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On generalized topological spaces II

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Abstract. This is the second part of A. Piękosz [Ann. Polon. Math. 107 (2013), 217–241]. The categories $\mathbf{GTS}(M)$, with M a non-empty set, are shown to be topological. Several related categories are proved to be finitely complete. Locally small and nice weakly small spaces can be described using certain sublattices of power sets. Some important elements of the theory of locally definable and weakly definable spaces are reconstructed in a wide context of structures with topologies.

1. Introduction. This paper is the second part of [P2], which described the origins of the notion of generalized topology in the sense of H. Delfs and M. Knebusch and gave the basic theory of generalized topological spaces and of the special case of small spaces, as well as the proof of the fact that both the categories **GTS** and **SS** are topological.

Here we prove that the subcategories **LSS**, **NWSS**, $\mathbf{WSS}_1$ of **GTS** are finitely complete. We find several interesting characterizations of our categories. The category $\mathbf{LSS}_{\mathrm{pt}}$ of partially topological locally small spaces is isomorphic to the category **UBorOB** of bornological universes having open bases. The whole category **LSS** is isomorphic to the category **Sublat** defined here, and the category **NWSS** of nice weakly small spaces admits an embedding into **Sublat**. We also deal here with paracompactness and Lindelöfness of locally small spaces.

Then the category **GTS** is used to build some natural categories of spaces over model-theoretic structures. A natural setting here for topological considerations is assuming a topology on the underlying set M of a model-theoretic structure $\mathcal{M}$, and putting the product topologies on the Cartesian powers of M. Such structures are called here *weakly topological*. They are more general than the *first order topological structures* of [Pil, M] and *structures with definable topologies* of [S]. We prove that the category $\mathbf{GTS}(M)$ of general-

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ized topological spaces over a set M is topological (as a construct), and its full subcategories $\mathbf{ADS}(\mathcal{M},\sigma)$, $\mathbf{DS}(\mathcal{M},\sigma)$, $\mathbf{LDS}(\mathcal{M},\sigma)$, $\mathbf{WDS}_1(\mathcal{M},\sigma)$ are finitely complete for a weakly topological $(\mathcal{M},\sigma)$ with underlying set M.

Separate subsections are devoted to the separation axioms (in the "definable" case) and completeness (in all cases).

We keep the notational conventions of [P2].

2. Generalized topological spaces

2.1. Locally small spaces. In this subsection we rebuild the theory of locally semialgebraic spaces from [DK] on a purely topological level.

Definition 2.1.1. A gts is *locally small* if there is an admissible covering of the whole space by small open subsets. In other words: a gts X is locally small iff $X = \bigcup^{a} \mathrm{Smop}_X$. Locally small gts's form a full subcategory **LSS** of **GTS**.

Definition 2.1.2 (cf. [DK, p. 31]). On locally small spaces we consider a topology, called in [DK] the *strong topology*, whose basis is the family of open sets of the gts. (The members of the strong topology are exactly the weakly open subsets of the space. The strong topology is the topology generated by the open sets of the generalized topology.)

Definition 2.1.3. A subset Y of a locally small space X is *locally constructible* if each intersection $Y \cap U$ with a small open $U \subseteq X$ is constructible in U (so also in X). A family $\mathcal{U}$ of subsets of X is *locally essentially finite* if this family is essentially finite on each small open set.

Remark 2.1.4. The Boolean algebra of locally constructible subsets of a locally small space may be strictly larger than the Boolean algebra of constructible subsets (to see this, one can construct a sequence X_n of constructible subsets of small spaces Z_n, each X_n needing at least n open sets in the description, and then take the direct sum of all the Z_n to get a needed locally small space; one may take for each Z_n the space from [P2, Example 2.3.7]).

Definition 2.1.5. We will say that a locally small space has the *closure property* (CPL) if the weak closure of any small locally closed subset is a closed subset.

Proposition 2.1.6.

(1) *Each locally essentially finite union of closed* (*open, respectively*) *subsets of a locally small space is closed* (*open, respectively*).
(2) *Each locally essentially finite union of locally constructible sets is locally constructible.*

Proof. (1) Let Z be a locally essentially finite union of closed sets. Take an admissible covering $\mathcal{U}$ of the space by small open subsets. For each element U of $\mathcal{U}$, the set $Z \cap U$ is a finite union of relatively closed subsets of U, so it is relatively closed. Each relative complement $Z^c \cap U$ is open in U. By regularity, the set Z^c is open, hence Z is closed. The case of open sets follows from regularity.

(2) Similar to the proof of the previous case. ■

A strong inverse to (2) of the previous proposition is the following

Proposition 2.1.7. *Each locally constructible subset of a locally small space is a locally essentially finite union of small locally closed subsets.*

Proof. If $X = \bigcup^{a}_{\alpha} X_\alpha$ with each X_α in Smop_X, and W is a locally constructible subset of X, then for each α we get $W \cap X_\alpha = W_{\alpha,1} \cup \cdots \cup W_{\alpha,k(\alpha)}$ with some locally closed $W_{\alpha,i} \subseteq X_\alpha$. Then the union $\bigcup_{\alpha,i} W_{\alpha,i} = W$ is locally essentially finite. ■

Proposition 2.1.8. *If a locally small space has the closure property* (CPL), *then the weak closure of each locally constructible set is a closed set.*

Proof. By Proposition 2.1.7, the weak closure of each locally constructible set is the union of the weak closures of some locally essentially finite family of small constructible sets. But the family of these closures is also locally essentially finite, and the assertion follows from Proposition 2.1.6. ■

Remark 2.1.9. By the above, if a locally small space X has the closure property (CPL), then the closure operator of the generated topology restricted to the class $\mathrm{LocConstr}_X$ of locally constructible sets may be considered as the *closure operator* of the generalized topology

$$\overline{\cdot} : \mathrm{LocConstr}_X \to \mathrm{Cl}_X .$$

Fact 2.1.10. *Each locally small space with the closure property* (CPL) *and with a regular* (*Hausdorff*) *strong topology is weakly regular.*

Example 2.1.11. Each topological discrete space is locally small (and it is not small, if infinite).

