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Koebe's general uniformisation theorem for planar Riemann surfaces

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Abstract. We give a complete and transparent proof of Koebe's General Uniformisation Theorem that every planar Riemann surface is biholomorphic to a domain in the Riemann sphere $\hat{\mathbb{C}}$, by showing that a domain with analytic boundary and at least two boundary components on a planar Riemann surface is biholomorphic to a circular-slit annulus in $\mathbb{C}$.

1. Introduction. A Riemann surface X is *planar* or *schlichtartig* if every smooth closed 1-form with compact support on X is exact. There is an equivalent topological condition that $X \setminus \gamma$ be disconnected for every smooth Jordan curve γ in X. In fact one can see that if γ is any Jordan curve on a Riemann surface X, then $X \setminus \gamma$ is either one region or the disjoint union of two regions. The General Uniformisation Theorem [GUT] of Koebe is the following:

Theorem 1.1 (Koebe 1909). *Every planar Riemann surface is biholomorphic to a domain in the Riemann sphere* $\hat{\mathbb{C}}$.

The Riemann mapping theorem for Riemann surfaces which classifies all the simply connected Riemann surfaces follows easily from Theorem 1.1 and the classical Riemann mapping theorem which classifies all the simply connected domains in $\hat{\mathbb{C}}$. Since an open subset of a planar Riemann surface is planar the converse of Theorem 1.1 follows trivially.

In this note we give a complete proof of Theorem 1.1 by showing that the method of proof used in Ahlfors [1] to prove that plane domains of finite connectivity with analytic boundary are biholomorphic to circular-slit annuli can be carried over to domains with analytic boundary on planar Riemann surfaces. However, our proof for the injectivity of the constructed mapping function seems to be new, and we feel it is more satisfactory than the one given in Ahlfors [1].

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One of the crucial results in the proof is the fact that for a relatively compact domain with good boundary, the planarity condition just means that the boundary curves generate the homology of the domain. In what follows, we use this result along with the beautiful construction of Weyl [5] to prove that the boundary curves form an *integral* basis for the homology of the domain.

2. Preliminaries. The main analytic tool in the proof is Perron's theorem guaranteeing the solvability of the Dirichlet problem for a domain with good boundary in a Riemann surface.

THEOREM 2.1. *For each compact subset of K of a Riemann surface X, and each open set U containing K, there exists an open set Ω in X with $K \subset \Omega \subset U$ such that*

(i) *$\bar{\Omega}$ is compact,*
(ii) *Ω has real analytic boundary.*

Proof. By using Sard's theorem we can find an open set V such that $K \subset V$, $\bar{V}$ is compact, $\bar{V} \subset U$ and V has $\mathcal{C}^\infty$ boundary.

Let $\partial V = \bigcup_{i=1}^{n} C_i$ be the decomposition of ∂V into connected components (we may assume $n \geq 2$!). Then by solving the Dirichlet problem for V, we can find a function h which is continuous on $\bar{V}$ with the properties:

- h is harmonic in V,
- $h|C_n = 1$ and $h|C_j = 0$ for $1 \leq j \leq n-1$.

Since h is non-constant, choose a non-critical value α of h such that $\alpha > \max\{h(x) : x \in K\}$, and a non-critical value $\beta < \min\{h(x) : x \in K\}$. We need not use Sard's theorem in this case since the critical points of a harmonic function are isolated. Then $\Omega = \{x : \beta < h(x) < \alpha\}$ has real analytical boundary, $\Omega \supset K$, and $\bar{\Omega}$ is compact. ∎

COROLLARY 2.2. *Let X be a non-compact Riemann surface. Then $X = \bigcup W_n$ for some sequence (W_n) of open sets in X with compact closure and real analytic boundary, and $\bar{W}_n \subset W_{n+1}$ for every n.*

THEOREM 2.3. *If the Riemann surface X is an increasing union of domains Ω_n biholomorphic to open sets in $\hat{\mathbb{C}}$, then X itself is biholomorphic to a domain in $\hat{\mathbb{C}}$.*

Proof. This is an easy consequence of Koebe's theorem on the compactness of the family of normalised schlicht functions in the unit disc. For a proof refer to [4]. ∎

THEOREM 2.4. *Let X be a planar Riemann surface and $\Omega \subset X$ be an open set in X with compact closure and $\mathcal{C}^\infty$ boundary. Then a closed $\mathcal{C}^\infty$*

one-form ω on Ω is exact if

$$\int_{C_i} \omega = 0$$

for each boundary curve C_i.

