

Lower bounds for Jung constants of Orlicz sequence spaces

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Abstract. A new lower bound for the Jung constant $JC(l^{(\Phi)})$ of the Orlicz sequence space $l^{(\Phi)}$ defined by an N -function Φ is found. It is proved that if $l^{(\Phi)}$ is reflexive and the function $t\Phi'(t)/\Phi(t)$ is increasing on $(0, \Phi^{-1}(1)]$, then

$$JC(l^{(\Phi)}) \geq \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)}.$$

Examples in Section 3 show that the above estimate is better than in Zhang's paper (2003) in some cases and that the results given in Yan's paper (2004) are not accurate.

1. Introduction. We begin by recalling some definitions.

DEFINITION 1 (Jung [6]). Let X be a normed linear space. The number $JC(X)$ of X , called the *Jung constant*, is defined by

$$JC(X) = \sup \left\{ \frac{r(A, X)}{d(A)} : A \subset X \text{ bounded and } d(A) > 0 \right\},$$

where $r(A, X) = \inf \{ \sup \{ \|x - z\| : x \in A \} : z \in X \}$ is the absolute Chebyshev radius of A and $d(A) = \sup \{ \|x - y\| : x, y \in A \}$, the diameter of A .

Clearly, $1/2 \leq JC(X) \leq 1$. Note that in Amir [1] and Franchetti [5], $2JC(X)$ is called the Jung constant of X and denoted by $J(X)$. Historical notes on the Jung constant can be found in Appell, Franchetti and Semenov [2] as well as in Rao and Ren [10, Ch. 4].

DEFINITION 2 (Bynum [3]). The *normal structure coefficient* of a Banach space X is defined as

$$N(X) = \inf \left\{ \frac{d(A)}{r(A)} : A \subset X \text{ bounded, closed and convex, } d(A) > 0 \right\},$$

where $r(A) = \inf \{ \sup \{ \|x - y\| : x \in A \} : y \in \overline{\text{co}}(A) \}$ is the relative Chebyshev radius of A with respect to $\overline{\text{co}}(A)$, and $\overline{\text{co}}(A)$ is the closure of the convex hull of A .

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It can be seen that $1 \leq N(X) \leq 2$. We say that X has the *uniform normal structure* provided $N(X) > 1$. If X is infinite-dimensional, then $N(X) \leq \sqrt{2}$ (cf. Maluta [8]). If X is nonreflexive, then $N(X) = 1$.

DEFINITION 3 (Maluta [8]). For a Banach space X , the *self-Jung constant* $\tilde{N}(X)$ is defined by $\tilde{N}(X) = \sup\{r(A)/d(A) : A \subset X \text{ is as in the definition of } N(X)\}$.

In [1], $2\tilde{N}(X)$ is called the self-Jung constant of X and denoted by $J_s(X)$. Since $\tilde{N}(X) = 1/N(X)$ and $JC(X) \leq \tilde{N}(X)$, for every Banach space X we have

$$(1) \quad 1/2 \leq JC(X) \leq 1/N(X) \leq 1,$$

which is used in Sections 4 and 5.

Now we turn to N -functions and the corresponding Orlicz sequence spaces.

DEFINITION 4. A function $\Phi : \mathbb{R} \rightarrow [0, \infty)$ is termed an N -function if (i) Φ is even and convex; (ii) $\Phi(u) = 0 \Leftrightarrow u = 0$, (iii) $\lim_{u \rightarrow 0} \Phi(u)/u = 0$ and (iv) $\lim_{u \rightarrow \infty} \Phi(u)/u = \infty$ (cf. Krasnosel'skiĭ and Rutickiĭ [7, pp. 6–9] as well as Rao and Ren [10, p. 1]).

An N -function $\Phi(u)$ is said to satisfy the Δ_2 -condition for small u [for large u], in symbols $\Phi \in \Delta_2(0)$ [$\Phi \in \Delta_2(\infty)$], if there exist $K > 2$ and $u_0 > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for $0 \leq u \leq u_0$ [for $u \geq u_0$]. $\Phi(u)$ is said to satisfy the ∇_2 -condition for small u [for large u], written $\Phi \in \nabla_2(0)$ [$\Phi \in \nabla_2(\infty)$], if there are $C > 1$ and $u_0 > 0$ such that $2C\Phi(u) \leq \Phi(Cu)$ for $0 \leq u \leq u_0$ [for $u \geq u_0$].

If Φ is an N -function, its *complementary function* Ψ is given by $\Psi(v) = \sup\{u|v| - \Phi(u) : u \geq 0\}$, which is also an N -function.

