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Lifting to the r-frame bundle by means of connections

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Dedicated to Professor Andrzej Zajtz on the occasion of his 75th birthday with respect and gratitude

Abstract. Let m and r be natural numbers and let $P^r : \mathcal{M}f_m \to \mathcal{FM}$ be the rth order frame bundle functor. Let $F : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ be natural bundles, where $k = \dim(P^r\mathbb{R}^m)$. We describe all $\mathcal{M}f_m$-natural operators A transforming sections σ of $FM \to M$ and classical linear connections ∇ on M into sections $A(\sigma,\nabla)$ of $G(P^rM) \to P^rM$. We apply this general classification result to many important natural bundles F and G and obtain many particular classifications.

0. Introduction. We fix natural numbers m and r. Let $P^r : \mathcal{M}f_m \to \mathcal{FM}$ be the rth order frame bundle functor, $k = \dim(P^r\mathbb{R}^m)$, and let $F : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ be natural bundles, where $\mathcal{M}f_m$ is the category of m-dimensional manifolds and their local diffeomorphisms and $\mathcal{FM}$ is the category of fibred manifolds and their fibred maps.

In the present note, we study the problem how a section $\sigma \in \underline{F}(M)$ of $FM \to M$ and a classical linear connection ∇ on M can induce a section $A(\sigma,\nabla) \in \underline{G}(P^rM)$ of $G(P^rM) \to P^rM$. This problem is reflected in the concept of $\mathcal{M}f_m$-natural operators $A : F \times Q \rightsquigarrow GP^r$ in the sense of [3]. We describe all $\mathcal{M}f_m$-natural operators $A : F \times Q \rightsquigarrow GP^r$ in question.

There are many "classical" examples of $\mathcal{M}f_m$-natural operators $A : F \times Q \to GP^r$ for particular F and G and r. For example, we know the so-called horizontal lifts of vector fields, forms, tensor fields, connections, differentiations from M to the linear frame bundle $LM = P^1M$ (see e.g. [1], [6]). In [10], using rather complicated computations in local coordinates, M. Sekizawa obtained an interesting classification of all first order $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow T^{(0,2)}P^1$ transforming classical linear connections ∇ on m-manifolds M into tensor fields $A(\nabla)$ of type $(0,2)$ on the linear frame bundle $LM = P^1M$ of M. A well-known example of an $\mathcal{M}f_m$-natural op-

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erator $A : Q \rightsquigarrow QP^r$ is the so-called complete lift in the sense of A. Morimoto [8] (see also [2]) of classical linear connections to the rth order frame bundle P^rM (which is an open subbundle in the bundle T^r_mM of (m,r)-velocities). In [7], using a normal coordinate technique, the second author extended (in a very simple way) the classification of [10] to all $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow T^{(p,q)}P^r$, and in particular obtained a classification of all $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow QP^r$. In [4], adapting the technique from [7], we classified all $\mathcal{M}f_m$-natural operators $A : T^{(p,q)} \times Q \rightsquigarrow T^{(p_1,q_1)}P^1$ transforming tensor fields τ of type (p,q) on M and classical linear connections ∇ on M into tensor fields $A(\tau,\nabla)$ of type (p_1,q_1) on LM. In [5], also adapting the technique from [7], we described all $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow \mathrm{Riem}\,P^r$ transforming classical linear connections ∇ on M into Riemannian structures $A(\nabla)$ on P^rM. Thus the main result of the present paper is a (maximal possible) generalization of the results mentioned above. To obtain this general result, we once more adapt the technique from [7]. Thanks to this technique, the proof of the main result seems to be almost obvious.

