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A decomposition of a set definable in an o-minimal structure into perfectly situated sets

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Dedicated to my wife Jolanta

Abstract. A definable subset of a Euclidean space X is called perfectly situated if it can be represented in some linear system of coordinates as a finite union of (graphs of) definable $\mathcal{C}^1$-maps with bounded derivatives. Two subsets of X are called simply separated if they satisfy the Łojasiewicz inequality with exponent 1. We show that every closed definable subset of X of dimension k can be decomposed into a finite family of closed definable subsets each of which is perfectly situated and such that any two different sets of the decomposition are simply separated and their intersection is of dimension $< k$.

Introduction. We will assume that there is given an o-minimal structure in the ordered field $\mathbb{R}$ of real numbers (see [1] for the definition and fundamental properties of o-minimal structures).

Let M be a $\mathcal{C}^1$-submanifold of $\mathbb{R}^n$ of dimension l and let V be a linear subspace of $\mathbb{R}^n$ of dimension $n-k$, where $k \geq l$. We will call M *perfectly situated relative to* V if the set of the tangents $\{T_aM \mid a \in M\}$ is a relatively compact subset of the set $\{W \in \mathbf{G}_l(\mathbb{R}^n) \mid W \cap V = \{0\}\}$, open in the Grassmann manifold of l-dimensional linear subspaces of $\mathbb{R}^n$. Let A now be a definable subset of $\mathbb{R}^n$ of dimension $\leq k$. Then A is a finite union $\bigcup_i M_i$ of definable $\mathcal{C}^1$-submanifolds. We will call A *perfectly situated relative to* V if so is each M_i. (This does not depend on the representation $A = \bigcup_i M_i$; cf. [1, Chap. 7, (3.2)].)

Proposition 0. *Let W be a linear complement of V in $\mathbb{R}^n$; i.e. $\mathbb{R}^n = W \oplus V$. The following conditions are equivalent*:

(1) *A is perfectly situated relative to V.*

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(2) *A is a finite disjoint union $\bigcup_i \widehat{\varphi}_i$ of graphs of definable $\mathcal{C}^1$-maps $\varphi_i : \Lambda_i \to V$ defined on $\mathcal{C}^1$-submanifolds $\Lambda_i \subset W$ with bounded derivatives* (*here $\widehat{\varphi}_i$ stands for the graph $\{w + \varphi_i(w) \mid w \in \Lambda_i\}$ of φ_i*).

(3) *There is $C > 0$ such that if $a \in A$, $(x_\nu)_{\nu\in\mathbb{N}}$ is a sequence of points of $A \setminus \{a\}$ convergent to a and $v = \lim_{\nu\to\infty}(x_\nu - a)/|x_\nu - a|$, then $d(v, V) \geq C$* (¹).

(4) *Every definable subset of $\overline{A}$ is perfectly situated relative to V.*

(5) *A is perfectly situated relative to V' for all V' from a neighbourhood of V in $\mathbf{G}_{n-k}(\mathbb{R}^n)$.*

(6) *A is perfectly situated relative to any linear subspace of V.*

Proof. (1)⇔(2) by [1, Chap. 7, (3.2)]. (1)⇔(3) by curve selection (cf. [1, Chap. 6, (1.5)] and the fact that a definable curve is $\mathcal{C}^1$ at its extremity. The others are simple consequences.

The notion of a perfectly situated subset was used by the author in [5, Chap. II].

Let P and Q be any two subsets of $\mathbb{R}^n$. We will say that P and Q are *simply separated* if there exists $C > 0$ such that for each $x \in P$, $d(x, Q) \geq Cd(x, P \cap Q)$. This condition is symmetric with respect to P and Q. Indeed, for each $y \in Q$ and $\varepsilon > 0$, there is $x \in P$ such that $d(y, P) + \varepsilon > |y - x|$; hence $(C+1)|y-x| \geq d(x, P) + C|y-x| \geq C(d(x, P\cap Q) + |y-x|) \geq Cd(y, P\cap Q)$; consequently, $d(y, P) \geq \frac{C}{C+1} d(y, P \cap Q)$. In other words, P and Q are simply separated if they satisfy the (global) Łojasiewicz inequality with exponent 1 (cf. [3, p. 139]).

The main result of the present paper is the following

THEOREM 0. *Let $\Sigma = \{\sigma \mid \sigma \subset \{1, \ldots, n\}, \operatorname{card}\sigma = n-k\} = \{\sigma_1, \ldots, \sigma_m\}$, where $m = \binom{n}{k}$. Let $V_i = \bigoplus_{\nu\in\sigma_i} \mathbb{R}e_\nu$ $(i = 1, \ldots, m)$, where $e_1, \ldots, e_n$ denote the canonical basis in $\mathbb{R}^n$. Any definable closed subset E of $\mathbb{R}^n$ of dimension k is the union $E = \bigcup_{i=1}^m S_i$ of definable closed subsets S_i such that for each i, S_i is perfectly situated relative to V_i and for each $j \neq i$, S_i and S_j are simply separated and* $\dim(S_i \cap S_j) < k$.

In the subanalytic case similar results have been formulated and proved in a different way by Parusiński [4]. We prove Theorem 0 by a construction based on Lemma 1 below and the Mean Value Theorem.

In the proof of Theorem 0 we will use the following

LEMMA 0. *Let V_i $(i = 1, \ldots, m)$ be as in Theorem 0. If E is a definable subset of $\mathbb{R}^n$ of constant dimension k (i.e., every nonempty open definable subset of E is of dimension k), then $E = \bigcup_{i=1}^m E_i$, where for each i, E_i is definable of constant dimension k, perfectly situated relative to V_i.*

(¹) $d(x, A) = \inf\{|x - a| \mid a \in A\}$ if $A \neq \emptyset$ and $d(x, \emptyset) = 1$.

Proof. It reduces to the case that E is a C^1-submanifold, when it follows from linear algebra and the fact that the Gauss mapping $E \ni x \mapsto T_x E \in \mathbf{G}_k(\mathbb{R}^n)$ is definable.

REMARK 0. If the set E is of constant dimension l, where $l < k$, then again $E = \bigcup_{i=1}^{m} E_i$, where for each i, E_i is definable of constant dimension l, perfectly situated relative to V_i. Indeed, if W_j $\big(j = 1, \ldots, p,\ p = \binom{n}{l}\big)$ are the corresponding linear subspaces of dimension $n - l$ and $E = \bigcup_{j=1}^{p} E'_j$, where for each j, E'_j is of constant dimension l perfectly situated relative to W_j, we put $E_i = \bigcup\{E'_j \mid V_i \subset W_j\}$.