It is known that $\Phi \in \nabla_2(0)$ [$\Phi \in \nabla_2(\infty)$] $\Leftrightarrow \Psi \in \Delta_2(0)$ [$\Psi \in \Delta_2(\infty)$], where Ψ is complementary to Φ . For a pair (Φ, Ψ) of complementary N -functions the Orlicz sequence space l^Φ is defined as

$$l^\Phi = \left\{ x = (x(1), x(2), \dots) : \rho_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda x(i)) < \infty \text{ for some } \lambda > 0 \right\},$$

on which the gauge norm (also called the Luxemburg norm in [7], [4] and [2]) and the Orlicz norm are given respectively by

$$(2) \quad \|x\|_{(\Phi)} = \inf\{c > 0 : \rho_\Phi(x/c) \leq 1\}$$

and

$$(3) \quad \|x\|_\Phi = \sup\left\{ \sum_{i=1}^{\infty} |x(i)y(i)| : \rho_\Psi(y) \leq 1 \right\}.$$

These norms are equivalent: $\|x\|_{(\Phi)} \leq \|x\|_{\Phi} \leq 2\|x\|_{(\Phi)}$. We set $l^{(\Phi)} = (l^{\Phi}, \|\cdot\|_{(\Phi)})$ and $l^{\Phi} = (l^{\Phi}, \|\cdot\|_{\Phi})$. Of course, $l^{(\Phi)}$ and l^{Φ} are Banach spaces. It is known that for an N -function Φ , $l^{(\Phi)}$ and l^{Φ} are reflexive $\Leftrightarrow \Phi \in \Delta_2(0) \cap \nabla_2(0)$. The Jung constants of $l^{(\Phi)}$ and l^{Φ} have been studied before by Zhang [15]. The main results in [15] can be summarized in the following two propositions.

PROPOSITION 1. *Let Φ be an N -function. Then*

- (i) $\Phi \notin \Delta_2(0) \cap \nabla_2(0) \Rightarrow JC(l^{(\Phi)}) = JC(l^{\Phi}) = 1$;
- (ii) $\Phi \in \Delta_2(0) \cap \nabla_2(0) \Rightarrow \max(JC(l^{(\Phi)}), JC(l^{\Phi})) < 1$.

The first part of Proposition 1 is nontrivial and the second part is a consequence of (1) and Theorem 3.1 in Chen [4, p. 107] (cf. [12] and [13]).

PROPOSITION 2. *Let $\Phi(u) = \int_0^{|u|} \varphi(t) dt$ and $\Psi(v) = \int_0^{|v|} \psi(s) ds$ denote a pair of complementary N -functions and let $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. Then*

$$(4) \quad JC(l^{(\Phi)}) \geq \max\left(\frac{1}{2\alpha_{\Phi}^0}, \beta_{\Phi}^0\right),$$

where

$$(5) \quad \alpha_{\Phi}^0 = \liminf_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_{\Phi}^0 = \limsup_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$

and

$$(6) \quad JC(l^{\Phi}) \geq \max\left(\beta_{\Psi}^0, \frac{1}{2\alpha_{\Psi}^0}\right),$$

where α_{Ψ}^0 and β_{Ψ}^0 are defined similarly to (5).

COROLLARY 1. *Let (Φ, Ψ) be as in Proposition 2. If the limit $C_{\Phi}^0 = \lim_{t \rightarrow 0} t\varphi(t)/\Phi(t)$ exists, then (4) and (6) reduce to*

$$(7) \quad \min(JC(l^{(\Phi)}), JC(l^{\Phi})) \geq \max(2^{1/C_{\Phi}^0-1}, 2^{-1/C_{\Phi}^0}).$$

Proof. Note that $1 < C_{\Phi}^0 < \infty$ because $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ (cf. Theorem 2 in [10, p. 3]). Hence (7) follows from Theorem 15 in [10, p. 11] and Corollary 9 in [10, p. 8]. ■

We often use (7) in concrete examples, e.g. in [10, p. 144, Example 7] it is proved that $JC(l^p) = \max(2^{1/p-1}, 2^{-1/p})$ for $1 < p < \infty$; see also Example 4 in Section 3.

2. Results. We start by introducing spaces having a symmetric norm.

DEFINITION 5. A basis $\{x_i\}$ of a real Banach space X is called *symmetric* if it is equivalent to the basis $\{x_{\pi(i)}\}$ for every rearrangement π of the set of natural numbers. An infinite-dimensional Banach space X with a symmetric

basis $\{x_i\}_{i=1}^\infty$ is said to have a *symmetric norm* if for every real sequence $\{a_i\}_{i=1}^\infty$ with $\sum_{i=1}^\infty a_i x_i \in X$,

$$\left\| \sum_{i=1}^\infty a_i x_i \right\|_X = \left\| \sum_{i=1}^\infty \theta_i a_i x_{\pi(i)} \right\|_X$$

for all choices of signs $\theta_i = \pm 1$ and all permutations π of the natural numbers.