We apply the main result of the present note to many important F and G, and obtain several interesting classifications (also different from those in [7], [4], [5]). Namely, for $F = \mathrm{id}_{\mathcal{M}f_m}$ and $G = E^{(k)} = (J^k(-,\cdot))^*$ we obtain a full classification of $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow E^{(k)}P^r$ of kth order linear differential operators $A(\nabla)$ on P^rM by means of classical linear connections ∇ on M. Similarly, for $F = \mathrm{Riem} : \mathcal{M}f_m \to \mathcal{FM}$ and $G = \mathrm{Riem} : \mathcal{M}f_k \to \mathcal{FM}$ we obtain a full classification of $\mathcal{M}f_m$-natural operators $A : \mathrm{Riem} \times Q \to \mathrm{Riem}\,P^r$ of Riemannian structures $A(g,\nabla)$ on P^rM from Riemannian structures g on M by means of classical linear connections ∇ on M. And similarly, for $F = Q : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ we obtain a full classification of $\mathcal{M}f_m$-natural operators $A : Q \times Q \to QP^r$ of classical linear connections $A(\nabla_1,\nabla)$ on P^rM from classical linear connections ∇_1 on M by means of classical linear connections ∇ on M.

All manifolds and maps are assumed to be smooth, i.e. of class C^∞.

1. Natural bundles and natural operators. The concept of natural bundles was introduced by A. Nijenhuis [9].

Definition 1 (see [3]). A *natural bundle* over m-manifolds is a covariant functor $F : \mathcal{M}f_m \to \mathcal{FM}$ with the following properties:

(i) *Base preservation.* $B \circ F = \mathrm{id}_{\mathcal{M}f_m}$, where $B : \mathcal{FM} \to \mathcal{M}f$ is the base functor.

(ii) *Locality.* Let $U \subset M$ be an open subset of an m-manifold M and let $i_U : U \to M$ denote the inclusion map. Then $FU = \pi_M^{-1}(U)$ and the induced map $F(i_U) : FU \to FM$ is the inclusion map of the inclusion $FU \subset FM$.

(iii) *Regularity.* F transforms smoothly parametrized families of local diffeomorphisms into smoothly parametrized families of fibred maps.

EXAMPLE 1. A simple example of a natural bundle over m-manifolds is the tangent functor $T : \mathcal{M}f_m \to \mathcal{FM}$ transforming any m-manifold M into its tangent bundle TM and any local diffeomorphism $\psi : M \to M_1$ into its tangent map $T\psi : TM \to TM_1$. Another example is the cotangent functor $T^* : Mf_m \to \mathcal{FM}$ sending any m-manifold M into its cotangent bundle $T^*M = (TM)^*$ and any local diffeomorphism $\psi : M \to M_1$ into its cotangent map $T^*\psi : T^*M \to T^*M_1$. Or more generally, given non-negative integers p and q, the functor $T^{(p,q)} : \mathcal{M}f_m \to \mathcal{FM}$ sending any m-manifold M into its bundle $T^{(p,q)}M = \bigotimes^p TM \otimes \bigotimes^q T^*M$ of tensors of type (p, q) and any local diffeomorphism $\psi : M \to M_1$ into its induced map $T^{(p,q)}\psi = \bigotimes^p T\psi \otimes \bigotimes^q T^*\psi : T^{(p,q)}M \to T^{(p,q)}M_1$ is a natural bundle over m-manifolds.

EXAMPLE 2. For any m-manifold M we have the Riemannian bundle $\mathrm{Riem}(M) = \bigcup_{x\in M} \mathrm{Met}(T_xM)$ over M, where given a vector space V we denote by $\mathrm{Met}(V)$ the set of inner products $G : V \times V \to \mathbb{R}$ on V. (We recall that $G : V \times V \to \mathbb{R}$ is an *inner product* if it is symmetric, bilinear and positive definite.) Clearly, $\mathrm{Riem}(M)$ is an open subbundle in the vector bundle $T^*M \odot T^*M$ of symmetric tensors of type $(0, 2)$ over M. Sections $g : M \to \mathrm{Riem}(M)$ are the so-called Riemannian structures on M. Every local diffeomorphism $\psi : M \to M_1$ induces $\mathrm{Riem}(\psi) : \mathrm{Riem}(M) \to \mathrm{Riem}(M_1)$, which is the restriction of $T^*\psi \odot T^*\psi : T^*M \odot T^*M \to T^*M_1 \odot T^*M_1$. The correspondence $\mathrm{Riem} : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