Example 2.1.12. Any metric space (X, d) has a natural generalized topology, where $\mathrm{Op}_Y = \tau(d)$ is the topology induced by the metric and Cov_X is the family of open families essentially finite on each bounded set in the sense of the metric. Then Sm_X is exactly the family $\mathcal{B}(d)$ of bounded sets, and $(X, \mathrm{Op}_X, \mathrm{Cov}_X)$ is locally small.

Example 2.1.13. Each topological space (X, τ) can be made a locally small space relative to a chosen open covering $\mathcal{U}$ of the whole of X in the following way: we take smallifications of members of $\mathcal{U}$, and the generalized topology on X is then given by the formula $X = \bigcup^a \mathrm{sm}(\mathcal{U})$.

Remark 2.1.14. Notice that the $\mathcal{T}$-space X of [EP] may be interpreted as a locally small space in the above sense with $\mathcal{T}$ being the family Smop_X. This fact is the cause of introducing the family $\mathcal{T}_{\mathrm{loc}}$ and the Grothendieck site $X_{\mathcal{T}_{\mathrm{loc}}}$ in [EP].

Definition 2.1.15. By a *small space of type* $\mathbb{R}^n_{\mathrm{sth}}$ we will understand a gts $(\mathbb{R}^n, \mathrm{Op}, \mathrm{EssFin}(\mathrm{Op}))$, where Op is some basis of the natural topology of $\mathbb{R}^n$ such that each of the open balls B_n centered at the origin with radius $n \in \mathbb{N} \setminus \{0\}$ belongs to Op.

For such spaces we will consider their *localization* $(\mathbb{R}^n_{\mathrm{sth}})_{\mathrm{loc}}$ in the following sense: the admissible union of the family of open small balls B_n in the space $\mathbb{R}^n_{\mathrm{sth}}$. In symbols we write

$$(\mathbb{R}^n_{\mathrm{sth}})_{\mathrm{loc}} = \bigcup_{n>0}^{a} B_n.$$

Example 2.1.16. The spaces $\mathbb{R}^n_{\mathrm{alg}}, \mathbb{R}^n_{\mathrm{salg}}, \mathbb{R}^n_{\mathrm{ts}}$ from Example 2.2.14 of [P2] are small of type $\mathbb{R}^n_{\mathrm{sth}}$. The spaces $\mathbb{R}^n_{\mathrm{san}}$, $\mathbb{R}^n_{\mathrm{suban}}$ as well as the localizations $(\mathbb{R}^n_{\mathrm{alg}})_{\mathrm{loc}}$, $(\mathbb{R}^n_{\mathrm{salg}})_{\mathrm{loc}}$, $(\mathbb{R}^n_{\mathrm{ts}})_{\mathrm{loc}}$ are locally small, but not small.

Definition 2.1.17. A family of subsets of a locally small space is called *locally finite* if each small open set of the space meets only finitely many members of the family.

Proposition 2.1.18. *An open family of a locally small space is admissible if and only if it is locally essentially finite.*

Proof. Each admissible family is locally essentially finite by the definition of a small subset. If an open family $\mathcal{V}$ is locally essentially finite, then it is essentially finite on members of an admissible covering $\mathcal{U}$ of the space by open small subspaces. This means that for each member U of $\mathcal{U}$, the family $U \cap_1 \mathcal{V}$ is admissible. By the transitivity axiom, the family $\mathcal{U} \cap_1 \mathcal{V}$ is admissible. By the saturation axiom, the family $\mathcal{V}$ is admissible. ∎

Corollary 2.1.19. *Each locally finite open family in a locally small space is admissible.*

Fact 2.1.20. *If* $\mathcal{U} = \{U_i\}_{i\in I}$ *is a locally finite open family in some locally small space, and for each* $i \in I$ *some open* $V_i \subseteq U_i$ *is given, then* $\{V_i\}_{i\in I}$ *is locally finite, thus admissible.*

Fact 2.1.21. *In a locally small gts each small set is contained in a small open set, that is: the family* Smop_X *is a basis* (*see* [H-N]) *of the bornology* Sm_X.

PROPOSITION 2.1.22. *Let X, Y be objects of* **LSS**. *For a mapping $f : X \to Y$ the following conditions are equivalent:*

(a) *f is strictly continuous,*
(b) $\mathrm{Smop}_X \preceq f^{-1}(\mathrm{Smop}_Y)$ *and $f|_U : U \to Y$ is continuous for each $U \in \mathrm{Smop}_X$,*
(c) *f is bounded continuous.*

Proof. Follows from of [P2, Propositions 2.2.26, 2.2.45, and Fact 2.3.8], and from Fact 2.1.21. ∎

FACT 2.1.23. *The preimage of a locally constructible set under a strictly continuous mapping of locally small spaces is locally constructible.*

PROPOSITION 2.1.24. *Each subset Y of a locally small space $X = \overset{a}{\bigcup}_\alpha X_\alpha$ (with each X_α small) forms an initial subobject in* **LSS** *given by the formula $Y = \overset{a}{\bigcup}_\alpha (X_\alpha \cap Y)$.*

Proof. Remember that $\langle \mathrm{Cov}_X \cap_2 Y\rangle$ is the generalized topology of the initial subobject on the set Y in **GTS**. On the other hand, the formula $Y = \overset{a}{\bigcup}_\alpha (X_\alpha \cap Y)$ defines a locally small space (Y, Cov_Y). We will prove that these two spaces are equal. By [P2, Propositions 2.2.44, 2.2.37], we get

$$\begin{aligned}
\mathrm{Cov}_Y &= \Big\langle \{X_\alpha\}_\alpha \cap_1 Y, \bigcup_\alpha \mathrm{Cov}_{X_\alpha \cap Y} \Big\rangle = \Big\langle \{X_\alpha\}_\alpha \cap_1 Y, \Big(\bigcup_\alpha \mathrm{Cov}_{X_\alpha}\Big) \cap_2 Y \Big\rangle \\
&\subseteq \Big\langle \{X_\alpha\}_\alpha \cap_1 Y, \Big\langle \{X_\alpha\}_\alpha, \bigcup_\alpha \mathrm{Cov}_{X_\alpha} \Big\rangle \cap_2 Y \Big\rangle \\
&= \langle \{X_\alpha\}_\alpha \cap_1 Y, \mathrm{Cov}_X \cap_2 Y\rangle = \langle \mathrm{Cov}_X \cap_2 Y\rangle.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathrm{Cov}_X \cap_2 Y &= \Big\langle \{X_\alpha\}_\alpha, \bigcup_\alpha \mathrm{Cov}_{X_\alpha} \Big\rangle \cap_2 Y \\
&\subseteq \Big\langle \{X_\alpha\}_\alpha \cap_1 Y, \Big(\bigcup_\alpha \mathrm{Cov}_{X_\alpha}\Big) \cap_2 Y \Big\rangle.
\end{aligned}$$