The following lemma will be used.

LEMMA 1 (Franchetti [5, Lemma 2]). *Let X be an infinite-dimensional Banach space with symmetric basis $\{e_i\}_{i=1}^\infty$ and symmetric norm. Define $f : \mathbb{N} = \{1, 2, \dots\} \rightarrow \mathbb{R}^+$ by $f(k, X) = \left\| \sum_{i=1}^k e_i \right\|_X$. Then*

$$(8) \quad JC(X) \geq \sup_{k \in \mathbb{N}} \frac{f(k, X)}{f(2k, X)}.$$

THEOREM 1. *Let $\Phi(u) = \int_0^{|u|} \varphi(t) dt$ and $\Psi(v) = \int_0^{|v|} \psi(s) ds$ denote a pair of complementary N -functions and let $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. Define*

$$(9) \quad \beta'_\Phi = \sup_{k \in \mathbb{N}} \frac{\Phi^{-1}(1/2k)}{\Phi^{-1}(1/k)}, \quad \alpha'_\Psi = \inf_{k \in \mathbb{N}} \frac{\Psi^{-1}(1/2k)}{\Psi^{-1}(1/k)}.$$

Then

$$(10) \quad JC(l^{(\Phi)}) \geq \beta'_\Phi,$$

$$(11) \quad JC(l^\Phi) \geq \frac{1}{2\alpha'_\Psi}.$$

Proof. Let $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots)$, etc. The canonical basis $\{e_i\}_{i=1}^\infty$ is symmetric in $l^{(\Phi)}$ as well as in l^Φ since $\Phi \in \Delta_2(0)$ (cf. [4, p. 47]). The norms (2) and (3) are symmetric. It can be seen from (2) that

$$(12) \quad f(k, l^{(\Phi)}) = \left\| \sum_{i=1}^k e_i \right\|_{(\Phi)} = \inf \{c > 0 : k\Phi(1/c) \leq 1\} = \frac{1}{\Phi^{-1}(1/k)}.$$

Thus (10) follows from (8), (12) and (9). Let λ_0 satisfy the equation

$$1 = \rho_\Psi \left[\varphi \left(\lambda_0 \sum_{i=1}^k e_i \right) \right] = k\Psi[\varphi(\lambda_0)].$$

Then $\varphi(\lambda_0) = \Psi^{-1}(1/k)$. By Proposition 14 in [9, p. 70] we have

$$(13) \quad f(k, l^\Phi) = \left\| \sum_{i=1}^k e_i \right\|_\Phi = \left\langle \sum_{i=1}^k e_i, \varphi \left(\lambda_0 \sum_{i=1}^k e_i \right) \right\rangle = k\varphi(\lambda_0) = k\Psi^{-1}(1/k).$$

Finally, (11) follows from (8), (13) and (9). ■

THEOREM 2. *Let $\Phi(u) = \int_0^{|u|} \varphi(t) dt$ and $\Psi(v) = \int_0^{|v|} \psi(s) ds$ be as in Theorem 1 and let $F_\Phi(t) = t\varphi(t)/\Phi(t)$, $t > 0$. Assume that φ is continuous and strictly increasing on $[0, \max\{\Phi^{-1}(1), \psi[\Psi^{-1}(1)]\}]$.*

(i) *If $F_\Phi(t)$ is increasing on $(0, \Phi^{-1}(1)]$, then*

$$(14) \quad JC(l^{(\Phi)}) \geq \max\left(2^{1/C_\Phi^0-1}, \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)}\right),$$

where

$$(15) \quad C_\Phi^0 = \lim_{t \rightarrow 0} F_\Phi(t).$$

(ii) *If $F_\Phi(t)$ is increasing on $(0, \psi[\Psi^{-1}(1)]]$, then*

$$(16) \quad JC(l^{(\Phi)}) \geq \max\left(2^{1/C_\Phi^0-1}, \frac{\Psi^{-1}(1)}{2\Psi^{-1}(1/2)}\right).$$

Proof. (i) Since $F_\Phi(t)$ is increasing on $(0, \Phi^{-1}(1)]$ and $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, the limit C_Φ^0 exists and $1 < C_\Phi^0 < \infty$ as noted before. Further, the function $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ for $u > 0$ is increasing on $(0, 1/2]$ (cf. [10, p. 93]) and $\gamma_\Phi^0 = \lim_{u \rightarrow 0} G_\Phi(u)$ exists. It follows from (9) and Corollary 9 in [10, p. 8] that

$$\beta'_\Phi = \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)} = G_\Phi(1/2) \geq \lim_{u \rightarrow 0} G_\Phi(u) = \gamma_\Phi^0 = 2^{-1/C_\Phi^0},$$

which proves (14) by (10) and (7).