REMARK 2.5. Since the C_i are not paths in Ω the integrals occurring in the theorem do not a priori make sense. But the C_i can be pushed into Ω slightly in a sense which the proof below makes clear. In what follows, we shall also treat the C_i as paths in Ω according to this convention.

Proof. First we observe that each boundary component C_i is a compact connected smooth submanifold of dimension one. Hence, by a standard result ([3]) there is a diffeomorphism $\varphi_i : C_i \to S^1$. We may extend it to a $\mathcal{C}^\infty$ map $\varphi_i' : V_i \to S^1$ where V_i is a neighbourhood of C_i.

There exists a $\mathcal{C}^\infty$ function $f_i : W_i \to \mathbb{R}$, where W_i is a neighbourhood of C_i in X, such that

(i) $C_i = \{p \in W_i : f_i(p) = 0\}$,
(ii) $W_i \cap \Omega = \{p \in W_i : f_i(p) < 0\}$,
(iii) df_i never vanishes.

This can be proved in general, but is true in our case by construction (by Theorem 2.1). Now consider

$$g_i = (\varphi_i', f_i) : V_i \cap W_i \to S^1 \times \mathbb{R}.$$

It is easy to see that this map has non-singular Jacobian at all points of C_i, and is one-one on the compact set C_i. Hence, by the Inverse Function Theorem, there exists a neighbourhood $V_i' \subset V_i \cap W_i$ and an $\epsilon > 0$ such that

$$g_i : V_i' \to S^1 \times (-\epsilon, \epsilon)$$

is a diffeomorphism, mapping $V_i' \cap \Omega$ onto $S^1 \times (-\epsilon, 0)$. Indeed this follows from Lemma 2.7 below. Clearly, all the $g_i^{-1}(S^1 \times \{t\})$, $t \in (-\epsilon, 0)$, are homologous curves in Ω, and we will consider any one of them as C_i pushed into Ω (cf. 2.5). Now, $V_i' \cap \Omega$ is diffeomorphic to $S^1 \times (-\epsilon, 0)$, and $\int_{C_i} \omega = 0$ by assumption. By the definition we have just given for $\int_{C_i} \omega = 0$, this means that, $\int_{C_{i,t}} \omega = 0$, for each $t \in (-\epsilon, 0)$, where $C_{i,t} = g_i^{-1}(S^1 \times \{t\})$. It follows that $\int_\gamma \omega = 0$, for each cycle γ in $V_i' \cap \Omega$. Hence $\omega |\, V_i' \cap \Omega$ is exact for all i. Therefore

$$\omega = df_i \quad \text{in } \Omega \cap V_i'$$

for some $f_i \in \mathcal{C}^\infty(\Omega \cap V_i')$. For each i, we can find a cut-off function $\mathcal{X}_i$ such that its support lies in V_i' and $\mathcal{X}_i \equiv 1$ in a neighbourhood of C_i. Therefore $\mathcal{X}_i f_i$ is a $\mathcal{C}^\infty$ function defined on Ω.

Now consider the one-form $\omega_1 = \omega - \sum_i d(\mathcal{X}_i f_i)$ whose value is equal to zero near the boundary $\partial\Omega$, hence ω_1 has compact support. By the planarity condition there exists $g_1 \in \mathcal{C}^\infty(\Omega)$ such that $\omega - \sum_i d(\mathcal{X}_i f_i) = dg_1$. This implies $\omega = d(g_1 + \sum_i \mathcal{X}_i f_i)$ and by writing $g = g_1 + \sum_i \mathcal{X}_i f_i$ we see that ω is exact. ∎

Remark 2.6. The referee has observed that we are essentially proving the collar theorem. In fact the argument used above is a special case of the argument (unpublished so far) due to R. R. Simha for proving the product neighbourhood theorem for a compact differentiable manifold embedded with trivial normal bundle in another differentiable manifold (Ehresmann Fibration Theorem).