Since Cov_Y is a generalized topology, $\mathrm{Cov}_Y = \langle \mathrm{Cov}_X \cap_2 Y\rangle$. ∎

QUESTION 2.1.25. Is such a Y always a strict subspace of X?

LEMMA 2.1.26. *The construct* **LSS** *has concrete finite products.*

Proof. For a family $X_1, \dots, X_k$ of locally small spaces assume that $\mathcal{U}_1, \dots, \mathcal{U}_k$ are their respective admissible coverings by small open subsets. Then the product space is given by the formula

$$X_1 \times \cdots \times X_k = \overset{a}{\bigcup} \mathcal{U}_1 \times_1 \cdots \times_1 \mathcal{U}_k.$$

The projections $\pi_i : X_1 \times \cdots \times X_k \to X_i$ are obviously strictly continuous, and for given strictly continuous $f_i : Y \to X_i$, the mapping $(f_1, \ldots, f_k) : Y \to X_1 \times \cdots \times X_k$ is also strictly continuous, since "admissible" means "locally essentially finite". ∎

The example below suggests **LSS** does not have concrete products.

EXAMPLE 2.1.27. Consider the topological discrete space $\mathbb{N}$. This space is locally small. The countable product $\mathbb{N}^{\mathbb{N}}$ exists in **Top**, but contains no non-empty small open sets, so is not an object of **LSS**. That is why the analogous concrete product in the category of locally noetherian spaces does not exist.

The equalizers for parallel pairs of morphisms exist in **LSS** by Proposition 2.1.24. Together with Lemma 2.1.26, this gives

THEOREM 2.1.28. *The construct* **LSS** *is finitely complete, and its finite limits are concrete.*

FACT 2.1.29. *The construct* **LSS** *has concrete direct sums.*

FACT 2.1.30. *Passing to the strong topology yields a functor* $()_{\mathrm{top}}$: **LSS** $\to$ **Top**, *which is a restriction of the functor* top : **GTS** $\to$ **Top**.

PROPOSITION 2.1.31. *The functor* ubor *restricted to the category* $\mathbf{LSS}_{\mathrm{pt}}$ *of partially topological locally small spaces has an inverse* lss : **UBorOB** $\to$ $\mathbf{LSS}_{\mathrm{pt}}$ *from the full subcategory of bornological universes having open bases* **UBorOB** *in* **UBor**.

Proof. Recall $\mathrm{ubor}(X, \tau, \mathrm{Cov}_X) = (X, \tau, \mathrm{Sm}_X)$ and $\mathrm{ubor}(f) = f$. Define lss as the restriction of gts to the category **UBorOB**, i.e. $\mathrm{lss}(X, \tau, \mathcal{B}) = (X, \tau, \mathrm{EF}(\tau, \mathcal{B}))$, where $\mathrm{EF}(\tau, \mathcal{B})$ is defined as in [P2, Proposition 2.2.71], and $\mathrm{lss}(f) = f$.

The composition gts $\circ$ ubor restricted to $\mathbf{LSS}_{\mathrm{pt}}$ is the identity functor. Indeed, $\mathrm{EF}(\tau, \mathrm{Sm}_X)$ is the family of open families that are essentially finite on small sets. By Proposition 2.1.18 and Fact 2.1.21, $\mathrm{EF}(\tau, \mathrm{Sm}_X) = \mathrm{EF}(\tau, \mathrm{Smop}_X) = \mathrm{Cov}_X$.

The composition ubor $\circ$ gts restricted to **UBorOB** is also the identity functor. Indeed, the family of small sets $\mathrm{Sm}(\mathrm{EF}(\tau, \mathcal{B}))$ always contains $\mathcal{B}$, and if $\tau \cap \mathcal{B}$ is a basis of $\mathcal{B}$ then $\tau \cap \mathcal{B} \in \mathrm{EF}(\tau, \mathcal{B})$. For a set $Z \in \mathrm{Sm}(\mathrm{EF}(\tau, \mathcal{B}))$ also $(\mathcal{B} \cap \tau) \cap_1 Z$ is essentially finite, hence $Z = \bigcup (\mathcal{B} \cap \tau) \cap Z = B \cap Z$ for some $B \in \mathcal{B} \cap \tau$, so $Z \in \mathcal{B}$. This means $\mathcal{B} = \mathrm{Sm}(\mathrm{EF}(\tau, \mathcal{B}))$. ∎

DEFINITION 2.1.32. Define the category **Sublat** as follows: objects are pairs $(X, \mathcal{L})$, where X is any set and $\mathcal{L}$ is a sublattice of $\mathcal{P}(X)$ containing the empty set and covering the set X, morphisms are mappings $f : X \to Y$ such that $\mathcal{L}_X \preceq f^{-1}(\mathcal{L}_Y)$ and $f^{-1}(\mathcal{L}_Y) \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X$.