(ii) Recall that the Young inequality $|uv| \leq \Phi(u) + \Psi(v)$ becomes an equality when $|v| = \varphi(|u|)$ or $|u| = \psi(|v|)$ (see [9, p. 10]). Note that $\varphi(t)$ is strictly increasing on $(0, \psi[\Psi^{-1}(1)]]$ if and only if $\psi(s)$ is continuous on $(0, \Psi^{-1}(1)]$. By letting $s = \varphi(t)$, $0 < t \leq \psi[\Psi^{-1}(1)]$, we see that $t = \psi(s)$ and

$$F_\Psi(s) = \frac{s\psi(s)}{\Psi(s)} = \frac{t\varphi(t)}{t\varphi(t) - \Phi(t)} = \frac{F_\Phi(t)}{F_\Phi(t) - 1},$$

or

$$(17) \quad \frac{1}{F_\Phi(t)} + \frac{1}{F_\Phi(s)} = 1, \quad 0 < s = \varphi(t) \leq \Psi^{-1}(1),$$

which implies

$$(18) \quad \frac{1}{C_\Phi^0} + \frac{1}{C_\Psi^0} = 1,$$

where $C_\Psi^0 = \lim_{s \rightarrow 0} F_\Psi(s)$. By the assumption on $F_\Phi(t)$ and (17), $F_\Psi(s)$ is decreasing on $(0, \Psi^{-1}(1)]$ so that $G_\Psi(v) = \Psi^{-1}(v)/\Psi^{-1}(2v)$ is decreasing on $(0, 1/2]$ and $\gamma_\Psi^0 = \lim_{v \rightarrow 0} G_\Psi(v)$ exists. Finally,

$$2\alpha'_\Psi = \frac{2\Psi^{-1}(1/2)}{\Psi^{-1}(1)} = 2G_\Psi(1/2) \leq 2\alpha_\Psi^0 = 2\gamma_\Psi^0 = 2^{1-1/C_\Psi^0},$$

so that (16) follows from (11), (7) and (18). ■

3. Examples. To illustrate Theorems 1 and 2 we present some examples, that were considered by Zhang [15] and Yan [14], except Example 4.

EXAMPLE 1. Consider the following pair of complementary N -functions:

$$\Phi(u) = e^{|u|} - |u| - 1, \quad \Psi(v) = (1 + |v|) \ln(1 + |v|) - |v|.$$

In [10, p. 9] it was proved that $F_\Phi(t) (= t\Phi'(t)/\Phi(t))$ is increasing on $(0, \infty)$ and $C_\Phi^0 = \lim_{t \rightarrow 0} F_\Phi(t) = 2$, so that $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. By Theorem 2 we obtain

$$JC(l^{(\Phi)}) \geq \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)} \approx 0.7483, \quad JC(l^{(\Psi)}) \geq \frac{\Psi^{-1}(1)}{2\Psi^{-1}(1/2)} \approx 0.7435,$$

which refine the inequalities $JC(l^{(\Phi)}), JC(l^{(\Psi)}) \geq 1/\sqrt{2}$ in [15, Example 2.12]. In fact, Franchetti [5] proved that $JC(X) \geq 1/\sqrt{2}$ for any infinite-dimensional Banach space X with a symmetric norm (see also [2, Theorem 1]).

EXAMPLE 2. For the space $l^{(\Phi_p)}$ generated by the N -function

$$(19) \quad \Phi_p(u) = e^{|u|^p} - 1, \quad 1 < p < \infty,$$

we have

$$(20) \quad JC(l^{(\Phi_p)}) \geq \left(\frac{\ln 3 - \ln 2}{\ln 2} \right)^{1/p} > \max(2^{1/p-1}, 2^{-1/p})$$

if $1 + a < p < \infty$, where

$$(21) \quad a = \frac{1}{\ln 2} \ln \left(\frac{\ln 2}{\ln 3 - \ln 2} \right) \approx 0.7736.$$