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1. Key lemma and consequences. The proof of Theorem 0 is based on the following elementary

LEMMA 1. *Let $f_i : E \to \mathbb{R}$ $(i = 1, \ldots, p)$ be a finite family of definable bounded functions on the same definable set $E \subset \mathbb{R}^m$ and let $\eta > 0$. Then E can be represented as a finite union $E = \bigcup_\mu A_\mu$ of definable sets $A_\mu \subset \mathbb{R}^m$ such that for each μ there exists $\varepsilon_\mu \in (0, \eta)$ such that for each i, either $|f_i| \le \varepsilon_\mu$ on A_μ or $|f_i| \ge 4\varepsilon_\mu$ on A_μ.*

Proof. Let $\Delta = \{\delta \mid \delta \subset \{1, \ldots, p\}\}$ and for each $\delta \in \Delta$ and $\varepsilon \in (0, \eta)$, let $\Omega(\delta, \varepsilon) = \{y = (y_1, \ldots, y_p) \in \mathbb{R}^p \mid |y_i| < \varepsilon \text{ if } i \in \delta,\ |y_i| > 4\varepsilon \text{ if } i \notin \delta\}$. Then the sets $\Omega(\delta, \varepsilon)$ form an open covering of $\mathbb{R}^p$. Let $f = (f_1, \ldots, f_p) : E \to \mathbb{R}^p$. Since $f(E)$ is bounded there is a finite family $\{\Omega(\delta_\mu, \varepsilon_\mu)\}$ covering $f(E)$ and the lemma follows.

LEMMA 2. *Let $f_i : E \to \mathbb{R}$ $(i = 1, \ldots, p)$ be a finite family of definable functions on the same set $E \subset \mathbb{R}^m$ and let $K > 0$. Then E can be represented as a finite union $E = \bigcup_\mu A_\mu$ of definable sets A_μ such that for each μ there exists $M_\mu \ge K$ such that for each i, either $|f_i| \le M_\mu$ on A_μ or $|f_i| \ge 4M_\mu$ on A_μ.*

Proof. Take $1/f_i$ in place of f_i in Lemma 1.

LEMMA 3. *Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ denote the projection $\pi(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1})$. Let $\mathcal{A}$ be any finite family of definable subsets of $\mathbb{R}^m$. Then there exists a definable cell decomposition $\mathcal{C}$ of $\mathbb{R}^m$ compatible with $\mathcal{A}$ and such that for each $C_1, C_2 \in \mathcal{C}$, if $\dim C_1 = \dim C_2 = m - 1$ and $\pi(C_1) = \pi(C_2)$ is open in $\mathbb{R}^{m-1}$, then there is $v \in \mathbb{R}^m \setminus \{0\}$ such that C_1 and C_2 are perfectly situated relative to $\mathbb{R}v$.*

Proof. We have $C_i = \{(u, \varphi_i(u)) \mid u \in \Omega\}$, $i = 1, 2$, where Ω is open in $\mathbb{R}^{m-1}$ and $u = (x_1, \ldots, x_{m-1})$. By [1, Chap. 7, (3.2)], we can assume φ_i are $\mathcal{C}^1$ and, by Lemma 2, that there is $M \geq 1$ such that, for each $i = 1, 2$, $j = 1, \ldots, m-1$, $|\partial\varphi_i/\partial x_j| \leq M$ on Ω or $|\partial\varphi_i/\partial x_j| \geq 4M$ on Ω. Moreover, one can assume that there exist $\mu, \nu \in \{1, \ldots, m-1\}$ such that $|\partial\varphi_1/\partial x_\mu| \geq |\partial\varphi_1/\partial x_j|$ and $|\partial\varphi_2/\partial x_\nu| \geq |\partial\varphi_2/\partial x_j|$ on Ω, for each $j = 1, \ldots, m-1$, and each of the functions $\partial\varphi_i/\partial x_j$ is of constant sign on Ω.

Case I: $|\partial\varphi_1/\partial x_\mu| \leq M$ and $|\partial\varphi_2/\partial x_\nu| \leq M$. We take $v = (0, \ldots, 0, 1)$.

Case II: $|\partial\varphi_1/\partial x_\mu| \geq 4M$ and $|\partial\varphi_2/\partial x_\nu| \leq M$. Put $v = (a_1, \ldots, a_m)$, where $a_j = 0$ for $j \neq \mu, m$, $a_\mu = \frac{1}{2}M^{-1}$ and $a_m = 1$. Then the sine of the angle α_1 between v and the tangent to C_1 is

$$\frac{|1 - a_\mu(\partial\varphi_1/\partial x_\mu)|}{|v|\sqrt{1 + |\operatorname{grad}\varphi_1|^2}} \geq \frac{\frac{1}{4}M^{-1}|\partial\varphi_1/\partial x_\mu|}{|v|\sqrt{m}\,|\partial\varphi_1/\partial x_\mu|} = \frac{1}{4|v|\sqrt{m}\,M}.$$

On the other hand, the sine of the angle α_2 between v and the tangent to C_2 is

$$\frac{|1 - a_\mu(\partial\varphi_2/\partial x_\mu)|}{|v|\sqrt{1 + |\operatorname{grad}\varphi_2|^2}} \geq \frac{1 - \frac{1}{2}M^{-1}M}{|v|\sqrt{m}\,M} = \frac{1}{2|v|\sqrt{m}\,M}.$$

Case III: $|\partial\varphi_1/\partial x_\mu| \geq 4M$ and $|\partial\varphi_2/\partial x_\nu| \geq 4M$, where $\mu = \nu$. Take the same v as in Case II.

Case IV: $|\partial\varphi_1/\partial x_\mu| \geq 4M$, $|\partial\varphi_2/\partial x_\nu| \geq 4M$, $\mu \neq \nu$ and $(\partial\varphi_1/\partial x_\mu) \times (\partial\varphi_1/\partial x_\nu) \geq 0$ on Ω. Put $a_\mu = \frac{1}{3}M^{-1}$, $a_\nu = \frac{2}{3}M^{-1}$, $a_m = 1$ and $a_j = 0$ if $j \neq \mu, \nu, m$. Then

$$\begin{aligned}
\sin\alpha_1 &= \frac{|1 - a_\mu(\partial\varphi_1/\partial x_\mu) - a_\nu(\partial\varphi_1/\partial x_\nu)|}{|v|\sqrt{1 + |\operatorname{grad}\varphi_1|^2}} \\
&\geq \frac{|a_\mu(\partial\varphi_1/\partial x_\mu) + a_\nu(\partial\varphi_1/\partial x_\nu)| - 1}{|v|\sqrt{1 + |\operatorname{grad}\varphi_1|^2}} \geq \frac{|a_\mu(\partial\varphi_1/\partial x_\mu)| - 1}{|v|\sqrt{1 + |\operatorname{grad}\varphi_1|^2}} \\
&\geq \frac{|\partial\varphi_1/\partial x_\mu|(a_\mu - |\partial\varphi_1/\partial x_\mu|^{-1})}{|v|\sqrt{m}\,|\partial\varphi_1/\partial x_\mu|} \geq \frac{1}{12|v|\sqrt{m}\,M}, \\
\sin\alpha_2 &\geq \frac{\frac{2}{3}M^{-1}|\partial\varphi_2/\partial x_\nu| - \frac{1}{3}M^{-1}|\partial\varphi_2/\partial x_\mu| - 1}{|v|\sqrt{1 + |\operatorname{grad}\varphi_2|^2}} \\
&\geq \frac{\frac{2}{3}M^{-1}|\partial\varphi_2/\partial x_\nu| - \frac{1}{3}M^{-1}|\partial\varphi_2/\partial x_\nu| - 1}{|v|\sqrt{1 + |\operatorname{grad}\varphi_2|^2}} \\
&= \frac{\frac{1}{3}M^{-1}|\partial\varphi_2/\partial x_\nu| - 1}{|v|\sqrt{1 + |\operatorname{grad}\varphi_2|^2}} \geq \frac{1}{12|v|\sqrt{m}\,M}.
\end{aligned}$$