THEOREM 2.1.33. *The categories* **LSS** *and* **Sublat** *are isomorphic.*

Proof. Define a functor $F : \mathbf{LSS} \to \mathbf{Sublat}$ by

$$F(X, \mathrm{Op}_X, \mathrm{Cov}_X) = (X, \mathrm{Smop}_X), \qquad F(f) = f.$$

Define a functor $G : \mathbf{Sublat} \to \mathbf{LSS}$ by $G(X, \mathcal{L}) = (X, \mathrm{Op}(\mathcal{L}), \mathrm{Cov}(\mathcal{L}))$, $G(f) = f$, where

$$\begin{aligned}
\mathrm{Op}(\mathcal{L}) &= \{U \subseteq X \mid U \cap_1 \mathcal{L} \subseteq \mathcal{L}\},\\
\mathrm{Cov}(\mathcal{L}) &= \mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L})\\
&= \{\mathcal{U} \subseteq \mathrm{Op}(\mathcal{L}) \mid \mathcal{U} \cap_1 L \in \mathrm{EssFin}(\mathcal{L}) \text{ for each } L \in \mathcal{L}\}.
\end{aligned}$$

Notice that

$$GF(X, \mathrm{Op}_X, \mathrm{Cov}_X) = (X, \mathrm{Op}(\mathrm{Smop}_X), \mathrm{Cov}(\mathrm{Smop}_X)) = (X, \mathrm{Op}_X, \mathrm{Cov}_X).$$

Indeed, for any locally small space a set is open iff all its intersections with small open sets are small open; and each open family is admissible iff it is essentially finite on all small open sets.

Moreover, $FG(X, \mathcal{L}) = (X, \mathrm{Smop}(\mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L}))) = (X, \mathcal{L})$. Indeed, it is clear that $\mathcal{L} \subseteq \mathrm{Smop}(\mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L}))$. Notice that $\mathcal{L} \in \mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L})$. For each small open subset S in $\mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L})$ we get an admissible family $S \cap_1 \mathcal{L}$ with union S. Since S is small, this means that S is a subset of some $L \in \mathcal{L}$, and $S \cap L = S \in \mathcal{L}$. The inclusion $\mathrm{Smop}(\mathrm{EF}(\mathrm{Op}(\mathcal{L}), \mathcal{L})) \subseteq \mathcal{L}$ is proved.

Finally, the notions of morphism in both categories agree. A morphism in **LSS** maps a small open set onto a small set contained in a small open set, and the preimage of a small open set is open, hence "locally" small open. Each morphism $f : X \to Y$ in **Sublat** is strictly continuous in the generalized topologies $\mathrm{Cov}(\mathcal{L}_X)$ and $\mathrm{Cov}(\mathcal{L}_Y)$. Indeed, notice first that $f^{-1}(\mathcal{L}_Y) \subseteq \mathrm{Op}(\mathcal{L}_X)$. We will prove that $f^{-1}(\mathrm{Op}(\mathcal{L}_Y)) \subseteq \mathrm{Op}(\mathcal{L}_X)$. If $U \in \mathrm{Op}(\mathcal{L}_Y)$, then consider the intersection $f^{-1}(U) \cap L$ where $L \in \mathcal{L}_X$. Since $f(L) \subseteq M \in \mathcal{L}_Y$, we get $f^{-1}(U) \cap L = f^{-1}(U) \cap f^{-1}(M) \cap L = f^{-1}(U \cap M) \cap L \in \mathrm{Op}(\mathcal{L}_X) \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X$, hence $f^{-1}(U)$ is in $\mathrm{Op}(\mathcal{L}_X)$. Now the families from $\mathrm{Cov}(\mathcal{L}_Y)$ are exactly subfamilies of $\mathrm{Op}(\mathcal{L}_Y)$ that are essentially finite on members of $\mathcal{L}_Y$. Their preimages under f are, in particular, essentially finite on every member of $\mathcal{L}_X$. We have proved that $f^{-1}(\mathrm{Cov}(\mathcal{L}_Y)) \subseteq \mathrm{Cov}(\mathcal{L}_X)$. ∎

PROPOSITION 2.1.34. *A locally finite subspace of a strongly T_1 locally small space is a closed subspace and a topological discrete space.*

Proof. Closedness is obvious. Since by [P2, Proposition 2.3.20], the subspace is topological discrete on each small open set, it is topological discrete. ∎

DEFINITION 2.1.35 (cf. [DK, I.4]). A locally small space is called *paracompact* if there is a locally finite covering of the space by small open subsets, and *Lindelöf* if there is a countable admissible covering of the space by small open subsets.

FACT 2.1.36. *Each topological discrete space is paracompact, and is Lindelöf if countable.*

REMARK 2.1.37. Each connected paracompact locally small space is Lindelöf (the proof of [DK, I.4.17] is purely topological).

PROPOSITION 2.1.38 (cf. [DK, I.4.6]). *For each paracompact locally small space X, the weak closure $\overline{Y}$ (that is, the closure in the strong topology) of a small set Y is small.*

Proof. Take a locally finite covering $\mathcal{U}$ of X by small open subsets. The set Y is covered by a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ of all members of $\mathcal{U}$ that meet Y. Then $\overline{Y}$ and the union of $\mathcal{U}\setminus\mathcal{U}_0$ are disjoint, and $\overline{Y}$ is contained in the union of $\mathcal{U}_0$, which is a small set. ∎

PROPOSITION 2.1.39. *Each locally small space satisfying* (AQC) *has a locally finite family of its quasi-components. Each locally small space satisfying* (ACC) *has a locally finite family of its connected components.*

Proof. Follows from [P2, Proposition 2.3.23]. ∎

PROPOSITION 2.1.40. *If X is a small space and Y is a locally small space, then the canonical projection $\pi_2 : X \times Y \to Y$ is an open and closed mapping.*

Proof. For each $U \in \mathrm{Smop}_Y$ the projections $\pi_2 : X \times U \to U$ and $\pi_2^c : X \times U \to U$ are open by [P2, Proposition 2.3.15]. Since Smop_Y is admissible, both $\pi_2 : X \times Y \to Y$ and $\pi_2^c : X \times Y \to Y$ are open by regularity. ∎

The following example shows that if X is only locally small, then $\pi_2 : X \times Y \to Y$ may be neither open nor closed.

EXAMPLE 2.1.41. Let $X = (\mathbb{R}_{\mathrm{salg}})_{\mathrm{loc}}$ and $Y = \mathbb{R}_{\mathrm{salg}}$. Consider the open set $C = \bigcup_{n=0}^{\infty}(n, n+1) \times (n, n+1)$ and the closed set $D = \bigcup_{n=0}^{\infty}\{n\} \times \{n\}$ in $X \times Y$. Then $\pi_2(C)$ is not open, and $\pi_2(D)$ is not closed in Y.