Lemma 2.7. *Let X be a locally compact metric space and $f : X \to Y$ a continuous map* (Y *is another metric space*)*. Let $K \subset X$ be a compact set such that*

(i) *$f|K$ is one-one,*
(ii) *each $p \in K$ has a neighborhood V_p such that $f|V_p$ is one-one.*

Then there exists an open set $V \supset K$ such that $f|V$ is one-one.

Proof. If there is no such V then for each $\epsilon_n > 0$, we can find x_n, y_n such that $x_n \neq y_n$, and $d(x_n, K), d(y_n, K) < \epsilon_n$, $f(x_n) = f(y_n)$. Since X is locally compact and K is compact there exists $\delta > 0$ such that $L = \{x \in X : d(x, K) \leq \delta\}$ is compact. Thus for $\epsilon_n \leq \delta$, we have $x_n, y_n \in L$. By passing to subsequences, we may assume $x_n \to x$ and $y_n \to y$. Then $x, y \in K$ and clearly $f(x) = f(y)$. This contradicts (ii) for $x = y = p$. ∎

Lemma 2.8 (Weyl). *For every path $\gamma_0 : [a, b] \to X$ in a Riemann surface X, and every open set U of X containing γ_0, there exists a closed one-form ω_{γ_0} in $X \setminus \{\gamma_0(a), \gamma_0(b)\}$ with support in $U \setminus \{\gamma_0(a), \gamma_0(b)\}$ such that*

(i) *$\int_\gamma \omega_{\gamma_0} \in \mathbb{Z}$ for every closed path γ in $X \setminus \{\gamma_0(a), \gamma_0(b)\}$,*
(ii) *if γ as in* (i) *meets γ_0 in only one point where γ and γ_0 'cross' each other, then*

$$\int_\gamma \omega_{\gamma_0} = \pm 1,$$

(iii) *if γ as in* (i) *does not meet γ_0, then*

$$\int_\gamma \omega_{\gamma_0} = 0.$$

Proof. If $\gamma_0([a, b])$ is contained in a coordinate disc (D_R, z), where D_R is a disc of radius R, and if $p = \gamma(a)$ and $q = \gamma(b)$ lie in D_r, $r \in (0, R)$, we note that $\varphi(z) = \frac{1}{2\pi}\{\arg(z - z_q) - \arg(z - z_p)\}$ is a well-defined smooth function on the annulus $r < |z| < R$. Choose $\mathcal{X} \in \mathcal{D}(D_R)$ with $\mathcal{X} \equiv 1$ on

$D_{(R+r)/2}$ and set

$$\omega_{\gamma_0} = \begin{cases} d\varphi & \text{for } z \in D_{(R+r)/2} \setminus \{p, q\}, \\ d(\mathcal{X}\varphi) & \text{for } |z| > r. \end{cases}$$

This ω_{γ_0} can be seen to have the desired properties (cf. [5]).

In the general case, we subdivide $[a, b]$ into subintervals $[t_i, t_{i+1}]$ each of which is mapped by γ_0 into a coordinate disc, contained in U, and take ω_{γ_0} to be the sum of the ω_i obtained as in the case above for $\gamma|[t_i, , t_{i+1}]$. We observe that the singularities at the common end points of two successive subpaths cancel out because we are on a Riemann surface (rather than a general differentiable surface); as is easy to see, if z and w are two local parameters at a point p on a Riemann surface, then $\frac{dz}{z} - \frac{dw}{w}$ has a removable singularity at p. Note that $w = a_1 z + a_2 z^2 + \cdots, a_1 \neq 0$. Refer to p. 82, Footnote 24 of [5]. ∎

THEOREM 2.9. *Let X be a Riemann surface and $\Omega \subset X$ be a domain in X with compact closure and analytic boundary curves $C_0, C_1, \ldots, C_n$ $(n \geq 1)$. Then for $p_1, \ldots, p_n \in \mathbb{R}$ there is a unique choice of $r_1, \ldots, r_n$ and a unique harmonic function u with*

$$\int_{C_j} * du = p_j$$

and $u \equiv r_i$ on each C_j, $j = 1, \ldots, n$, and $u \equiv 0$ on C_0.