EXAMPLE 3. For any m-manifold M we have the extended sth order vector tangent bundle $E^{(s)}M = (J^s(M, \mathbb{R}))^*$ of M. Sections $D : M \to E^{(s)}M$ of $E^{(s)}M$ are in bijection with sth order linear differential operators $\widetilde{D} : C^\infty(M) \to C^\infty(M)$. (More precisely, we put $\widetilde{D}(f)(x) := \langle D(x), j^s_x f\rangle$, $f : M \to \mathbb{R}$, $x \in M$.) Every $\mathcal{M}f_m$-map $\psi : M \to M_1$ induces $E^{(s)}\psi : E^{(s)}M \to E^{(s)}M_1$, $\langle E^{(s)}\psi(\omega), j^s_{\psi(x)}g\rangle = \langle \omega, j^s_x(g \circ \psi)\rangle$ for $\omega \in E^{(r)}_x M$, $x \in M$, $g : M_1 \to \mathbb{R}$. The correspondence $E^{(s)} : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

EXAMPLE 4. For any m-manifold M we have the bundle $F^{(p,q,s)}M = (J^s(\bigwedge^p T^*M))^* \otimes \bigwedge^q T^*M$ over M. Sections $D : M \to F^{(p,q,s)}M$ are in bijection with sth order linear differential operators $\widetilde{D} : \Omega^p(M) \to \Omega^q(M)$ from p-forms on M into q-forms on M. (More precisely, we put $\widetilde{D}(\omega)(x) = \langle D(x), j^s_x\omega\rangle$ for $\omega \in \Omega^p(M)$, $x \in M$). Every $\mathcal{M}f_m$-map $\psi : M \to M_1$ induces $F^{(p,q,s)}\psi : F^{(p,q,s)}M \to F^{(p,q,s)}M_1$ in an obvious way. The correspondence $F^{(p,q,s)} : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

Example 5. For any m-manifold M we have the rth order frame bundle $P^rM = \mathrm{inv}\, J^r_0(\mathbb{R}^m, M)$ of M. This is a principal bundle with the corresponding Lie group $G^r_m = J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ acting on the right on P^rM via compositions of jets. Every $\mathcal{M}f_m$-map $\psi : M \to M_1$ induces a principal bundle map $P^r\psi : P^rM \to P^rM_1$ by $P^r\psi(j^r_0\varphi) = j^r_0(\psi \circ \varphi)$, where $\varphi : \mathbb{R}^m \to M$ is an $\mathcal{M}f_m$-map. The correspondence $P^r : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle.

Example 6. For any m-manifold M we have the classical linear connection bundle $QM := (\mathrm{id}_{T^*M} \otimes \pi^1)^{-1}(\mathrm{id}_{TM}) \subset T^*M \otimes J^1TM$ of M, where $\pi^1 : J^1TM \to TM$ is the projection of the first jet prolongation $J^1TM = \{j^1_xX \mid X \in \mathcal{X}(M), x \in M\}$ of the tangent bundle TM of M. Sections $\tilde{\nabla} : M \to QM$ correspond bijectively to classical linear connections on M. Every local diffeomorphism $\psi : M \to M_1$ induces (in an obvious way) a fibred map $Q\psi : QM \to QM_1$ over ψ. The correspondence $Q : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle.

Remark 1. A classical linear connection on a manifold M is an $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that (1) $\nabla_{fX}Y = f\nabla_XY$ and (2) $\nabla_XfY = XfY + f\nabla_XY$ for any vector fields $X, Y \in \mathcal{X}(M)$ on M and any map $f : M \to \mathbb{R}$. The classical linear connection ∇ corresponding to a section $\tilde{\nabla} : M \to QM$ is defined by $(\nabla_XY)_x = TY(X_x) - \mathcal{T}(Y_x, \langle\tilde{\nabla}, X_x\rangle) \in V_{Y_x}TM = T_xM$, where $\mathcal{T} : TM \times_M J^1TM \to TTM$ is given by $\mathcal{T}(v, j^1_xZ) = \mathcal{T}(Z)_v$ (here $\mathcal{T}(Z)$ means the flow lifting of $Z \in \mathcal{X}(M)$ to TM).