QUESTION 2.1.42. Does **LSS** have (concrete) coequalizers?

2.2. Weakly small spaces. In this subsection we reintroduce the theory of weakly semialgebraic spaces from [K] on a topological level.

DEFINITION 2.2.1 (cf. [K, IV.1, Definition 6]). A *weakly* (or *piecewise*) *small space* is a gts X having a family $(X_\alpha)_{\alpha \in A}$ of closed small (so strict) subspaces indexed by a partially ordered set A such that the following conditions hold:

(W1) X is the union of all X_α's as sets,

(W2) if $\alpha \leq \beta$ then X_α is a (closed, small) subspace of X_β,
(W3) for each $\alpha \in A$ there are only finitely many $\beta \in A$ such that $\beta < \alpha$,
(W4) for each pair $\alpha, \beta \in A$ there is $\gamma \in A$ such that $X_\alpha \cap X_\beta = X_\gamma$,
(W5) for each pair $\alpha, \beta \in A$ there is $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$,
(W6) the gts X is the inductive limit in **GTS** of the direct family $(X_\alpha)_{\alpha \in A}$, which means:
   (a) a subset U of X is *open* iff each set $X_\alpha \cap U$ is open in X_α,
   (b) an open family $\mathcal{U}$ is *admissible* iff for each $\alpha \in A$ the family $\mathcal{U} \cap_1 X_\alpha$ is admissible (= essentially finite) in X_α.

Such a family $(X_\alpha)_{\alpha \in A}$ is called an *exhaustion* of X. Members of an exhaustion will be called *pieces* of X. If $(X_\alpha)_{\alpha \in A}$ is an exhaustion of X, then we will write $X = \overset{e}{\bigcup}_{\alpha \in A} X_\alpha$. The family of all small closed subsets of X will be denoted by $\mathrm{Smcl}_{\mathrm{X}}$.

The condition (W6) has the following consequences:

Fact 2.2.2. *An open family is admissible in a weakly small space iff it is "piecewise essentially finite"* (*i.e. essentially finite on every member of the exhaustion considered*). *In other words: if* $X = \overset{e}{\bigcup}_\alpha X_\alpha$, *then* $\langle \mathrm{Cov}_{X_\alpha} \rangle_\alpha^* = \mathrm{Cov}_X$.

Fact 2.2.3. *A subset Y of a weakly small space X is open* (*closed, respectively*) *if and only if it is piecewise open* (*closed, respectively*).

Definition 2.2.4 (cf. [K, p. 13]). The *index function* for the exhaustion $(X_\alpha)_{\alpha \in A}$ is the function $\eta : X \to A$ given by the formula

$$\eta(x) = \inf\{\alpha \in A \mid x \in X_\alpha\}.$$

Here the infimum exists thanks to (W3). The index function η gives a decomposition of the space X into small locally closed subspaces $X^0_\alpha = \eta^{-1}(\alpha) = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$.

Definition 2.2.5. The full subcategory in **GTS** of weakly small spaces will be denoted by **WSS**.

Definition 2.2.6 (cf. [K, p. 25]). The *strong topology* on $X = \overset{e}{\bigcup}_{\alpha \in A} X_\alpha$ is the topology on the set X that makes X the inductive limit of the direct system of the topological spaces $(X_\alpha)_{\mathrm{top}}$. Its members are all the piecewise weakly open subsets, not only the weakly (piecewise) open subsets.

Remark 2.2.7. The open sets from the generalized topology may not form a basis of the strong topology (see [K, Appendix C]). Another unpleasant fact about weakly small spaces (compared with locally small spaces) is that points may not have small neighbourhoods (consider an infinite wedge of circles as in Example 4.1.8 of [K]). Notice that the strong topology and the generated topology coincide on every piece.

Definition 2.2.8 (cf. [K, IV.3, Definition 1]). A *weakly* (or *piecewise*) *constructible subset* is a subset $Y \subseteq X$ that has constructible intersections with all members of any chosen exhaustion $(X_\alpha)_{\alpha \in A}$.

Proposition 2.2.9.

(1) *A piecewise essentially finite union of closed* (*open, respectively*) *subsets of a weakly small space is closed* (*open, respectively*).
(2) *A piecewise essentially finite union of piecewise constructible subsets of a weakly small space is piecewise constructible.*

Proof. (1) An essentially finite union of closed (resp. open) subsets is closed (resp. open). Hence a piecewise essentially finite union of closed (resp. open) sets is piecewise closed (resp. open), thus closed (resp. open).

(2) Similar to the proof of the previous case. ∎

Remark 2.2.10. Notice that each exhaustion of a weakly small space is a piecewise essentially finite (relative to this exhaustion) family of closed sets, and remember that a constructible subset of a piece is a finite union of locally closed subsets of the piece.

Proposition 2.2.11. *Piecewise constructible subsets of a weakly small space are exactly piecewise essentially finite unions of locally closed subsets of pieces.*

Proof. The proof is similar to the proof of Proposition 2.1.7. Each piecewise constructible subset is a piecewise essentially finite union of locally closed subsets of pieces. A piecewise essentially finite union of locally closed subsets of pieces has a constructible trace on each piece, hence is piecewise constructible. ∎

Definition 2.2.12. We will say that a weakly small space has the *closure property* (CPW) if the following holds: the weak closure of a locally closed subset of a piece is a closed subset.

Remark 2.2.13. If the space X has the closure property (CPW), then the topological closure operator restricted to the class $\mathrm{ConstrP}_X$ of constructible subsets of pieces of X may be treated as the *closure operator*

$$\bar{\cdot} : \mathrm{ConstrP}_X \to \mathrm{Cl}_X$$

of the generalized topology.

Example 2.2.14. Each topological discrete space is weakly small. An exhaustion is the family of finite subsets of the space with its natural partial ordering.