Proof. Since $\partial\Omega$ consists of analytic curves $C_0, C_1, \ldots, C_n$, a barrier at each point on C_i exists and the Dirichlet problem can be solved with the given arbitrary boundary values. The proof follows now by using the Schwarz Reflection Principle and some point set topology! Also refer to [1]. ∎

The following lemma plays an important role in our proof to show that the constructed holomorphic map from a domain with analytic boundary on a planar Riemann surface onto a circular-slit domain in $\mathbb{C}$ is indeed injective.

LEMMA 2.10. *Let X be a Riemann surface, and $f : X \to \mathbb{C}$ a non-constant holomorphic map. Then*

$$\frac{i}{2} \int_X df \wedge d\bar{f} \geq \operatorname{Area} f(\Omega)$$

with equality if and only if f is one-one.

3. Proof of Koebe's GUT. Suppose X is compact. Then for any $p \in X$, $X^* = X \setminus p$ is non-compact and planar. If we have found a one-one holomorphic function f^* on X^*, then p cannot be an essential singularity for f by Weierstrass' theorem, and it is clear that the extended holomorphic map $f : X \to \hat{\mathbb{C}}$ is one-one.

Given Theorem 2.3, and Corollary 2.2, it is clear that, in order to prove the General Uniformisation Theorem for non-compact Riemann surfaces, it is sufficient to prove the following theorem.

Theorem 3.1. *Let X be a non-compact planar Riemann surface, and $\Omega \subset X$ a domain with compact closure and analytic boundary. Then Ω is biholomorphic to a domain in $\mathbb{C}$.*

Proof. We construct a biholomorphic map from Ω onto a circular-slit domain in $\mathbb{C}$ (i.e. an open annulus $\{A_R : 1 < |z| < R\} \setminus \bigcup \gamma_j$ where γ_j are disjoint circular arcs lying on circles $\{z : |z| = R_j\}$, $1 < R_j < R$; the R_j need not be distinct). See Figure 1.

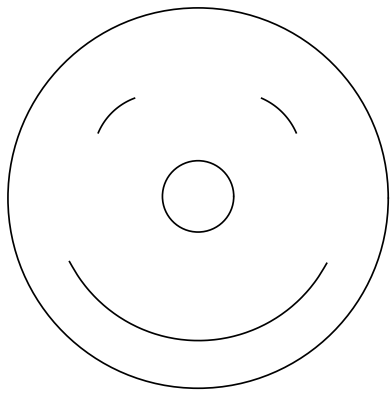

Fig. 1. Circular-slit domain of connectivity 5

We may assume that Ω has at least two boundary components. Otherwise, we replace Ω by $\Omega \setminus \bar{D}_\epsilon = \Omega_\epsilon$ where D_ϵ is a small disc around a fixed point $p \in \Omega$. If Ω_ϵ is biholomorphic to a domain in $\mathbb{C}$, by Theorem 2.2 so is $\Omega \setminus p$, hence so is Ω.

Suppose Ω is mapped by F biholomorphically onto a circular-slit domain in $\mathbb{C}$ then $\log |F|$ is a harmonic function in the domain with constant values on the boundary of Ω. Therefore the idea is to look for a harmonic function in the domain Ω with suitable constant values on the boundary and try to make it the logarithm of the absolute value of a one-one holomorphic function F.

Step 1. Let C_j, $0 \leq j \leq n$, be the boundary components of Ω. The harmonic function h is determined by the R_j, which in turn are determined by the integrals of $*dh$ along the C_j, $0 \leq j \leq n$. By construction, F will be of the form

$$F(q) = \exp\left\{\left(\int_p^q (dh + i{*}dh)\right) + h(p)\right\}$$

so that $dh + i{*}dh = \frac{dF}{F}$. If we assume that F maps C_0 onto the unit circle and C_n onto $|z| = R_n$, we see that

$$\int_{C_j} {*}dh = 2\pi \frac{1}{2\pi i} \int_{C_j} (dh + i{*}dh) = 2\pi \frac{1}{2\pi i} \int_{C_j} \frac{dF}{F} = 2\pi \frac{1}{2\pi i} \int_{F(C_j)} \frac{dz}{z}.$$

Hence, if the C_j are oriented properly we must have

$$\int_{C_j} {*}dh = \begin{cases} 2\pi & \text{if } j = n, \\ -2\pi & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So we make this choice for the periods of ${*}dh$, and obtain a harmonic function h on Ω with uniquely determined constant boundary values λ_j on $C_j, 1 \leq j \leq n$ say, where $\lambda_0 = 0$ by convention.