CASE V: $|\partial\varphi_1/\partial x_\mu| \ge 4M$, $|\partial\varphi_2/\partial x_\nu| \ge 4M$, $\mu \ne \nu$ and $(\partial\varphi_1/\partial x_\mu) \times (\partial\varphi_1/\partial x_\nu) \le 0$ on Ω. One easily modifies Case IV, putting $a_\mu = \frac{1}{3}M^{-1}$, $a_\nu = -\frac{2}{3}M^{-1}$, $a_m = 1$ and $a_j = 0$ for $j \ne \mu, \nu, m$.

Let X be a subset of $\mathbb{R}^m$ and let $\alpha > 0$. As in [6, p. 79], we call X *α-regular* if there exists $C > 0$ such that any two points a, b of X can be joined in X by a rectifiable arc $\gamma : [0,1] \to X$ of length $|\gamma| \le C|a-b|^\alpha$.

THEOREM 1 (Kurdyka [2], Parusiński [4]). *If Ω is any definable open subset of $\mathbb{R}^m$, then there exists a finite family $(G_i)_i$ of disjoint, definable, open, 1-regular subsets of Ω such that* $\dim(\Omega \setminus \bigcup_i G_i) < m$.

Proof. Consider the following two assertions:

(A_m) *For any definable subset Ω of $\mathbb{R}^m$ and any nonempty open subset V of $\mathbb{R}^m \setminus \{0\}$, there exists a finite family $(G_i)_i$ of disjoint, definable, open, 1-regular subsets of Ω such that* $\dim(\Omega \setminus \bigcup_i G_i) < m$ *and, for each i, there is $v_i \in V$ such that ∂G_i is perfectly situated relative to $\mathbb{R}v_i$.*

(B_m) *For any definable open subset D of $\mathbb{R}^m$ there exists a finite family $(H_j)_j$ of disjoint, definable open subsets of D such that* $\dim(D \setminus \bigcup_j H_j) < m$ *and, for each j, there is $v_j \in \mathbb{R}^m \setminus \{0\}$ such that ∂H_j is perfectly situated relative to $\mathbb{R}v_j$.*

$(A_{m-1}) \Rightarrow (B_m)$. By Lemma 3, we can assume that D is an open cell $D = \{(u, x_m) \mid u \in \Omega,\ \varphi_1(u) < x_m < \varphi_2(u)\}$ such that $C_1 = \widehat{\varphi}_1$ and $C_2 = \widehat{\varphi}_2$ are perfectly situated relative to a common line $\mathbb{R}v$ (the cases $\varphi_1 \equiv -\infty$ or $\varphi_2 \equiv +\infty$ can also occur but they will follow by a modification). By Proposition 0 and (A_{m-1}), we can assume that $\pi(v) \ne 0$ and $\partial\Omega$ is perfectly situated relative to $\mathbb{R}\pi(v)$. Then $\partial D \subset C_1 \cup C_2 \cup (\partial\Omega \times \mathbb{R})$ is perfectly situated relative to $\mathbb{R}v$.

$(A_{m-1} \& B_m) \Rightarrow (A_m)$. Using (B_m), Proposition 0 and a linear change of coordinates, we reduce to the case $\Omega = \{(u, x_m) \mid u \in Q,\ \varphi_1(u) < x_m < \varphi_2(u)\}$, where Q is open in $\mathbb{R}^{m-1}$, $\varphi_i : Q \to \mathbb{R}$ $(i = 1, 2)$ are definable $\mathcal{C}^1$-functions such that $\varphi_1 < \varphi_2$ on Q, and $|\partial\varphi_i/\partial x_j| \le M$ on Q for $i = 1, 2$, $j = 1, \ldots, m-1$, for some $M \ge 1$ (or $\varphi_1 \equiv -\infty$ or $\varphi_2 \equiv +\infty$). We can assume that $V = \Delta \times (\alpha - \varepsilon, \alpha + \varepsilon)$, where Δ is open bounded in $\mathbb{R}^{m-1} \setminus \{0\}$, $\alpha, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Take $L > 0$ such that $|u| \le L$ for each $u \in \Delta$. Dividing Q we can assume that, for each i, j, there exists $\theta_{ij} \in \mathbb{R}$ such that $|\partial\varphi_i/\partial x_j - \theta_{ij}| \le \eta$ on Q, where $0 < \eta \le \varepsilon/(8L\sqrt{m-1})$. Moreover, by (A_{m-1}), we can assume that Q is 1-regular and ∂Q is perfectly situated relative to some $u \in \Delta$.

Put $v = (u, a_m)$. The sine of the angle between v and the tangent to $C_i = \widehat{\varphi}_i$ is

$$\frac{|a_m - \langle u, \operatorname{grad}\varphi_i\rangle|}{|v|\sqrt{1+|\operatorname{grad}\varphi_i|^2}} \ge \frac{|a_m - \langle u, \theta_i\rangle - \langle u, \operatorname{grad}\varphi_i - \theta_i\rangle|}{|v|\sqrt{m}\,M}$$
$$\ge \frac{|a_m - \langle u, \theta_i\rangle| - |u|\cdot|\operatorname{grad}\varphi_i - \theta_i|}{|v|\sqrt{m}\,M} \ge \frac{\varepsilon/4 - L\sqrt{m-1}\,\eta}{|v|\sqrt{m}\,M} \ge \frac{\varepsilon}{8|v|\sqrt{m}\,M},$$

where $\theta_i = (\theta_{i1}, \ldots, \theta_{i,m-1})$ and $a_m \in \{\alpha - \varepsilon/2, \alpha, \alpha + \varepsilon/2\}$ is such that

$$|a_m - \langle u, \theta_i\rangle| \ge \varepsilon/4 \quad (i = 1, 2).$$

In order to prove that Ω is 1-regular, we first observe that φ_i are Lipschitz (because Q is 1-regular and all first derivatives of φ_i are bounded; cf. [6, p. 76]). Taking the image of Ω under the Lipschitz automorphism

$$Q \times \mathbb{R} \ni (u, x_m) \mapsto (u, x_m - \varphi_1(u)) \in Q \times \mathbb{R}$$