Proof. The function $F_{\Phi_p}(t) = pt^p e^{t^p} / (e^{t^p} - 1)$ is increasing on $(0, \infty)$ and $C_{\Phi_p}^0 = \lim_{t \rightarrow 0} F_{\Phi_p}(t) = p$, so that $\Phi_p \in \Delta_2(0) \cap \nabla_2(0)$. We first consider $2 \leq p < \infty$. Since $\Phi_p^{-1}(u) = [\ln(1 + u)]^{1/p}$ for $u > 0$, from (14) in Theorem 2 we have

$$JC(l^{(\Phi_p)}) \geq \frac{\Phi_p^{-1}(1/2)}{\Phi_p^{-1}(1)} > 2^{-1/p} \geq 2^{1/p-1}, \quad 2 \leq p < \infty,$$

which proves (20) when $2 \leq p < \infty$. Next we consider $1 < p < 2$. It can be seen that $\Phi_p^{-1}(1/2)/\Phi_p^{-1}(1) > 2^{1/p-1}$ if and only if $1 + a < p < 2$, where a is given by (21). Thus,

$$(22) \quad JC(l^{(\Phi_p)}) \geq \frac{\Phi_p^{-1}(1/2)}{\Phi_p^{-1}(1)} > 2^{1/p-1} > 2^{-1/p}, \quad 1 + a < p < 2,$$

which completes the proof of (20). ■

REMARK 1. The estimate (20) improves the estimate in [15, Example 2.11] when $1 + a < p < \infty$. The estimate (22) shows that the exact

value for $JC(l^{(\Phi_p)})$ given in [14, Example 2.7], $JC(l^{(\Phi_p)}) = 2^{1/p-1}$, $1 < p < 2$, is impossible at least when $1.7736 \approx 1 + a < p < 2$.

EXAMPLE 3. Consider the N -function

$$(23) \quad M_p(u) = |u|^{2p} + 2|u|^p, \quad 1 < p < \infty.$$

Then

$$(24) \quad JC(l^{(M_p)}) \geq \left(\frac{\sqrt{3} - \sqrt{2}}{2 - \sqrt{2}} \right)^{1/p} > \max(2^{1/p-1}, 2^{-1/p})$$

if $1 + b < p < \infty$, where

$$(25) \quad b = \frac{1}{\ln 2} \ln \left(\frac{2 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \right) \approx 0.8812.$$

Proof. $F_{M_p}(t)$ is increasing on $(0, \infty)$ from p to $2p$ and $C_{M_p}^0 = p$, so that $M_p \in \Delta_2(0) \cap \nabla_2(0)$. The inverse of M_p is $M_p^{-1}(u) = (\sqrt{1+u} - 1)^{1/p}$, $u > 0$. Thus, the proof is similar to that of Example 2. ■

REMARK 2. The estimate (24) and (25) show that the exact value for $JC(l^{(M_p)})$ in [14, Example 2.6] is not true when $1 + b < p < 2$.

Let us make a comparison between Corollary 1 and Theorem 2 when C_Φ^0 exists to estimate the lower bounds of $JC(l^{(\Phi)})$ and $JC(l^\Phi)$. If $F_\Phi(t)$ is increasing on $(0, \Phi^{-1}(1)]$ or on $(0, \psi[\Psi^{-1}(1)]]$, then Theorem 2 is an improvement of Corollary 1, as shown by Examples 1–3. On the other hand, Theorem 1 is not better than Proposition 2 if $F_\Phi(t)$ is decreasing.

EXAMPLE 4. Let $M_{p,\lambda}(u) = |u|^p[\ln(1 + |u|)]^r$ with $1 \leq p < \infty$ and $0 < \lambda < \infty$, which is considered in [2, p. 184] when $p = 2$ and $\lambda \geq 0$. The function $F_{M_{p,\lambda}}(t) = p + \{\lambda t / [(1 + t) \ln(1 + t)]\}$ is decreasing on $(0, \infty)$ and $C_{M_{p,\lambda}}^0 = p + \lambda > 1$, so that $M_{p,\lambda} \in \Delta_2(0) \cap \nabla_2(0)$. It can be seen that $\beta'_{M_{p,\lambda}} \leq \lim_{u \rightarrow 0} G_{M_{p,\lambda}}(u) = \beta_{M_{p,\lambda}}^0$. Therefore, Proposition 2 improves Theorem 1. Finally, from Corollary 1 we obtain

$$\min(JC(l^{(M_{p,\lambda})}), JC(l^{M_{p,\lambda}})) \geq \max(2^{1/p+\lambda-1}, 2^{-1/p+\lambda}) = JC(l^{p+\lambda})$$

(see the end of Section 1 for $JC(l^p)$).

4. Intermediate Orlicz sequence spaces. Now we turn to certain intermediate N -functions and the corresponding intermediate Orlicz sequence spaces.