Step 2. We show that

$$\frac{1}{2\pi} \int_{\gamma} {*}dh \in \mathbb{Z} \tag{3.1}$$

for *all* closed paths γ in Ω. We apply Lemma 2.8 to our situation; we fix a point p_j inside each C_j and define γ_j to be a path from p_0 to p_j, for $1 \leq j \leq n$. We can choose the γ_j so that γ_j does not meet C_k if $k \neq j$, and γ_j meets C_j exactly once, where it crosses C_j. All this can be carried out rigorously if we recall that the C_j, $0 \leq j \leq n$, have disjoint neighbourhoods V_j such that each $V_j \cap \bar{\Omega}$ is diffeomorphic to a product $S^1 \times (-\epsilon, 0]$.

Now consider

$$\omega = {*}dh - \sum_{j=1}^{n} \pi_j \, \omega_{\gamma_j},$$

where $\pi_j = \int_{C_j} {*}dh$. Then $\int_{C_j} \omega = 0$ for $0 \leq j \leq n$ by construction, hence ω is exact by Theorem 2.4. Hence $\int_\gamma \omega = 0$ for all closed paths in Ω, which shows that

$$\int_{\gamma} {*}dh = \sum \pi_j \int_{\gamma} \omega_{\gamma_j}$$

is an integral multiple of 2π for all γ, as asserted.

Then

$$F(q) = \exp\left\{ \Big(\int_{p}^{q} (dh + i{*}dh) \Big) + h(p) \right\}$$

is a well-defined holomorphic function in Ω with $|F| = e^h$. The Schwarz Reflection Principle shows that h (hence F) extends harmonically (resp. holomorphically) to a neighbourhood of $\bar{\Omega}$.

STEP 3. We compute $\int_\Omega dF \wedge d\bar{F}$:

$$\begin{aligned}\int_\Omega dF \wedge d\bar{F} &= \int_\Omega d(F \wedge d\bar{F}) = \int_{\partial\Omega} F\, d\bar{F} \\ &= \int_{\partial\Omega} F\,\bar{F}(dh - i{*}dh) = \sum (e^{\lambda_i})^2 \int_{\partial\Omega} (dh - i{*}dh) \\ &= -2\pi i\{(e^{\lambda_n})^2 - 1\} \quad \text{by (3.1).}\end{aligned}$$

Since $dF \wedge d\bar{F} = -2i|F'(z)|^2 dxdy$ locally, we see that

$$\frac{i}{2}\int_\Omega dF \wedge d\bar{F} = \int_\Omega |F'(z)|^2\, dx\, dy = \pi\{(e^{\lambda_n})^2 - 1\} > 0.$$

This shows that $R_i = e^{\lambda_n} > 1$, as expected. We set generally

$$R_j = e^{\lambda_j}, \quad 1 \le j \le n.$$

STEP 4. We examine the image domain $F(\Omega)$. By the Open Mapping Theorem, $\partial F(\Omega) \subset F(\partial\Omega) \subset \bigcup_{j=0}^n \{|z| = R_j\}$. Clearly, $F(\Omega)$ meets each circle $\{|z| = R\}$ with $R \in (1, R_n)$ ($|F|$ is continuous!). Also, if $\{|z| = R\}$ is not entirely contained in $F(\Omega)$, we must have $\{|z| = R\} \cap \partial F(\Omega) \neq \emptyset$, so that R must be one of the R_j. It follows that $F(\Omega)$ contains the annulus $\{1 < |z| < R_n\}$ except for a subset of $\bigcup_j \{|z| = R_j\}$. In particular we get

$$\operatorname{Area} F(\Omega) \ge \pi(R_n^2 - 1) = \int_\Omega |F'(z)|^2\, dx\, dy \quad \text{(by Step 3).}$$

But it is obvious from the change of variable formula for double integrals that

$$\int_\Omega |F'(z)|^2\, dx\, dy \ge \operatorname{Area} F(\Omega) \tag{3.2}$$

with equality only if F is one-one (refer to Lemma 2.10). Thus we have proved that F is one-one. This completes the proof. ■

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