EXAMPLE 2.2.15. Each topological space (X,τ) may be made weakly small relative to any covering $\{F_\beta\}_{\beta\in B}$ by closed sets satisfying the condition $F_{\beta_0}\setminus\bigcup_{\beta\neq\beta_0}F_\beta\neq\emptyset$. Then we consider $X=\overset{e}{\bigcup}_{\alpha\in A}X_\alpha$ with the exhaustion $X_\alpha=\bigcup_{\beta\in\alpha}\mathrm{sm}(F_\beta)$ with $\alpha\in\mathrm{Fin}(B)$.

EXAMPLE 2.2.16. For the space from [P2, Example 2.3.7] consider the exhaustion $X_n=\{0,\dots,n\}$, $n\in\mathbb{N}$. Then $\mathbb{N}$ becomes a weakly small space. The whole space is small, but is not contained in a piece.

PROPOSITION 2.2.17. *Each object of* **WSS** *is a direct limit* (*taken in* **GTS**) *of the diagram of all its small closed subsets with inclusions as morphisms.*

Proof. Notice that $\mathrm{Cov}_X=\langle\mathrm{Cov}_X\cap_2X_\alpha\rangle^*_\alpha=\langle\mathrm{Cov}_X\cap_2F\rangle^*_{F\in\mathrm{Smcl_X}}$. ■

COROLLARY 2.2.18. *Let* $X=\overset{e}{\bigcup}_{\alpha\in A}X_\alpha$, Y *be weakly small spaces. For a mapping* $f:X\to Y$, *the following conditions are equivalent:*

(a) f *is strictly continuous,*
(b) $f|X_\alpha$ *is strictly continuous for each* $\alpha\in A$,
(c) $f|F$ *is strictly continuous for each* $F\in\mathrm{Smcl_X}$.

FACT 2.2.19. *The preimage of a constructible subset of a piece by a strictly continuous mapping of weakly small spaces is always constructible, but may not be small.*

FACT 2.2.20 (cf. [K, Example IV.1.10]). *The construct* **WSS** *has concrete direct sums.*

PROPOSITION 2.2.21. *If a weakly small space* X *satisfies* (AQC), *then the family of its quasi-components is piecewise finite. If a weakly small space* X *satisfies* (ACC), *then the family of its connected components is piecewise finite.*

Proof. Follows from [P2, Proposition 2.3.23]. ■

DEFINITION 2.2.22. A weakly small space X will be called *nice* if some of its exhaustions is a basis of the bornology Sm_X, i.e. $\mathrm{Sm}_X=\downarrow\{X_\alpha\}_\alpha$ in $\mathcal{P}(X)$. We will denote by **NWSS** the full subcategory of **WSS** consisting of nice weakly small spaces.

REMARK 2.2.23. The space from Example 2.2.16 is an object of **NWSS**, since one can take a one-element exhaustion of this space.

REMARK 2.2.24. In any object X of **NWSS** the family $\mathrm{Smcl_X}$ is a closed basis of the bornology Sm_X.

Proposition 2.2.25. *Let X, Y be objects of* **NWSS**, *with* $X = \overset{e}{\bigcup}_{\alpha\in A} X_\alpha$ *giving a basis of the bornology* Sm_X. *For a mapping* $f : X \to Y$, *the following conditions are equivalent:*

(a) *f is strictly continuous,*
(b) *f is bounded continuous,*
(c) *$f|_F$ is bounded continuous for each* $F \in \mathrm{Smcl}_X$,
(d) *$f|_{X_\alpha}$ is bounded continuous for each* $\alpha \in A$.

Proof. The downward implications are trivial. If f is bounded continuous on each piece of the chosen exhaustion, then each such restriction both in the domain and codomain is strictly continuous by [P2, Fact 2.3.8]. By Corollary 2.2.18, the whole f is strictly continuous. ■

Proposition 2.2.26. *Each subset Y of an object* $X = \overset{e}{\bigcup}_\alpha X_\alpha$ *(with this exhaustion generating* Sm_X*) of* **NWSS** *forms, by the formula* $Y = \overset{e}{\bigcup}_\alpha (X_\alpha \cap Y)$, *an initial subobject in* **NWSS**.

Proof. Consider an object $Z = \overset{e}{\bigcup}_\gamma Z_\gamma$ (with this exhaustion generating Sm_Z) of **NWSS** and a mapping $h : Z \to Y$ such that $i_{YX} \circ h$ is strictly continuous. Then for every γ_0 the restriction $i_{YX} \circ h|_{Z_{\gamma_0}}$ has image its in some $X_{\alpha_0} \cap Y$ by the assumption. We will see that $h|_{Z_{\gamma_0}} : Z_{\gamma_0} \to Y \cap X_{\alpha_0}$ is strictly continuous. It suffices to check continuity. But $(h|_{Z_{\gamma_0}})^{-1}(\mathrm{Op}_{X_{\alpha_0}\cap Y}) = (h|_{Z_{\gamma_0}})^{-1}(\mathrm{Op}_{X_{\alpha_0}}) \subseteq \mathrm{Op}_{Z_{\gamma_0}}$, since $i_{YX} \circ h|_{Z_{\gamma_0}} : Z_{\gamma_0} \to X_{\alpha_0}$ is continuous.
Now apply Corollary 2.2.18 to get strict continuity of h. ■

Question 2.2.27. When is $(Y, \langle \mathrm{Cov}_{X_\alpha \cap Y}\rangle^*_\alpha)$ an initial subobject of (X, Cov_X) in **GTS**?

Theorem 2.2.28. *The construct* **NWSS** *is finitely complete, and its finite limits are concrete.*

Proof. Since each subset of an object in **NWSS** with the induced exhaustion forms an initial subobject in **NWSS**, the existence of equalizers for pairs of parallel mappings is clear.

It is enough to consider binary products. Assume X, Y have exhaustions $(X_\alpha)_{\alpha\in A}$, $(Y_\beta)_{\beta\in B}$ generating the bornologies Sm_X, Sm_Y, respectively. Then $(X_\alpha \times Y_\beta)_{(\alpha,\beta)\in A\times B}$ is an exhaustion defining the weakly small space $X \times Y$ and the canonical projections are strictly continuous. Hence the chosen exhaustion generates the bornology $\mathrm{Sm}_{X\times Y}$, and $X \times Y$ is an object of **NWSS** with the canonical projections being morphisms.