DEFINITION 6. Let Φ be an N -function and let $\Phi_0(u) = u^2$. For each $0 < s \leq 1$ we define an intermediate N -function Φ_s between Φ and Φ_0 at s to be the inverse of

$$(26) \quad \Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s} [\Phi_0^{-1}(u)]^s = [\Phi^{-1}(u)]^{1-s} (\sqrt{u})^s, \quad u \geq 0.$$

The corresponding Orlicz sequence space $l^{(\Phi_s)}$ [l^{Φ_s}] is called an *intermediate space* between $l^{(\Phi)}$ [l^Φ] and l^2 at $s \in (0, 1]$ in the sense of Calderón (cf. [9, p. 223]).

The author [11] proved that $\Phi_s \in \Delta_2 \cap \nabla_2$ if $0 < s \leq 1$, where $\Delta_2 = \Delta_2(0) \cap \Delta_2(\infty)$ and $\nabla_2 = \nabla_2(0) \cap \nabla_2(\infty)$, that is, $\Phi_s(u)$ satisfies both the Δ_2 - and ∇_2 -conditions for all $u \geq 0$ (cf. [10, p. 40]).

The following result enables us to find the exact values of $JC(l^{(\Phi_s)})$ and $JC(l^{\Phi_s})$ as well as the normal structure coefficients $N(l^{(\Phi_s)})$ and $N(l^{\Phi_s})$.

LEMMA 2 (Ren [11]). *Let Φ be an N -function and let Φ_s be the inverse of (26). Then for every $s \in (0, 1]$ we have*

$$(27) \quad 2^{s/2} \leq \min(N(l^{(\Phi_s)}), N(l^{\Phi_s})).$$

The proof of (27) can be found also in [10, p. 85]. The main result of this section is given by:

THEOREM 3. *Let Φ and Φ_s be as in Lemma 2. Then*

$$(28) \quad \max\left(\beta'_{\Phi_s}, \frac{1}{2\alpha_{\Phi_s}^0}, \beta_{\Phi_s}^0\right) \leq JC(l^{(\Phi_s)}) \leq 2^{-s/2},$$

$$(29) \quad \max\left(\frac{1}{2\alpha'_{\Psi_s^+}}, \frac{1}{2\alpha_{\Phi_s}^0}, \beta_{\Phi_s}^0\right) \leq JC(l^{\Phi_s}) \leq 2^{-s/2},$$

where Ψ_s^+ is complementary to Φ_s . In particular, if $\Phi \notin \Delta_2(0) \cap \nabla_2(0)$, then $JC(l^{(\Phi_s)}) = 2^{-s/2} = JC(l^{\Phi_s})$.

Proof. (28) and (29) follow from (27), (1), Proposition 2, Theorem 1 and the fact that $2\alpha_{\Phi_s}^0\beta_{\Psi_s^+}^0 = 1 = 2\alpha'_{\Psi_s^+}\beta_{\Phi_s}^0$ (cf. [10, p. 15]). The proof of the second part is the same as that of Theorem 9 in [10, p. 145]. ■

REMARK 3. Theorem 3 partly refines Theorem 6 in [10, p. 144].

For computing the exact value of the Jung constant in a special Banach space we now present a simple example. Consider the N -function

$$M_p(u) = |u|^p e^{-1/|u|}, \quad 1 < p < \infty.$$

Its inverse M_p^{-1} on $[0, \infty)$ is not obtained explicitly. Note that

$$C_{M_p}^0 = \lim_{t \rightarrow 0} tM_p'(t)/M_p(t) = \lim_{t \rightarrow 0} (p + 1/t) = \infty,$$

so that $M_p \notin \Delta_2(0)$. Let Φ_s be the inverse of the function $\Phi_s^{-1}(u) = [M_p^{-1}(u)]^{1-s}(\sqrt{u})^s$, $u \geq 0$ and $0 < s \leq 1$. By the second part of Theorem 3 we obtain the exact values $JC(l^{(\Phi_s)}) = 2^{-s/2} = JC(l^{\Phi_s})$.

For the upper bounds of the normal structure coefficients $N(l^{(\Phi)})$ and $N(l^\Phi)$ we have the following.

PROPOSITION 3. *Let (Φ, Ψ) be a pair of complementary N -functions with $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. Then*

$$(30) \quad N(l^{(\Phi)}) \leq \min\left(2\alpha_{\Phi}^0, \frac{1}{\beta_{\Phi}^0}, \frac{1}{\beta'_{\Phi}}\right),$$

$$(31) \quad N(l^{\Phi}) \leq \min\left(2\alpha_{\Phi}^0, \frac{1}{\beta_{\Phi}^0}, 2\alpha'_{\Psi}\right),$$

where $\alpha_{\Phi}^0, \beta_{\Phi}^0, \beta'_{\Phi}$ and α'_{Ψ} are as in (5) and (9).

