F) + S_0(r, G),$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

SUBCASE 2.2: $A \neq 0$ and $C = 0$. Then $D \neq 0$ and from (10) we get

(16) \[ F = \alpha G + \beta, \]

where $\alpha = A/D$, $\beta = B/D$.

If $1$ is an e.v.P. of $F$ and $G$, by the second fundamental theorem we get on integration

$$2T_0(r, F) \leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + S_0(r, F),$$

which implies (9) by (11) and Lemma 8 and so contradicts (3).
Suppose 1 is not an e.v.P. of $F$ and $G$. Then from (16) we get $\alpha + \beta = 1$ and so

$$F = \alpha G + 1 - \alpha.$$ 

If $\alpha \neq 1, 1 - b$, by the second fundamental theorem we get on integration

$$2T_0(r, F) \leq N_0(r, \infty; F) + N_0(r, 0; F) + N_0(r, b; F) + N_0(r, 1 - \alpha; F) + S_0(r, F)$$

$$= N_0(r, \infty; F) + N_0(r, 0; F) + N_0(r, b; F) + N_0(r, 0; G) + S_0(r, F),$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

If $\alpha = 1$ then $F \equiv G$ and so $f - g \equiv s$, where $s = s(z)$ is a solution of the differential equation $\Psi(D)w = 0$.

If $\alpha = 1 - b$ then

$$F = (1 - b)G + b.$$ 

Since $b \neq 2$, by the second fundamental theorem we get on integration

$$2T_0(r, G) \leq N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, b/(b - 1); G) + S_0(r, G)$$

$$= N_0(r, \infty; G) + N_0(r, 0; G) + N_0(r, b; G) + N_0(r, 0; F) + S_0(r, G),$$

which by (11) and Lemma 8 implies (9) and so contradicts (3). This proves the theorem.

4. Applications. In this section we discuss two applications of the main theorem, the first of which improves a result of Yi and Yang [13] and the second gives a better answer to the question of Yang [9] mentioned in the introduction.

**Theorem 2.** Let $f, g$ be two nonconstant meromorphic functions with $\Theta(\infty; f) = \Theta(\infty; g) = 1$. If for $n \geq 1$ the derivatives $f^{(n)}, g^{(n)}$ share $(1, 2)$ and

\begin{align*}
(i) \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) + \min\{\delta_2(b; f^{(n)}), \delta_2(b; g^{(n)})\} &> 1 \\
(ii) \Theta(\alpha; f) + \Theta(\alpha; g) &> 1
\end{align*}

for some $b \neq 0, 1, \infty, 1/2, 2, -\omega, -\omega^2$, and

\begin{align*}
(i) \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) + \min\{\delta_2(b; f^{(n)}), \delta_2(b; g^{(n)})\} &> 1 \\
(ii) \Theta(\alpha; f) + \Theta(\alpha; g) &> 1
\end{align*}

for some $\alpha \neq \infty$, then either (I) $f^{(n)}g^{(n)} \equiv 1$ or (II) $f \equiv g$.

**Proof.** From the given condition it follows that $f, g$ are transcendental and so $f^{(n)}, g^{(n)}$ are transcendental. Choosing $\Psi(D) = D^n$ in Theorem 1 we get either $f^{(n)}g^{(n)} \equiv 1$ or $f - g \equiv Q$, where $Q$ is a polynomial of degree at most $n - 1$. If possible let $Q \neq 0$. Then by Nevanlinna’s theorem on three
small functions [2, p. 47] we get
\[ T(r, f) \leq \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha + Q; f) + \overline{N}(r, \infty; f) + S(r, f) \]
\[ = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) + \overline{N}(r, \infty; f) + S(r, f). \]
Since \( f - g \equiv Q \), it follows that \( T(r, f) = T(r, g) + O(\log r) \). So \( \Theta(\alpha; f) + \Theta(\alpha; g) \leq 1 \), which is a contradiction. Therefore \( Q \equiv 0 \) and so \( f \equiv g \). This proves the theorem. 

The following examples show that the condition \( \Theta(\alpha; f) + \Theta(\alpha; g) > 1 \) is necessary for the validity of case (II).