Remark 2. One can show (see e.g. [3]) that any natural bundle $F : \mathcal{M}f_m \to \mathcal{FM}$ is associated with $P^r : \mathcal{M}f_m \to \mathcal{FM}$ for some r. Namely, $FM = P^rM \times_{G^r_m} S$ and $F\psi = P^r\psi \times_{G^r_m} \mathrm{id}_S$ for some r and some action of G^r_m on a manifold S.

A general concept of natural operators can be found in the fundamental monograph [3]. We only need the following special case of the definition of natural operators.

Definition 2. Let $F : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ be natural bundles, where $k = \dim(P^r\mathbb{R}^m)$. An $\mathcal{M}f_m$-*natural operator* $A : F \times Q \rightsquigarrow GP^r$ is a family of $\mathcal{M}f_m$-invariant regular operators (functions)

$$A = A_M : \underline{F}(M) \times \underline{Q}(M) \to \underline{G}(P^rM)$$

for any $\mathcal{M}f_m$-object M, where $\underline{F}(M)$ is the set of all sections of $FM \to M$, $\underline{Q}(M)$ is the set of all classical linear connections on M (sections of $Q(M) \to M$) and $\underline{G}(P^rM)$ is the set of all sections of $G(P^rM) \to P^rM$. The invariance means that if $(\sigma_1, \nabla_1) \in \underline{F}(M_1) \times \underline{Q}(M_1)$ and $(\sigma_2, \nabla_2) \in \underline{F}(M_2) \times \underline{Q}(M_2)$ are related by an $\mathcal{M}f_m$-map $\psi : M_1 \to M_2$ (i.e. $F\psi \circ \sigma_1 = \sigma_2 \circ \psi$ and $Q\psi \circ \nabla_1 = \nabla_2 \circ \psi$) then $A(\sigma_1, \nabla_1)$ and $A(\sigma_2, \nabla_2)$ are $P^r\psi$-related (i.e. $G(P^r\psi) \circ A(\sigma_1, \nabla_1) = A(\sigma_2, \nabla_2) \circ P^r\psi$). The regularity means that A

transforms smoothly parametrized families of pairs of sections into smoothly parametrized families of sections.

2. A general example of natural operators $A : F \times Q \rightsquigarrow GP^r$. Let $F : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ be natural bundles, where $k = \dim(P^r\mathbb{R}^m)$. We are going to present a general example of $\mathcal{M}f_m$-natural operators $A : F \times Q \rightsquigarrow GP^r$. We start with the following notations.

For $s = 0, 1, \ldots, \infty$, let Z^s be the set of all s-jets $j_0^s\nabla \in J_0^s(Q(\mathbb{R}^m))$ of classical linear connections ∇ on $\mathbb{R}^m$ with

$$\sum_{j,k=1}^{m} \nabla^i_{jk}(x)x^j x^k = 0 \quad \text{for } i = 1, \ldots, m.$$

We see that Z^s is a finite-dimensional manifold (diffeomorphic to a finite-dimensional vector space) if s is finite, and Z^∞ is a topological space with respect to the inverse limit topology given by the inverse system $\cdots \to Z^{s+1} \to Z^s \to \cdots \to Z^0$ of jet projections.