Proof. In [10, p. 108] it was proved that

$$(32) \quad \max(N(l^{(\Phi)}), N(l^{\Phi})) \leq \min\left(2\alpha_{\Phi}^0, \frac{1}{\beta_{\Phi}^0}\right).$$

It should be noted that (32) can also be deduced immediately from Proposition 2 and (1). By Theorem 1 we get

$$(33) \quad N(l^{(\Phi)}) \leq 1/\beta'_{\Phi}, \quad N(l^{\Phi}) \leq 2\alpha'_{\Psi}.$$

Finally, (30) and (31) follow from (32) and (33). ■

COROLLARY 2. *Let (Φ, Ψ) and $F_{\Phi}(t)$ be as in Theorem 2.*

(i) *If $F_{\Phi}(t)$ is increasing on $(0, \Phi^{-1}(1)]$, then*

$$(34) \quad N(l^{(\Phi)}) \leq \min\left(2^{1-1/C_{\Phi}^0}, \frac{\Phi^{-1}(1)}{\Phi^{-1}(1/2)}\right).$$

(ii) *If $F_{\Phi}(t)$ is increasing on $(0, \psi[\Psi^{-1}(1)]]$, then*

$$(35) \quad N(l^{\Phi}) \leq \min\left(2^{1-1/C_{\Phi}^0}, \frac{2\Psi^{-1}(1/2)}{\Psi^{-1}(1)}\right).$$

By Lemma 2 and Proposition 3 we get the desired result.

THEOREM 4. *Let Φ and Φ_s be as in Lemma 2. Then*

$$(36) \quad 2^{s/2} \leq N(l^{(\Phi_s)}) \leq \min\left(2\alpha_{\Phi_s}^0, \frac{1}{\beta_{\Phi_s}^0}, \frac{1}{\beta'_{\Phi_s}}\right),$$

$$2^{s/2} \leq N(l^{\Phi_s}) \leq \min\left(2\alpha_{\Phi_s}^0, \frac{1}{\beta_{\Phi_s}^0}, 2\alpha'_{\Psi_s^+}\right).$$

In particular, if $\Phi \notin \Delta_2(0) \cap \nabla_2(0)$, we get $N(l^{(\Phi_s)}) = 2^{s/2} = N(l^{\Phi_s})$.

REMARK 4. Theorem 4 partly refines Theorem 8 in [10, p. 114].

5. Supplements to examples. Now we apply Corollary 2, Theorems 3 and 4 to Examples 1–3 in Section 3.

EXAMPLE 5. Let (Φ, Ψ) be as in Example 1. By (34) and (35) we obtain

$$N(l^{(\Phi)}) \leq \frac{\Phi^{-1}(1)}{\Phi^{-1}(1/2)} \approx 1.3364, \quad N(l^{(\Psi)}) \leq \frac{2\Psi^{-1}(1/2)}{\Psi^{-1}(1)} \approx 1.3450.$$

Before studying the next two examples we have to prove the following.

LEMMA 3. Let $M(u) = \int_0^{|u|} p(t) dt$ be an N -function and let $\Phi_s(u) = \int_0^{|u|} \varphi_s(t) dt$ be the inverse of $\Phi_s^{-1}(u) = [M^{-1}(u)]^{1-s}(\sqrt{u})^s$, $u \geq 0$ and $0 < s \leq 1$. Consider $F_M(t) = tp(t)/M(t)$, $t > 0$, and $F_{\Phi_s}(t) = t\varphi_s(t)/\Phi_s(t)$, $t > 0$. Then $F_M(t)$ is increasing on $(0, M^{-1}(1)]$ if and only if $F_{\Phi_s}(t_s)$ is increasing on $(0, \Phi_s^{-1}(1)]$.