we can assume that $\varphi_1 \equiv 0$. Since Q is 1-regular and φ_2 is Lipschitz, $\widehat{\varphi}_2$ is 1-regular. Let now $a = (u, a_m) \in \Omega$ and $b = (w, b_m) \in \Omega$, where $a_m \le b_m$. Take an arc $\gamma : [0,1] \to Q$ such that $\gamma(0) = u, \gamma(1) = w$ and $|\gamma| \le C|u - w|$. Then the arc $\delta = (\gamma, a_m) \cup (\{w\} \times [a_m, b_m])$ joins a and b, lies in $Q \times (0, +\infty)$ and $|\delta| \le (C+1)|a - b|$. If $\delta \not\subseteq \Omega$, let c be the first and d the last point of δ that lies on $\widehat{\varphi}_2$. Take an arc λ joining c and d on $\widehat{\varphi}_2$ such that $|\lambda| \le C'|c - d| \le C'|\delta| \le C'(C+1)|a - b|$. Replacing the part of δ between c and d by λ, moving the resulting arc slightly downwards and adding suitable small vertical line segments, we obtain the required arc.

2. Admissible arcs. Let $\lambda = (\lambda_1, \ldots, \lambda_m) : (\alpha, \beta) \to \mathbb{R}^m$ be $\mathcal{C}^1$ on (α, β), where $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. We will call λ an *admissible arc* in $\mathbb{R}^m$ if it satisfies the following conditions:

1) each of the functions λ_i and each of the derivatives λ_i' is of constant sign;

2) for each i, either $|\lambda_i'| \ge 1$ on (α, β) or $|\lambda_i'| < 1$ on (α, β);

3) for each i and j, either $|\lambda_i'| \le |\lambda_j'|$ on (α, β) or $|\lambda_i'| \ge |\lambda_j'|$ on (α, β).

For any admissible arc λ, we put

$$\nu(\lambda) = \min\{i \mid |\lambda_i'| \ge |\lambda_j'| \text{ on } (\alpha, \beta),\ j = 1, \ldots, m\} \quad \text{and} \quad f_\lambda = \lambda_{\nu(\lambda)}.$$

For each $s, t \in (\alpha, \beta)$ and each $j = 1, \ldots, m$,

$$(*) \qquad |f_\lambda(t) - f_\lambda(s)| \ge |\lambda_j(t) - \lambda_j(s)|.$$

To see this we can assume that $f_\lambda' \ge 0$, replacing perhaps λ by $\lambda(\alpha + \beta - t)$. Then, for any fixed $s \in (\alpha, \beta)$, consider the functions $\theta_j(t) = f_\lambda(t) - f_\lambda(s) - |\lambda_j(t) - \lambda_j(s)|$ for $t \in [s, \beta)$. Since $\theta_j'(t) = f_\lambda'(t) \pm |\lambda_j'(t)| \ge 0$ and $\theta_j(s) = 0$, we have $\theta_j \ge 0$ and $f_\lambda(t) - f_\lambda(s) \ge |\lambda_j(t) - \lambda_j(s)|$.

We will say that λ is an *admissible arc of the first kind* if $|f'_\lambda| \geq 1$; otherwise λ is *of the second kind*. For any admissible arc λ of the first kind, we put $c_\lambda = \alpha$ if $|f_\lambda|$ is increasing and $c_\lambda = \beta$ if $|f_\lambda|$ is decreasing. Since the limit $\lim_{t \to c_\lambda} f_\lambda(t) \in \mathbb{R}$ exists, it follows from $(*)$ that the limit $\lim_{t \to c_\lambda} \lambda(t) \in \mathbb{R}^m$ also exists; it will be denoted by $\lambda(c_\lambda)$.

LEMMA 4. *Let $\lambda : (\alpha, \beta) \to \mathbb{R}^m$ be an admissible arc of the first kind. Let $\widetilde{\lambda}(t) = (t, \lambda(t))$ and $T = \mathbb{R} \times \{0\} \subset \mathbb{R}^{1+m}$. Then, for each $t \in (\alpha, \beta)$,*

$$d(\widetilde{\lambda}(t), T) \geq \frac{1}{\sqrt{m+1}} |\widetilde{\lambda}(t) - \widetilde{\lambda}(c_\lambda)|.$$

Proof. Replacing perhaps λ by $-\lambda$ or by $\mp\lambda(\alpha+\beta-t)$, we reduce to the case $f_\lambda > 0$ and $f'_\lambda \geq 1$ on (α, β). Then $c_\lambda = \alpha$. Apart from $(*)$, we have $|f_\lambda(t) - f_\lambda(s)| \geq |t - s|$; hence,

$$d(\widetilde{\lambda}(t), T) = |\lambda(t)| \geq f_\lambda(t) \geq f_\lambda(t) - f_\lambda(\alpha) \geq \frac{1}{\sqrt{m+1}} |\widetilde{\lambda}(t) - \widetilde{\lambda}(\alpha)|.$$

All the above definitions and Lemma 4 extend to arcs $\lambda : (\alpha, \infty) \to \mathbb{R}^m$ ($\alpha \in \mathbb{R}$), when $c_\lambda = \alpha$, and to arcs $\lambda : (-\infty, \beta) \to \mathbb{R}^m$ ($\beta \in \mathbb{R}$), when $c_\lambda = \beta$.

3. Simple separation relative to a set. Let P, Q and Z be any subsets of $\mathbb{R}^n$. We will say that P and Q are *simply separated relative to Z* (or *simply Z-separated*) if there exists $C > 0$ such that $d(x, Q) \geq Cd(x, Z)$ for each $x \in P$.

PROPOSITION 1. *The following conditions are equivalent*:

(i) *P and Q are simply separated relative to Z*;
(ii) *$\overline{P} \cap \overline{Q} \subset \overline{Z}$ and $\overline{P} \cup \overline{Z}$, $\overline{Q} \cup \overline{Z}$ are simply separated.*

Proof. (i)$\Rightarrow$(ii). If $z \in \overline{P} \cap \overline{Q}$, $d(z, Q) = 0 \geq Cd(z, Z) = 0$, so $z \in \overline{Z}$. Therefore $(\overline{P} \cup \overline{Z}) \cap (\overline{Q} \cup \overline{Z}) = \overline{Z}$. Let $x \in \overline{P}$. Then either $d(x, \overline{Q} \cup \overline{Z}) = d(x, Q) \geq Cd(x, Z) \geq \min(C, 1) d(x, \overline{Z})$ or $d(x, \overline{Q} \cup \overline{Z}) = d(x, \overline{Z}) \geq \min(C, 1) d(x, \overline{Z})$.

(ii)$\Rightarrow$(i). If $x \in P$, then $d(x, Q \cup Z) \geq Cd(x, Z)$ and either $d(x, Q) = d(x, Q \cup Z) \geq \min(C, 1) d(x, Z)$ or $d(x, Q) \geq d(x, Q \cup Z) = d(x, Z) \geq \min(C, 1) d(x, Z)$.