Remark 3. It is known that the condition defining Z^s is equivalent to saying that the usual coordinates $x^1, \ldots, x^m$ on $\mathbb{R}^m$ are ∇-normal with centre 0. To see this equivalence, apply the well-known system of partial differential equations

$$\frac{d^2\gamma^i}{dt^2} + \nabla^i_{jk}(\gamma)\,\frac{d\gamma^j}{dt}\,\frac{d\gamma^k}{dt} = 0, \qquad i = 1, \ldots, m,$$

on ∇-geodesics and apply the well-known fact that the ∇-geodesics passing through the centre of ∇-normal coordinates are straight lines.

Let $\theta := j_0^r(\mathrm{id}_{\mathbb{R}^m}) \in P^r\mathbb{R}^m$.

Let $H^r_m := \ker(G^r_m \to GL(m))$ be the kernel of the Lie group epimorphism $G^r_m \to GL(m)$, the jet projection.

Definition 3. We say that a function $\mu : J_0^\infty(F\mathbb{R}^m) \times Z^\infty \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ has the *local finite determination property* if for any $u_1 \in J_0^\infty(F\mathbb{R}^m)$, $u_2 \in Z^\infty$ and $u_3 \in H^r_m$ we can find an open neighbourhood $U_1 \subset J_0^\infty(F\mathbb{R}^m)$ of u_1, an open neighbourhood $U_2 \subset Z^\infty$ of u_2, an open neighbourhood $U_3 \subset H^r_m$ of u_3, a natural number s and a smooth map $f : \tilde{\pi}_s(U_1) \times \pi_s(U_2) \times U_3 \to G_\theta(P^r\mathbb{R}^m)$ such that $\mu = f \circ (\tilde{\pi}_s \times \pi_s \times \mathrm{id}_{U_3})$ on $U_1 \times U_2 \times U_3$, where $\tilde{\pi}_s : J_0^\infty(F\mathbb{R}^m) \to J_0^s(F\mathbb{R}^m)$ and $\pi_s : Z^\infty \to Z^s$ are the jet projections.

For example, if s is finite and $f : J_0^s(F\mathbb{R}^m) \times Z^s \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ is a smooth map, then $\mu = f \circ (\tilde{\pi}_s \times \pi_s \times \mathrm{id}_{W^r}) : J_0^\infty(F\mathbb{R}^m) \times Z^\infty \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ has the local finite determination property.

Now, we are in a position to present the following general example of $\mathcal{M}f_m$-natural operators $A : F \times Q \rightsquigarrow GP^r$.

EXAMPLE 7. Let $\mu : J^\infty_0(F\mathbb{R}^m) \times Z^\infty \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ be a function with the local finite determination property. Let ∇ be a classical linear connection on an m-manifold M and $\sigma \in \underline{F}(M)$ be a section of $FM \to M$. Define a section $A^{(\mu)}(\sigma,\nabla) \in \underline{G}(P^rM)$ of $G(P^rM) \to P^rM$ by

$$A^{(\mu)}(\sigma,\nabla)(\varrho) \\ := G(R_{J^r\psi(\varrho)})(G(P^r(\psi^{-1}))(\mu(j^\infty_0(\psi_*\sigma), j^\infty_0(\psi_*\nabla), J^r\psi(\varrho))))$$

$\varrho \in (P^r_xM)$, $x \in M$, where ψ is a ∇-normal coordinate system on M with centre x such that $P^1\psi(\pi^r_1(\varrho)) = j^1_0(\mathrm{id}_{\mathbb{R}^m})$ and $R_\xi : P^rM \to P^rM$ is the right translation by $\xi \in G^r_m$. Of course, $\psi_*\sigma = F\psi \circ \sigma \circ \sigma^{-1}$ is the image of σ under ψ. Similarly, $\psi_*\nabla = Q\psi \circ \nabla \circ \psi^{-1}$.

The definition of $A^{(\mu)}(\sigma,\nabla)(\varrho)$ is correct because $J^r\psi(\varrho) \in H^r_m \subset G^r_m = P^r_0\mathbb{R}^m$ and $\mathrm{germ}_x(\psi)$ is uniquely determined by ∇ and $\pi^r_1(\varrho)$. The map $A^{(\mu)}(\sigma,\nabla) : P^rM \to G(P^rM)$ is smooth by the local finite determination property of μ. It is easy to see that $A(\sigma,\nabla)$ is a section of $G(P^rM) \to P^rM$. Because of the canonical character of the construction of $A^{(\mu)}(\sigma,\nabla)$ we have the following lemma.