For strictly continuous $f : Z \to X$, $g : Z \to Y$, the mapping $(f, g) : Z \to X \times Y$ is strictly continuous, since if $Z = \overset{e}{\bigcup}_{\gamma\in\Gamma} Z_\gamma$, then for each Z_{γ_0} there are X_{α_0} and Y_{β_0} such that $f(Z_{\gamma_0}) \subseteq X_{\alpha_0}$ and $g(Z_{\gamma_0}) \subseteq Y_{\beta_0}$. The mapping

$(f,g)|_{Z_{\gamma_0}}$ is strictly continuous from [P2, Fact 2.3.12]. Now (f,g) is strictly continuous by Corollary 2.2.18. Thus $X \times Y$ with the canonical projections is the product of X and Y in **NWSS**. ∎

THEOREM 2.2.29. *The category* **NWSS** *admits a full embedding into* **Sublat**.

Proof. Define a functor $I : \mathbf{NWSS} \to \mathbf{Sublat}$ by $I(X, \mathrm{Cl}_X, \mathrm{Int}_X) = (X, \mathrm{Smcl_X})$ and $I(f) = f$. Obviously $\mathrm{Smcl_X}$ is a sublattice of $\mathcal{P}(X)$ containing the empty set and covering X. If $f : X \to Y$ is a strictly continuous mapping, then $f^{-1}(\mathrm{Smcl_Y}) \succeq \mathrm{Smcl_X}$ and $f^{-1}(\mathrm{Smcl_Y}) \cap_1 \mathrm{Smcl_X} \subseteq \mathrm{Smcl_X}$. Hence f is a morphism in **Sublat**. The functor I is obviously faithful. Notice that I is injective on objects, since the family $\mathrm{Smcl_X}$ uniquely determines Int_X (and Cl_X). Namely,

$$\mathrm{Cl}_X = \{G \subseteq X \mid G \cap_1 \mathrm{Smcl_X} \subseteq \mathrm{Smcl_X}\}, \quad \mathrm{Int_X} = \langle \mathrm{Noeth}(\mathrm{Cl_F})\rangle^*_{\mathrm{F} \in \mathrm{Smcl_X}}.$$

Assume that two objects $(X, \mathrm{Cl}_X, \mathrm{Int}_X)$, $(Y, \mathrm{Cl}_Y, \mathrm{Int}_Y)$ of **NWSS** are given. If $g : (X, \mathrm{Smcl_X}) \to (\mathrm{Y}, \mathrm{Smcl_Y})$ is a morphism in **Sublat**, then g is bounded. Similarly to the proof of Theorem 2.1.33, the preimage of a closed set is closed, hence g is continuous. By Proposition 2.2.25, g is strictly continuous. This proves I is full. ∎

THEOREM 2.2.30 (cf. [K, IV.2.1]). *If a weakly small space* $X = \overset{e}{\bigcup}_{\alpha \in A} X_\alpha$ *is strongly* T_1, *and* L *is a small space, then for each strictly continuous mapping* $f : L \to X$ *there is* $\alpha_0 \in A$ *such that* $f(L) \subseteq X_{\alpha_0}$.

Proof. Let $\eta : X \to A$ denote the index function for the exhaustion $(X_\alpha)_{\alpha \in A}$. If $\eta(f(L))$ were infinite, then for each $\alpha \in \eta(f(L))$ we could choose $x_\alpha \in f(L)$ with $\eta(x) = \alpha$ and some $y_\alpha \in f^{-1}(x_\alpha)$. Set $S = \{x_\alpha : \alpha \in \eta(f(L))\}$. For each $\gamma \in A$, the set $S \cap X_\gamma$ is finite. Since X is strongly T_1, the set S is closed, as also is each of its subsets, so S is topological discrete. Then $\{y_\alpha \mid \alpha \in \eta(f(L))\} \subseteq f^{-1}(S)$ is a topological discrete, infinite, and small subset of L. This is a contradiction.

Hence $\eta(f(L))$ is finite, and there is $\alpha_0 \geq \eta(f(L))$. We get $f(L) \subseteq X_{\alpha_0}$. ∎

THEOREM 2.2.31 (cf. [K, IV.2.2]). *If a weakly small space* X *with an exhaustion* $(X_\alpha)_{\alpha \in A}$ *is strongly* T_1, *then each small subspace* L *of* X *is contained in some* X_{α_0}. *In particular, each member* X_β *of any exhaustion* $(X_\beta)_{\beta \in B}$ *of* X *is contained in some member* X_{α_0} *of the initial exhaustion.*

Proof. For each such L, the inclusion mapping $i : L \to X$ is strictly continuous. By Theorem 2.2.30, the set $i(L) = L$ is contained in a member of the exhaustion $(X_\alpha)_{\alpha \in A}$. ∎

DEFINITION 2.2.32. Denote by $\mathbf{WSS}_1$ the full subcategory of **WSS** composed of the strongly T_1 objects of **WSS**.

Fact 2.2.33. $\mathbf{WSS}_1$ *is a full subcategory of* $\mathbf{NWSS}$.

Remark 2.2.34. In $\mathbf{WSS}_1$ the term "piecewise" may be reinterpreted as "when restricted to a closed small set" (and then it does not depend on the choice of an exhaustion). Passing to the strong topology yields a functor $()_{\mathrm{stop}} : \mathbf{WSS}_1 \to \mathbf{Top}$.

Corollary 2.2.35. *Each small set in an object X of* $\mathbf{WSS}_1$ *is contained in an element of any exhaustion.*

Proposition 2.2.36. *Each subset Y of an object $X = \overset{e}{\bigcup}_\alpha X_\alpha$ of* $\mathbf{WSS}_1$ *provides, by the formula $Y = \overset{e}{\bigcup}_\alpha (X_\alpha \cap Y)$, an initial subobject in* $\mathbf{WSS}_1$.