We will use the following easy

PROPOSITION 2.

(1) *If P, Q are simply Z-separated, $P' \subset P$, $Q' \subset Q$, $Z \subset Z'$, then P', Q' are simply Z'-separated.*

(2) *If P_i, Q_i are simply Z_i-separated for $i = 1, \ldots, s$, then $\bigcup_i P_i, \bigcup_i Q_i$ are simply $\bigcup_i Z_i$-separated.*

(3) *If* P, Q *are simply* S*-separated and* S, Q *are simply* T*-separated, then* P, Q *are simply* T*-separated.*

(4) *If* $Q' \subset Q$, $d(x, Q) = d(x, Q')$ *for each* $x \in P$, *and* P, Q' *are simply* Z*-separated, then* P, Q *are simply* Z*-separated.*

Proof. It is left to the reader.

LEMMA 5. *Let* $C = \{x = (u, x_k) \mid u = (x_1, \ldots, x_{k-1}) \in D,\ \alpha(u) < x_k < \beta(u)\}$ *be an open definable cell in* $\mathbb{R}^k$, *possibly with* $\alpha \equiv -\infty$ *or* $\beta \equiv +\infty$ *but not both at the same time. Let* $\varphi = (\varphi_1, \ldots, \varphi_m), \psi = (\psi_1, \ldots, \psi_m) : C \to \mathbb{R}^m$ *be* $\mathcal{C}^1$ *definable mappings and* φ *be Lipschitz. Assume that there is* $M \geq 1$ *such that* $|\partial\varphi_i/\partial x_k| \leq M$ *for each* $i \in \{1, \ldots, m\}$ *and* $|\partial\psi_j/\partial x_k| \geq 2M$ *for some* $j \in \{1, \ldots, m\}$. *Assume that, for each* $u \in D$,

$$(\alpha(u), \beta(u)) \ni x_k \mapsto \psi(u, x_k) - \varphi(u, x_k) \in \mathbb{R}^m$$

is an admissible arc (of the first kind necessarily). Then (the graphs [2] *of)* φ *and* ψ *are simply separated relative to* $\overline{\psi} \setminus \psi$.

Proof. Let $x = (u, x_k) \in C$. By Lemma 4 we have

$$d((x, \psi(x) - \varphi(x)), \overline{C} \times \{0\}) \geq \frac{1}{\sqrt{m+1}} |(x, \psi(x) - \varphi(x)) - (u, c_u, \psi(u, c_u) - \varphi(u, c_u))|,$$

where $c_u \in \{\alpha(u), \beta(u)\}$. Now, it is enough to apply to this inequality the Lipschitz automorphism

$$\overline{C} \times \mathbb{R}^m \ni (x, y) \mapsto (x, y + \varphi(x)) \in \overline{C} \times \mathbb{R}^m.$$

LEMMA 6. *Let* $\varphi : \Omega \to \mathbb{R}^m$ *be a Lipschitz mapping on an open subset* Ω *of* $\mathbb{R}^k$. *Then* $\overline{\varphi}$ *and* $\mathbb{R}^{k+m} \setminus (\Omega \times \mathbb{R}^m)$ *are simply separated (i.e., they are simply* $(\overline{\varphi} \setminus \varphi)$*-separated).*

Proof. Let $a \in \Omega$ and $b \in \partial\Omega$ be such that $|a - b| = d(a, \partial\Omega)$. Then

$$d((a, \varphi(a)), \mathbb{R}^{k+m} \setminus (\Omega \times \mathbb{R}^m)) = |a - b| \geq L^{-1} |\varphi(a) - \overline{\varphi}(b)|,$$

hence

$$d((a, \varphi(a)), \mathbb{R}^{k+m} \setminus (\Omega \times \mathbb{R}^m)) \geq \frac{1}{L+1} |(a, \varphi(a)) - (b, \overline{\varphi}(b))|.$$

COROLLARY. *If* S *is any subset of* $\mathbb{R}^{k+m} \setminus (\Omega \times \mathbb{R}^m)$, *then* φ *and* S *are simply* $(\overline{\varphi} \setminus \varphi)$*-separated.*

LEMMA 7. *Let* $(\Omega_\mu)_\mu$ *be a finite family of open definable disjoint subsets of* $\mathbb{R}^k$. *For every* μ, *let* $\varphi_{\mu\nu} : \Omega_\mu \to \mathbb{R}^m\ (\nu \in J_\mu)$ *be a finite family of* $\mathcal{C}^1$ *definable disjoint (as graphs) mappings such that there exists* $M_\mu \geq 1$ *such*

[2] Here and in what follows we will identify a mapping with its graph.

that for each $\nu \in J_\mu$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,k\}$, *either* $|\partial\varphi_{\mu\nu i}/\partial x_j| \le M_\mu$ *or* $|\partial\varphi_{\mu\nu i}/\partial x_j| \ge 2M_\mu$ *on* Ω_μ. *Put*

$$A = \bigcup\{\varphi_{\mu\nu} \mid \forall i,j : |\partial\varphi_{\mu\nu i}/\partial x_j| \le M_\mu\},$$
$$B = \bigcup\{\varphi_{\mu\nu} \mid \exists i,j : |\partial\varphi_{\mu\nu i}/\partial x_j| \ge 2M_\mu\}.$$

Then there exists $M > 0$ *such that for each pair of definable sets* $A' \subset A$ *and* $B' \subset B$ *and any set* $S \subset (\mathbb{R}^k \setminus \bigcup_\mu \Omega_\mu) \times \mathbb{R}^m$ *there exists a definable set* $Z \subset \overline{A'} \cup \overline{B'}$ *of dimension* $< k$ *such that* $B' \cup S$ *and* A' *are simply* Z*-separated with constant* M, *i.e., for each* $a \in B' \cup S$, $d(a, A') \ge M d(a, Z)$.

Proof. Special case: $A' = A$ and $B' = B$. Let

$$\Gamma = \{(\mu,\nu) \mid \forall i,j : |\partial\varphi_{\mu\nu i}/\partial x_j| \le M_\mu\},$$
$$\Delta_j = \{(\mu,\nu) \mid \exists i : |\partial\varphi_{\mu\nu i}/\partial x_j| \ge 2M_\mu\} \quad (j = 1,\dots,k).$$

Then $B = \bigcup_j B_j$, where $B_j = \bigcup\{\varphi_{\mu\nu} \mid (\mu,\nu) \in \Delta_j\}$.

It suffices to prove the lemma for each B_j in place of B; then we will take $Z = \bigcup_j Z_j$, where Z_j corresponds to B_j. Of course, it is enough to consider the case $j = k$. Consequently, we will assume that $B = B_k$. By Theorem 1, we can assume that each Ω_μ is 1-regular; thus, all $(\varphi_{\mu\nu})$ $((\mu,\nu) \in \Gamma)$ are Lipschitz with a common constant L.