LEMMA 1. *The family $A^{(\mu)} : F \times Q \rightsquigarrow GP^r$ of operators*

$$A^{(\mu)}_M : \underline{F}(M) \times \underline{Q}(M) \to \underline{G}(P^rM), \quad A^{(\mu)}_M(\sigma,\nabla) = A^{(\mu)}(\sigma,\nabla),$$

is an $\mathcal{M}f_m$-natural operator.

3. A classification of natural operators $A : F \times Q \rightsquigarrow GP^r$. The main result of the present note is the following theorem.

THEOREM 1. *Let r and m be natural numbers and let $F : \mathcal{M}f_m \to \mathcal{FM}$ and $G : \mathcal{M}f_k \to \mathcal{FM}$ be natural bundles, where $k = \dim(P^r\mathbb{R}^m)$. Any $\mathcal{M}f_m$-natural operator $A : F \times Q \rightsquigarrow GP^r$ is of the form*

$$A_M(\sigma,\nabla) = A^{(\mu)}(\sigma,\nabla), \quad \sigma \in \underline{F}(M),\ \nabla \in \underline{Q}(M),$$

for some uniquely determined (by A) function $\mu : J^\infty_0(F\mathbb{R}^m) \times Z^\infty \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ with the local finite determination property.

In the special case $F = \mathrm{id}_{\mathcal{M}f_m}$, we see that $J^\infty_0(F\mathbb{R}^m)$ is a one-point set, and then any $\mathcal{M}f_m$-natural operator $A : Q \rightsquigarrow GP^r$ transforming classical linear connections ∇ on m-manifolds M into sections $A(\nabla)$ of $G(P^rM) \to P^rM$ is of the form

$$A_M(\nabla) = A^{(\mu)}(\nabla), \quad \nabla \in \underline{Q}(M),$$

for some uniquely determined function $\mu : Z^\infty \times H^r_m \to G_\theta(P^r\mathbb{R}^m)$ with the local finite determination property.

In the special case $r = 1$, we see that H^1_m is the trivial group and $P^1M = LM$ is the linear frame bundle, and then any $\mathcal{M}f_m$-natural operator A :

$F \times Q \rightsquigarrow GL$ is of the form

$$A_M(\sigma, \nabla) = A^{(\mu)}(\sigma, \nabla), \qquad \sigma \in \underline{F}(M),\ \nabla \in \underline{Q}(M),$$

for some function $\mu : J_0^\infty(F\mathbb{R}^m) \times Z^\infty \to G_{l_0}(L\mathbb{R}^m)$ with the local finite determination property, where $l_0 \in L\mathbb{R}^m$ is the usual basis in $T_0\mathbb{R}^m$.

Proof. Let $A : F \times Q \rightsquigarrow GP^r$ be an $\mathcal{M}f_m$-natural operator. We must define $\mu = \mu^A : J_0^\infty(F\mathbb{R}^m) \times Z^\infty \times H_m^r \to G_\theta(P^r\mathbb{R}^m)$ by

$$\mu(j_0^\infty\sigma, j_0^\infty\nabla, \varrho) = G(R_{\varrho^{-1}})(A(\sigma, \nabla)(\varrho)).$$

Then using the non-linear Peetre theorem and the Boman theorem (see e.g. [3]), we easily see that μ has the local finite determination property. Then by the definition of μ and A^μ we deduce that

$$A(\sigma, \nabla)(\varrho) = A^{(\mu)}(\sigma, \nabla)(\varrho)$$

for any section $\sigma \in \underline{F}(\mathbb{R}^m)$, any classical linear connection ∇ on $\mathbb{R}^m$ such that the identity map $\mathrm{id}_{\mathbb{R}^m}$ is a ∇-normal coordinate system with centre 0, and any $\varrho \in H_m^r$. Then by the invariance of A and $A^{(\mu)}$ with respect to normal coordinates we deduce that $A = A^{(\mu)}$. ∎

4. Some important corollaries. We present some corollaries of Theorem 1.

(a) *The case of $F = \mathrm{id}_{\mathcal{M}f_m}$ and $G = Q : \mathcal{M}f_k \to \mathcal{FM}$.* In this case we recover (in another form) the following result of [7].