**Example 3.** Let \( f = 1 + e^z \) and \( g = e^z \). Then
\[ \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) + \min\{\delta_2(b; f^{(n)}), \delta_2(b; g^{(n)})\} = 2 \]
for any \( b \neq 0, \infty, \Theta(\infty; f) = \Theta(\infty; g) = 1 \), \( \Theta(0; f) + \Theta(0; g) = 1 \), \( \Theta(1; f) + \Theta(1; g) = 1 \), \( \Theta(\alpha; f) + \Theta(\alpha; g) < 1 \) for \( \alpha \neq 0, 1, \infty \) and \( f^{(n)} \), \( g^{(n)} \) share \((1, 2)\) but \( f - g \equiv 1 \).

**Example 4.** Let \( f = 1 + e^z \) and \( g = (-1)^n e^{-z} \). Then
\[ \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) + \min\{\delta_2(b; f^{(n)}), \delta_2(b; g^{(n)})\} = 2 \]
for any \( b \neq 0, \infty, \Theta(\infty; f) = \Theta(\infty; g) = 1 \), \( \Theta(0; f) + \Theta(0; g) = 1 \), \( \Theta(1; f) + \Theta(1; g) = 1 \), \( \Theta(\alpha; f) + \Theta(\alpha; g) < 1 \) for \( \alpha \neq 0, 1, \infty \) and \( f^{(n)} \), \( g^{(n)} \) share \((1, 2)\) but \( f^{(n)}g^{(n)} \equiv 1 \).

**Remark 1.** Theorem 2 improves Theorem C, a result of Yi and Yang [13] and also a recent result of Lahiri [3].

In the following theorem we provide a better answer to a question of Yang [9] than those given in Theorems F and G.

**Theorem 3.** Let \( f \) and \( g \) be two meromorphic functions such that \( f^{(n)} \), \( g^{(n)} \) \( (n \geq 1) \) share \((1, 2)\), \( f \), \( g \) share \((\alpha, 0)\) for some \( \alpha \neq \infty \) and
\[ \frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} + \min\{\delta_2(b; f^{(n)}), \delta_2(b; g^{(n)})\} \]
\[ > 1 + \frac{2(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{2(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)} \]
for some \( b \neq 0, 1, \infty, 1/2, 2, -\omega, -\omega^2 \), with \( \sum_{a \neq \infty} \delta_p(a; f) > 0 \), \( \sum_{a \neq \infty} \delta_p(a; g) > 0 \) and \( \omega \) being the imaginary cube root of unity. Then either \( f^{(n)}g^{(n)} \equiv 1 \) or \( f \equiv g \).

**Proof.** From the assumption it follows that \( f \) and \( g \) are transcendental and so \( f^{(n)} \) and \( g^{(n)} \) are transcendental. Choosing \( \Psi(D) = D^n \) we see from Theorem 1 that either \( f - g \equiv Q \) or \( f^{(n)}g^{(n)} \equiv 1 \), where \( Q \) is a polynomial of degree at most \( n - 1 \). If possible, let \( Q \neq 0 \). Since \( f, g \) share \((\alpha, 0)\), it follows
that $\overline{N}(r, \alpha; f) = \overline{N}(r, 0; Q) = O(\log r)$. Now by Nevanlinna’s theorem on three small functions [2, p. 47] we get
\[
T(r, f) \leq \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha + Q; f) + \overline{N}(r, \infty; f) + S(r, f)
\]
\[
= \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) + \overline{N}(r, \infty; f) + S(r, f)
\]
which implies that $\Theta(\infty; f) = 0$. Similarly we see that $\Theta(\infty; g) = 0$. Since this contradicts the assumption, it follows that $Q \neq 0$ and so $f \equiv g$. This proves the theorem.

The following example shows that Theorem 3 is sharp.

**Example 5.** Let $f = -2^{-n}e^{2z} + (-1)^{n+1}2^{-n}e^{z} - 2^{-n}e^{-z}$ and $g = (-1)^{n+1}2^{-n}e^{2z} - 2^{-n}e^{-z}$. Then $f^{(n)}$, $g^{(n)}$ share $(1, 2)$, $f, g$ share $(0, 0)$, $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $\sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) + \min\{\delta_2(b; f^{(n)}), \delta_2(b; b^{(n)})\} = 1$ for any $b \neq 0, \infty$ but neither $f \equiv g$ nor $f^{(n)}g^{(n)} \equiv 1$.

**Concluding Remark.** Since Example 1 shows that in Theorem 1 sharing $(1, 2)$ cannot be relaxed to sharing $(1, 0)$, we conclude the paper with the following question: *Is it possible in Theorem 1 to relax sharing $(1, 2)$ to sharing $(1, 1)$?*

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