By a suitable cell decomposition compatible with all Ω_μ, we can assume that each Ω_μ is an open definable cell $C = \{x = (u, x_k) \mid u \in D,\ \alpha(u) < x_k < \beta(u)\}$, and for each $u \in D$, $(\mu,\nu) \in \Gamma$ and $(\mu,\sigma) \in \Delta_k$,

$$(\alpha(u),\beta(u)) \ni x_k \mapsto \varphi_{\mu\sigma}(u,x_k) - \varphi_{\mu\nu}(u,x_k) \in \mathbb{R}^m$$

is an admissible arc. Now, by Lemma 5 and Corollary to Lemma 6, we obtain the required conclusion with $Z = \bigcup_{\mu,\nu}(\overline{\varphi}_{\mu,\nu} \setminus \varphi_{\mu\nu})$ and M depending only on L, M_μ, m and k.

General case. This reduces to the special case by taking a cell decomposition $\mathcal{C}$ of $\mathbb{R}^k$ compatible with all sets Ω_μ, $\pi(\varphi_{\mu\nu} \cap A')$ and $\pi(\varphi_{\mu\nu} \cap B')$, where $\pi : \mathbb{R}^{k+m} \to \mathbb{R}^k$ is the projection $\pi(x_1,\dots,x_{k+m}) = (x_1,\dots,x_k)$, and considering the family $\varphi_{\mu\nu}|C$, where $C \in \mathcal{C}$ open is contained in Ω_μ. Then $\varphi_{\mu\nu}|C$ $((\mu,\nu) \in \Gamma)$ are Lipschitz with the same constant L as in the special case and the argument of the special case follows.

4. Decompositions

Proposition 3. *Let* E *be a definable subset of* $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ *of dimension* $l \le k$. *Let* C *be a definable subset of* E *of constant dimension* l *perfectly situated relative to* $\mathbb{R}^{n-k}$. *Then* $E = A \cup B$, *where* A *and* B *are definable*, A *is of constant dimension* l *perfectly situated relative to* $\mathbb{R}^{n-k}$, $C \subset A$ *and there is* $M > 0$ *such that for each pair of definable sets* $A' \subset A$

and $B' \subset B$, there is a definable set $Z \subset \overline{A'} \cup \overline{B'}$ of dimension $< l$ such that A', B' are simply Z-separated with constant M.

Proof. CASE I: $l = k$. By a cell decomposition, Proposition 0 and Lemma 1, E can be represented in the form

$$E = \bigcup_{\mu,\nu} \varphi_{\mu\nu} \cup S,$$

where $\varphi_{\mu\nu}$ and S are as in Lemma 7 (where $m = n - k$) and $\bigcup\{\varphi_{\mu\nu} \mid \varphi_{\mu\nu} \subset C,\ (\mu,\nu) \in \Gamma\}$ is dense in C. Lemma 7 concludes the proof.

CASE II: $l < k$. By Proposition 0 and Lemma 0, $C = C_1 \cup \ldots \cup C_s$, where each C_i is definable of constant dimension l and there exists a permutation of variables $\alpha_i : \mathbb{R}^k \to \mathbb{R}^k$ such that $\widetilde{C}_i = (\alpha_i \times \mathrm{id}|\mathbb{R}^{n-k})(C_i)$ is perfectly situated relative to $\mathbb{R}^{n-l}$.

If now $(\alpha_i \times \mathrm{id}|\mathbb{R}^{n-k})(E) = A_i \cup B_i$ are appropriate decompositions following from Case I, it is enough to put

$$A = \bigcup_{i=1}^{s} (\alpha_i^{-1} \times \mathrm{id}|\mathbb{R}^{n-k})(A_i) \quad \text{and} \quad B = \bigcap_{i=1}^{s} (\alpha_i^{-1} \times \mathrm{id}|\mathbb{R}^{n-k})(B_i).$$

Now we will modify the set Z; in particular, we will be able to have Z perfectly situated relative to $\mathbb{R}^{n-k}$.

LEMMA 8. *Let $A, B, A_*, B_*, Z, Z_*, C, S$ and T be subsets of $\mathbb{R}^n$ such that A, B are simply Z-separated, $Z \subset S \cup T$ and $C \subset \overline{A}$ is such that $d(y, A) = d(y, C)$ for each $y \in T$. Assume that $T \cup C = A_* \cup B_*$, where A_*, B_* are simply Z_*-separated. Then*:

(1) *if $C \subset A_*$, then A, B are simply $S \cup (A_* \cap T) \cup Z_*$-separated*;

(2) *if $T \subset A_*$, then A, B are simply $S \cup (A_* \cap C) \cup Z_*$-separated.*

Proof. (1) Let $x \in A$. There exists $y \in Z$ such that $d(x, B) \ge 2Md(x, Z) \ge M|x - y|$. Suppose $y \notin S \cup (A_* \cap T)$. Then $y \in B_* \cap T$; hence, $|x - y| \ge d(x, A) = d(x, C) \ge d(y, A_*) \ge M|y - z|$, where $z \in Z_*$. Consequently,

$$\begin{aligned} |x - z| &\le |x - y| + |y - z| \le (1 + 1/M)|x - y| \\ &\le (1/M)(1 + 1/M)d(x, B). \end{aligned}$$

(2) Let $x \in A$. There is $y \in Z$ such that $d(x, B) \ge M|x - y|$. Suppose $y \notin S$. Then $y \in T$, and so $y \in A_*$. There is $z \in C$ such that $|x - y| \ge d(y, A) = d(y, C) \ge \frac{1}{2}|y - z|$.

If $z \in A_*$, then $z \in A_* \cap C$ and $|x - z| \le |x - y| + |y - z| \le \frac{3}{2}|x - y| \le \frac{3}{2}M^{-1}d(x, B)$.

Suppose now that $z \notin A_*$. Consequently, $z \in B_*$ and $|y-z| \geq d(z, A_*) \geq M|z-t|$, for some $t \in Z_*$. Then

$$\begin{aligned}|x-t| &\leq |x-z| + |z-t| \leq |x-z| + M^{-1}|y-z| \\ &\leq |x-y| + |y-z| + M^{-1}|y-z| \leq |x-y| + 2(1+M^{-1})|x-y| \\ &= (3+2M^{-1})|x-y| \leq M^{-1}(3+2M^{-1})d(x,B).\end{aligned}$$

LEMMA 9. *If $P \subset Q$ are two definable subsets of $\mathbb{R}^n$, Q is closed of constant dimension q ($q \geq 1$) and $\dim P < q$, then there exists a definable set $P' \subset Q$ of constant dimension $q-1$ such that $P \subset P'$.*

Proof. Use a triangulation [1, Chap. 8, (2.9)] compatible with P and Q.