Corollary 1. *The $\mathcal{M}f_m$-natural operators $A : Q \rightsquigarrow QP^r$ transforming classical linear connections ∇ on m-manifolds M into classical linear connections $A(\nabla)$ on P^rM are in bijection with the functions $\mu^A : Z^\infty \times H_m^r \to \mathbb{R}^{k*} \otimes \mathbb{R}^{k*} \otimes \mathbb{R}^k$ having the local finite determination property, where $k = \dim(P^r\mathbb{R}^m)$.*

Proof. We have $Q_\theta(P^r\mathbb{R}^m) = T_\theta^*P^r\mathbb{R}^m \otimes T_\theta^*P^r\mathbb{R}^m \otimes T_\theta P^r\mathbb{R}^m = \mathbb{R}^{k*} \otimes \mathbb{R}^{k*} \otimes \mathbb{R}^k$. ∎

(b) *The case of $r = 1$ and $F = T^{(p,q)} : \mathcal{M}f_m \to \mathcal{FM}$ and $G = T^{(p_1,q_1)} : \mathcal{M}f_k \to \mathcal{FM}$.* In this case we recover (in another form) the following result of [4].

Corollary 2. *The $\mathcal{M}f_m$-natural operators $A : T^{(p,q)} \times Q \rightsquigarrow T^{(p_1,q_1)}P^1$ transforming tensor fields τ of type (p,q) on m-manifolds M and classical linear connections ∇ on M into tensor fields $A(\tau, \nabla)$ of type (p_1, q_1) on the linear frame bundle $LM = P^1M$ are in bijection with the functions $\mu^A : J_0^\infty(T^{(p,q)}\mathbb{R}^m) \times Z^\infty \to \bigotimes^{p_1}\mathbb{R}^k \otimes \bigotimes^{q_1}\mathbb{R}^{k*}$ having the local finite determination property, where $k = m + m^2 = \dim(L\mathbb{R}^m)$.*

Proof. We have $T^{(p_1,q_1)}_{j_0^1(\mathrm{id}_{\mathbb{R}^m})}(P^1\mathbb{R}^m) = \bigotimes^{p_1}\mathbb{R}^k \otimes \bigotimes^{q_1}\mathbb{R}^{k*}$. ∎

(c) *The case of* $F = \mathrm{id}_{\mathcal{M}f_m}$ *and* $G = \mathrm{Riem} : \mathcal{M}f_k \to \mathcal{FM}$. In this case we recover (in another form) the following result of [5].

Corollary 3. *The* $\mathcal{M}f_m$*-natural operators* $A : Q \rightsquigarrow \mathrm{Riem}\, P^r$ *transforming classical linear connections* ∇ *on* m*-manifolds* M *into Riemannian structures* $A(\nabla)$ *on* P^rM *are in bijection with the functions* $\mu^A : Z^\infty \times H^r_m \to \mathrm{Met}(\mathbb{R}^k)$ *having the local finite determination property, where* $k = \dim(P^r\mathbb{R}^m)$.

Proof. We have $\mathrm{Riem}_\theta(P^r\mathbb{R}^m) = \mathrm{Met}(\mathbb{R}^k)$. ∎

(d) *The case of* $F = Q : \mathcal{M}f_m \to \mathcal{FM}$ *and* $G = Q : \mathcal{M}f_k \to \mathcal{FM}$. In this case we obtain the following corollary of Theorem 1.