Proof. Since $\ln \Phi_s^{-1}(u) = (1-s)\ln M^{-1}(u) + (s/2)\ln u$ for $u > 0$ we get by differentiation

$$\frac{1}{\Phi_s^{-1}(u)\varphi_s[\Phi_s^{-1}(u)]} = \frac{1-s}{M^{-1}(u)p[M^{-1}(u)]} + \frac{s}{2u},$$

so that

$$\frac{\Phi_s[\Phi_s^{-1}(u)]}{\Phi_s^{-1}(u)\varphi_s[\Phi_s^{-1}(u)]} = \frac{(1-s)M[M^{-1}(u)]}{M^{-1}(u)p[M^{-1}(u)]} + \frac{s}{2}, \quad u > 0.$$

By letting $u = M(t)$ and $t_s = \Phi_s^{-1}(u) = \Phi_s^{-1}[M(t)] = t^{1-s}[M(t)]^{s/2}$ in the above, we see that $0 < t_s \leq [M^{-1}(1)]^{1-s} = \Phi_s^{-1}(1)$ if $0 < t \leq M^{-1}(1)$ and that

$$(37) \quad \frac{1}{F_{\Phi_s}(t_s)} = \frac{1-s}{F_M(t)} + \frac{s}{2},$$

which proves the lemma. ■

EXAMPLE 6. Let Φ_p be given by (19) in Example 2.

(i) It follows from (34) in Corollary 2 that

$$N(l^{(\Phi_p)}) \leq \left(\frac{\ln 2}{\ln 3 - \ln 2} \right)^{1/p} < \min(2^{1-1/p}, 2^{1/p})$$

if $1 + a < p < \infty$, where a is in (21).

(ii) Let $\Phi_s(u)$ be the intermediate N -function between $\Phi_p(u)$ and u^2 at $s \in (0, 1]$, i.e.,

$$(38) \quad \Phi_s^{-1}(u) = [\ln(1+u)]^{(1-s)/p}(\sqrt{u})^s, \quad u \geq 0.$$

By Lemma 3, $F_{\Phi_s}(t)$ is increasing on $(0, \Phi_s^{-1}(1)]$ and

$$\beta'_{\Phi_s} = \frac{\Phi_s^{-1}(1/2)}{\Phi_s^{-1}(1)} = (\beta'_{\Phi_p})^{1-s} \left(\frac{1}{\sqrt{2}} \right)^s.$$

From Theorem 3 and Theorem 2 for $1 + a < p < \infty$ with a being from (21), we have

$$(39) \quad 2^{-s/2} \geq JC(l^{(\Phi_s)}) \geq \beta'_{\Phi_s} = 2^{-s/2} \left(\frac{\ln 3 - \ln 2}{\ln 2} \right)^{(1-s)/p} > \max(2^{(1-s)/p+s/2-1}, 2^{(s-1)/p-s/2}).$$

(iii) Let Φ_s be the inverse of (38). By Theorem 4 we have $2^{s/2} \leq N(l^{(\Phi_s)}) \leq 1/\beta'_{\Phi_s}$ if $1 + a < p < \infty$.

REMARK 5. (39) shows that the exact value for $JC(l^{(\Phi_s)})$ in [14, Example 2.7, equality (20)] is not valid when $1 + a < p < 2$.

Also, (39) is an improvement of Example 8 in [10, p. 144].

(iii) in Example 6 refines Example 9 in [10, p. 115].

EXAMPLE 7. Let M_p be given by (23) of Example 3. Similarly to Example 6, we have the following assertions:

(i) Let $1 + b < p < \infty$ with b as in (25). Then

$$N(l^{(M_p)}) \leq \frac{1}{\beta'_{M_p}} = \left(\frac{2 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \right)^{1/p} < \min(2^{1-1/p}, 2^{1/p}).$$

(ii) Let M_s be the inverse of $M_s^{-1}(u) = (\sqrt{1+u} - 1)^{(1-s)/p} (\sqrt{u})^s$, $u \geq 0$ and $0 < s \leq 1$. Then note that

$$\beta'_{M_s} = (\beta'_{M_p})^{1-s} \left(\frac{1}{2} \right)^{s/2} = 2^{-s/2} \left(\frac{\sqrt{3} - \sqrt{2}}{2 - \sqrt{2}} \right)^{(1-s)/p}$$

and that (37) implies

$$\frac{1}{C_{M_s}^0} = \frac{1-s}{C_{M_p}^0} + \frac{s}{2} = \frac{1-s}{p} + \frac{s}{2}.$$

Moreover, it follows from Theorems 2 and 3 that for $1 + b < p < \infty$,

$$(40) \quad 2^{-s/2} \geq JC(l^{(M_s)}) \geq \beta'_{M_s} > \max(2^{1/C_{M_s}^0-1}, 2^{-1/C_{M_s}^0}).$$

(iii) Let M_s be as in (ii). Then $2^{s/2} \leq N(l^{(M_s)}) \leq 1/\beta'_{M_s}$ whenever $1 + b < p < \infty$.

REMARK 6. (40) shows that the exact value in [14, Example 2.6] is not true when $1.8821 \approx 1 + b < p < 2$. Also, (40) is an improvement of Example 3.7 in [15, p. 43].

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