PROPOSITION 4. *Let A and B be definable subsets of $\mathbb{R}^n$ of constant dimension $l \leq k$ simply separated relative to a definable set $Z \subset \overline{A} \cup \overline{B}$ of dimension $< l$. Suppose that A is perfectly situated relative to $\mathbb{R}^{n-k}$. Then there exists a definable set $\widetilde{Z} \subset \overline{A} \cup \overline{B}$ of dimension $< l$ perfectly situated relative to $\mathbb{R}^{n-k}$ such that A, B are simply separated relative to $\widetilde{Z}$.*

Proof. Induction on l. By Lemma 9, $Z \subset S \cup T$, where S, T are definable of constant dimension $l-1$ such that $S \subset A$ and $T \subset B$, and there exists a definable set $C \subset \overline{A}$ of constant dimension $l-1$ such that for each $y \in T$, $d(y, A) = d(y, C)$. By Proposition 3, $T \cup C = A_* \cup B_*$, where A_*, B_* are definable of constant dimension $l-1$, A_* is perfectly situated relative to $\mathbb{R}^{n-k}$, $C \subset A_*$ and A_*, B_* are simply separated relative to a definable set $Z_* \subset \overline{A}_* \cup \overline{B}_*$ of dimension $< l-1$. By the induction hypothesis we can assume Z_* is perfectly situated relative to $\mathbb{R}^{n-k}$ and, by Lemma 8(1), A, B are simply separated relative to the set $S \cup (A_* \cap T) \cup Z_*$, perfectly situated relative to $\mathbb{R}^{n-k}$.

PROPOSITION 5. *Let A and B be definable subsets of $\mathbb{R}^n$ of constant dimension $l \leq k$ simply separated relative to a definable set $Z \subset \overline{A} \cup \overline{B}$ of dimension $< l$. Suppose that A is perfectly situated relative to $\mathbb{R}^{n-k}$. Then there exists a definable set $\widetilde{Z} \subset \overline{B}$ of dimension $< l$ perfectly situated relative to $\mathbb{R}^{n-k}$ such that A, B are simply separated relative to $\widetilde{Z}$.*

Proof. Induction on l. By Proposition 4, we can assume that Z is perfectly situated relative to $\mathbb{R}^{n-k}$. Put $S = Z \cap \overline{B}$ and let T be a definable subset of $\overline{A}$ of constant dimension $l-1$ such that $Z \cap \overline{A} \subset T$. Let C be a definable subset of $\overline{B}$ of constant dimension $l-1$ such that $d(x, B) = d(x, C)$ for each $x \in T$. By Proposition 3 and the induction hypothesis, $T \cup C = A_* \cup B_*$, where A_*, B_* are definable sets of constant dimension $l-1$, A_* is perfectly situated relative to $\mathbb{R}^{n-k}$, $T \subset A_*$ and A_*, B_* are simply separated relative to a definable set $Z_* \subset \overline{B}_*$ of dimension $< l-1$, perfectly situated relative to $\mathbb{R}^{n-k}$. Since $A_* \cap B_*$ is nowhere dense in B_* and B_* is of constant dimension, we have $\overline{B}_* \subset \overline{C} \subset \overline{B}$ and $Z_* \subset \overline{B}$. By Lemma 8(2), A, B are simply separated relative to $S \cup (A_* \cap C) \cup Z_*$, which is a subset of $\overline{B}$ perfectly situated relative to $\mathbb{R}^{n-k}$.

Proposition 6. *Let P and Q be closed definable subsets of $\mathbb{R}^n$ of dimensions $\leq k$ and let P be perfectly situated relative to $\mathbb{R}^{n-k}$. Then there exists a closed definable set $S \subset Q$ perfectly situated relative to $\mathbb{R}^{n-k}$, of dimension $\leq \min(\dim P, \dim Q)$, such that P, Q are simply S-separated.*

Proof. Case I: P and Q are both of constant dimension l. By Propositions 3 and 5, $P \cup Q = A \cup B$, where A, B are closed definable sets of constant dimension l, A is perfectly situated relative to $\mathbb{R}^{n-k}$, $P \subset A$ and A, B are simply separated relative to a closed definable set $Z \subset B$ of dimension $< l$, perfectly situated relative to $\mathbb{R}^{n-k}$. Since $B \setminus Z \subset B \setminus A \subset Q$ and B is of constant dimension l, we have $B \subset Q$. By Proposition 2(2), $(Q \setminus B) \cup A = A$ and $(Q \setminus B) \cup B = Q$ are simply $(Q \setminus B) \cup Z$-separated; hence, P and Q are S-separated, where $S = \overline{Q \setminus B} \cup Z$ $(\subset (A \cap Q) \cup Z)$.

Case II: P and Q are both of constant dimensions p and q, respectively, and $p \neq q$. This reduces to Case I by Lemma 9 and Proposition 2(4).

Case III: general, reduces to the previous ones by representing P and Q as finite unions of sets of constant dimension and using Proposition 2(2).

5. Proof of Theorem 0

Part 1. We have $E = E^\circ \cup E^*$, where E° is closed of constant dimension k and E^* is closed of dimension $< k$. By Lemma 0,

$$E^\circ = \bigcup_{i=1}^{m} E_i^\circ,$$

where E_i° is definable closed of constant dimension k, perfectly situated relative to V_i. By Proposition 3,

$$E^\circ = A_1 \cup B_1,$$

where A_1, B_1 are closed definable of constant dimension k, A_1 is perfectly situated relative to V_1, $E_1^\circ \subset A_1$, and any pair of definable subsets A_1' and B_1' of A_1 and B_1, respectively, is simply separated relative to some set $Z_1 \subset \overline{A_1'} \cup \overline{B_1'}$ of dimension $< k$.

Then $E_2^\circ \setminus A_1 \subset B_1$ is of constant dimension k, perfectly situated relative to V_2. By Proposition 3,

$$B_1 = A_2 \cup B_2,$$

where A_2, B_2 are closed definable of constant dimension k, A_2 is perfectly situated relative to V_2, $E_2^\circ \setminus A_1 \subset A_2$, and any pair of definable subsets A_2' and B_2' of A_2 and B_2, respectively, is simply separated relative to some set $Z_2 \subset \overline{A_2'} \cup \overline{B_2'}$ of dimension $< k$.

Then $E_3^\circ \setminus (A_1 \cup A_2) \subset B_2$ is of constant dimension k, perfectly situated relative to V_3. By Proposition 3,

$$B_2 = A_3 \cup B_3,$$

where A_3, B_3 are closed definable of constant dimension k, A_3 is perfectly situated relative to V_3, $E_3^\circ \setminus (A_1 \cup A_2) \subset A_3$, and any pair of definable subsets A_3' and B_3' of A_3 and B_3, respectively, is simply separated relative to some set $Z_3 \subset \overline{A_3'} \cup \overline{B_3'}$ of dimension $< k$.

We continue this process by induction up to the mth step, when

$$B_{m-1} = A_m \cup B_m.$$

Since $E^\circ = E_1^\circ \cup \ldots \cup E_m^\circ \subset A_1 \cup \ldots \cup A_m$, we have $E^\circ = A_1 \cup \ldots \cup A_m$ (and since B_m is of constant dimension k and $\dim B_m = \dim(B_m \cap (A_1 \cup \ldots \cup A_m)) \le \dim((B_1 \cap A_1) \cup \ldots \cup (B_m \cap A_m)) < k$, we have $B_m = \emptyset$).