Corollary 4. *The* $\mathcal{M}f_m$*-natural operators* $A : Q \times Q \to QP^r$ *transforming pairs* (∇_1, ∇) *of classical linear connections on* m*-manifolds* M *into classical linear connections* $A(\nabla_1, \nabla)$ *on* P^rM *are in bijection with the functions* $\mu^A : J^\infty_0(Q\mathbb{R}^m) \times Z^\infty \times H^r_m \to \mathbb{R}^{k*} \otimes \mathbb{R}^{k*} \otimes \mathbb{R}^k$ *having the local finite determination property, where* $k = \dim(P^r\mathbb{R}^m)$.

Proof. This is clear. ∎

(e) *The case of* $F = \mathrm{Riem} : \mathcal{M}f_m \to \mathcal{FM}$ *and* $G = \mathrm{Riem} : \mathcal{M}f_k \to \mathcal{FM}$. In this case we have the following corollary of Theorem 1.

Corollary 5. *The* $\mathcal{M}f_m$*-natural operators* $A : \mathrm{Riem} \times Q \to \mathrm{Riem}(P^r)$ *transforming Riemannian structures* g *on* m*-manifolds* M *and classical linear connections* ∇ *on* M *into Riemannian structures* $A(g, \nabla)$ *on* P^rM *are in bijection with the functions* $\mu^A : J^\infty_0(\mathrm{Riem}(\mathbb{R}^m)) \times Z^\infty \times H^r_m \to \mathrm{Met}(\mathbb{R}^k)$ *having the local finite determination property, where* $k = \dim(P^r\mathbb{R}^m)$.

Proof. This is clear. ∎

(f) *The case of* $F = \mathrm{id}_{\mathcal{M}f_m}$ *and* $G = E^{(s)} : \mathcal{M}f_k \to \mathcal{FM}$. In this case we get the following corollary of Theorem 1.

Corollary 6. *The* $\mathcal{M}f_m$*-natural operators* $A : Q \rightsquigarrow E^{(s)}P^r$ *transforming classical linear connections* ∇ *on* m*-manifolds* M *into* s*th order linear differential operators* $A(\nabla) : C^\infty(P^rM) \to C^\infty(P^rM)$ *on* P^rM *are in bijection with the functions* $\mu^A : Z^\infty \times H^r_m \to \bigoplus_{l=0}^s S^l\mathbb{R}^k$ *having the local finite determination property, where* $k = \dim(P^r\mathbb{R}^m)$.

Proof. We have $E^{(s)}_\theta(P^r\mathbb{R}^m) = \bigoplus_{l=0}^s S^l\mathbb{R}^k$. ∎

(g) *The case of* $F = \mathrm{id}_{\mathcal{M}f_m}$ *and* $G = F^{(p,q,s)} : \mathcal{M}f_k \to \mathcal{FM}$. In this case we have

Corollary 7. *The* $\mathcal{M}f_m$*-natural operators* $A : Q \rightsquigarrow F^{(p,q,s)}P^r$ *transforming classical linear connections* ∇ *on* m*-manifolds* M *into* s*th order linear differential operators* $A(\nabla) : \Omega^p(P^rM) \to \Omega^q(P^rM)$ *are in bijection* *with the functions* $\mu^A : Z^\infty \times H^r_m \to \bigoplus_{l=0}^s S^l\mathbb{R}^k \otimes \bigwedge^p \mathbb{R}^k \otimes \bigwedge^q \mathbb{R}^{k*}$ *having the local finite determination property, where* $k = \dim(P\mathbb{R}^m)$.

Proof. We have $F_\theta^{(p,q,s)}(P^r\mathbb{R}^m) = \bigoplus_{l=0}^s S^l\mathbb{R}^k \otimes \bigwedge^p \mathbb{R}^k \otimes \bigwedge^q \mathbb{R}^{k*}$. ∎

REMARK 4. The above list of corollaries of Theorem 1 is not complete. Many other corollaries can be obtained in a similar way.

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