By Proposition 5, for each pair $i, j \in \{1, \ldots, m\}$ such that $i < j$ there exists a closed definable set $Z_{ij} \subset A_i$ of dimension $< k$, perfectly situated relative to V_j, such that A_i and A_j are simply Z_{ij}-separated.

By Remark 0,

$$E^* = \bigcup_{i=1}^{m} E_i^*,$$

where E_i^* is closed definable perfectly situated relative to V_i.

Put $P_i = A_i \cup E_i^*$ $(i = 1, \ldots, m)$. Then P_i is closed perfectly situated relative to V_i. By Propositions 6 and 2(2), for any $i, j \in \{1, \ldots, m\}$ such that $i < j$, there exists a closed definable set $T_{ij} \subset P_j$ of dimension $< k$, perfectly situated relative to V_j, such that P_i, P_j are simply T_{ij}-separated.

Part 2. Now we define a family $(C_{i_1 \ldots i_\mu \nu})$ of closed definable sets, where $1 \le i_1 < \ldots < i_\mu < \nu \le m$ are integers. We use induction on ν.

If $\nu = 1$, we put $C_1 = P_1$. If $\nu = 2$, we put $C_2 = P_2$ and $C_{12} = T_{12}$.

Let $\nu > 1$. We define $C_{i_1 \ldots i_\mu \nu}$ by induction on μ.

If $\mu = 0$, we put $C_\nu = P_\nu$. If $\mu = 1$, we put $C_{i_1 \nu} = T_{i_1 \nu}$.

Suppose $1 < \mu < \nu$. Then the set D_ν^μ defined by

$$D_\nu^\mu = \bigcup \{ C_{j_1 \ldots j_\sigma \nu} \mid 1 \le j_1 < \ldots < j_\sigma < \nu,\ \sigma < \mu \}$$

is perfectly situated relative to V_ν.

If now $1 \le i_1 < \ldots < i_\mu < \nu$ are integers, there exists a closed definable set $C_{i_1 \ldots i_\mu \nu} \subset C_{i_1 \ldots i_\mu}$ of dimension $< k$, perfectly situated relative to V_ν, such that D_ν^μ and $C_{i_1 \ldots i_\mu}$ are simply $C_{i_1 \ldots i_\mu \nu}$-separated.

LEMMA 10. *Let $1 \le j_1 < \ldots < j_\sigma < \lambda \le m$ and $1 \le i_1 < \ldots < i_\mu < \nu \le m$ be integers and $\lambda < \nu$.*

(1) *If $\mu \le \sigma$, then $C_{j_1 \ldots j_\sigma \lambda}$ and $C_{i_1 \ldots i_\mu \nu}$ are simply separated relative to $C_{j_1 \ldots j_\sigma \lambda \nu}$.*

(2) *If $\mu \ge \sigma$ and $i_{\sigma+1} > \lambda$, then $C_{j_1 \ldots j_\sigma \lambda}$ and $C_{i_1 \ldots i_\mu \nu}$ are simply separated relative to $C_{j_1 \ldots j_\sigma \lambda i_{\sigma+1} \ldots i_\mu \nu}$.*

Proof. (1) This follows from $C_{i_1\ldots i_\mu\nu} \subset D_\nu^{\sigma+1}$.

(2) We use induction on $\mu-\sigma$. If $\mu=\sigma$, see (1). Suppose $\mu>\sigma$. By (1), $C_{j_1\ldots j_\sigma\lambda}$ and $C_{i_1\ldots i_\sigma i_{\sigma+1}}$ are simply separated relative to $C_{j_1\ldots j_\sigma\lambda i_{\sigma+1}}$. Hence, $C_{j_1\ldots j_\sigma\lambda}$ and $C_{i_1\ldots i_\mu\nu}$ are simply separated relative to $C_{j_1\ldots j_\sigma\lambda i_{\sigma+1}}$. By the induction hypothesis $C_{j_1\ldots j_\sigma\lambda i_{\sigma+1}}$ and $C_{i_1\ldots i_\mu\nu}$ are simply separated relative to $C_{j_1\ldots j_\sigma\lambda i_{\sigma+1}\ldots i_\mu\nu}$ and we conclude by Proposition 2(3).

Part 3. Put $S_\nu = \bigcup\{C_{i_1\ldots i_\mu\nu} \mid 1 \le i_1 < \ldots < i_\mu < \nu\}$ for each $\nu \in \{1,\ldots,m\}$. Then S_ν is perfectly situated relative to V_ν.

We will show that if $1 \le \lambda < \nu \le m$, then S_λ and S_ν are simply separated.

By Proposition 2(2), it suffices to check that if we have two sequences $1 \le j_1 < \ldots < j_\sigma < \lambda$ and $1 \le i_1 < \ldots < i_\mu < \nu$, then $C_{j_1\ldots j_\sigma\lambda}$ and $C_{i_1\ldots i_\mu\nu}$ are simply $S_\lambda \cap S_\nu$-separated. If $\mu \le \sigma$, this follows from Lemma 10(1); and if $\mu > \sigma$ and $i_{\sigma+1} > \lambda$, this follows from Lemma 10(2).

Suppose now that $\mu > \sigma$ and $i_{\sigma+1} \le \lambda$. If λ occurs among $i_{\sigma+1},\ldots,i_\mu$, then $C_{i_1\ldots i_\mu\nu} \subset S_\lambda \cap S_\nu$ and clearly $C_{j_1\ldots j_\sigma\lambda}$ and $C_{i_1\ldots i_\mu\nu}$ are simply $S_\lambda\cap S_\nu$-separated. Otherwise, take $\varrho \in \{1,\ldots,\mu\}$ such that $i_\varrho < \lambda$ and $i_\omega > \lambda$ if $\varrho < \omega \le \mu$. By Lemma 10(1), $C_{i_1\ldots i_\varrho}$ and $C_{j_1\ldots j_\sigma\lambda}$ are simply $C_{i_1\ldots i_\varrho\lambda}$-separated; hence, $C_{i_1\ldots i_\mu\nu}$ and $C_{j_1\ldots j_\sigma\lambda}$ are simply $C_{i_1\ldots i_\varrho\lambda}$-separated. By Lemma 10(2), $C_{i_1\ldots i_\varrho\lambda}$ and $C_{i_1\ldots i_\mu\nu}$ are simply $C_{i_1\ldots i_\varrho\lambda i_{\varrho+1}\ldots i_\mu\nu}$-separated. By Proposition 2(3), $C_{i_1\ldots i_\mu\nu}$ and $C_{j_1\ldots j_\sigma\lambda}$ are simply $C_{i_1\ldots i_\varrho\lambda i_{\varrho+1}\ldots i_\mu\nu}$-separated; hence, simply $S_\lambda\cap S_\nu$-separated. This ends the proof.

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