Proof. Follows from Proposition 2.2.26. ∎

Theorem 2.2.37. *The construct* $\mathbf{WSS}_1$ *is finitely complete, and its finite limits are concrete.*

Proof. Since each subset taken with the induced exhaustion provides an initial subobject in $\mathbf{WSS}_1$, the existence of equalizers for pairs of parallel mappings is clear. It is enough to consider binary products. If X, Y are objects of $\mathbf{WSS}_1$, then their product in $\mathbf{NWSS}$ is an object with a pair of morphisms of $\mathbf{WSS}_1$. Hence it is their product in $\mathbf{WSS}_1$. ∎

Proposition 2.2.38. *A piecewise finite subspace of an object of* $\mathbf{WSS}_1$ *is a closed subspace and a topological discrete space.*

Proof. Closedness is obvious. Since by [P2, Proposition 2.3.20] the subspace is topological discrete on each piece, it is topological discrete. ∎

Proposition 2.2.39. *If X is a small space and Y is an object of* $\mathbf{WSS}_1$, *then the canonical projection $\pi_2 : X \times Y \to Y$ is an open and closed mapping.*

Proof. Similar to the proof of Proposition 2.1.40. ∎

Example 2.1.41 shows that if X is only an object of $\mathbf{WSS}_1$, then $\pi_2 : X \times Y \to Y$ may be neither open nor closed. (One may take the closed covering $[n, n+1]$, $n \in \mathbb{Z}$, to get an exhaustion of $(\mathbb{R}_{\mathrm{salg}})_{\mathrm{loc}}$ as in Example 2.2.15.)

Question 2.2.40. Do $\mathbf{WSS}_1$ and $\mathbf{NWSS}$ have (concrete) infinite products?

Question 2.2.41. Do $\mathbf{WSS}_1$ and $\mathbf{NWSS}$ have (concrete) coequalizers?

3. Spaces over structures. In this section, we deal with locally definable and weakly definable spaces over structures (in the sense of model theory).

3.1. Generalized topological spaces over sets. Let M be any non-empty set.

Definition 3.1.1 (cf. [P1]). A *function sheaf over* M on a gts X is a family $\mathcal{F}$ of functions $h : U \to M$, where $U \in \mathrm{Op}_X$, which is closed under:

(a) restrictions to open subsets $V \subseteq U$,
(b) gluings of compatible families of functions defined on members of any admissible family $\mathcal{U} \in \mathrm{Cov}_X$.

For such a function sheaf, define $\mathcal{F}(U) = \{h : U \to M \mid h \in \mathcal{F}\}$. Denote by $\mathrm{FSh}(X, M)$ the family of all function sheaves on a gts X over a set M, and by $\mathrm{FSh}_{\mathbf{GTS}}(M)$ the class of all function sheaves on generalized topological spaces over set M.

Remark 3.1.2. Notice that we may identify $\mathrm{FSh}(X, M)$ with a subset of $\mathcal{P}^2(X \times M)$, and each function sheaf $\mathcal{F}$ with the family of graphs of its members. The function sheaves of $\mathrm{FSh}(X, M)$ are partially ordered by inclusion, which may be understood as inclusion in $\mathcal{P}^2(X \times M)$.

Definition 3.1.3. We will call a function sheaf $\mathcal{F} \in \mathrm{FSh}(X, M)$ *empty* if for each non-empty open subset V of X the family $\mathcal{F}(V)$ is empty. In other words: $\mathcal{F}$ is empty iff it is a singleton of the empty function (defined on the empty set). The empty sheaf on X will be denoted $\mathrm{Empty}(X)$. We will call a function sheaf $\mathcal{F} \in \mathrm{FSh}(X, M)$ **full** if for each $V \in \mathrm{Op}_X$ the family $\mathcal{F}(V)$ equals M^V (the family of all functions from V to M). The full sheaf on X will be denoted $\mathrm{Full}(X)$.

Definition 3.1.4. For function sheaves $\mathcal{F} \in \mathrm{FSh}(X, M)$, $\mathcal{G} \in \mathrm{FSh}(Y, M)$ and a strictly continuous mapping $f : X \to Y$, define the following families

$$f_*\mathcal{F} = \{h : V \to M \mid V \in \mathrm{Op}_Y,\ h \circ f \in \mathcal{F}\},$$
$$f^{-1}\mathcal{G} = \mathcal{G} \circ f = \{h \circ f \mid h \in \mathcal{G}\}.$$

They will be called the *image* of $\mathcal{F}$, and the *preimage* of $\mathcal{G}$, respectively.

Remark 3.1.5. The image of a function sheaf is always a function sheaf, but the preimage of a function sheaf may not be a function sheaf.

Fact 3.1.6. *For each family of function sheaves on the same gts X, the intersection of this family is a function sheaf.*

Definition 3.1.7. For each family $\mathcal{F}$ of functions defined on open sets in a gts X the smallest function sheaf containing $\mathcal{F}$ will be called the *function sheaf generated by* $\mathcal{F}$ or the *sheafification* of $\mathcal{F}$ and denoted by $\mathcal{F}^+$.

Fact 3.1.8. *For each strictly continuous function $f : X \to Y$ if $\mathcal{G}$ is a family of functions defined on open sets in Y with values in M, then* $f^{-1}(\mathcal{G}^+) \subseteq (f^{-1}\mathcal{G})^+$.

Definition 3.1.9 (cf. [P1, Section 2]). A *generalized topological space* (*gts*) *over* M is a pair $(X, \mathcal{O}_X)$, where X is a gts and $\mathcal{O}_X$ is a function sheaf over M on X. A *morphism* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of gts's over M is a strictly continuous mapping $f : X \to Y$ such that $\mathcal{O}_Y \subseteq f_* \mathcal{O}_X$ (equivalently: $f^{-1}\mathcal{O}_Y \subseteq \mathcal{O}_X$).

We get a category $\mathbf{GTS}(M)$ of generalized topological spaces over M and their morphisms. (Objects of $\mathbf{GTS}(M)$ may be identified with members of $\mathrm{FSh}_{\mathbf{GTS}}(M)$.)

Fact 3.1.10. *If* $X = \overset{a}{\bigcup}_\alpha X_\alpha$, *then the object* $(X, \mathcal{O}_X)$ *of* $\mathbf{GTS}(M)$ *is uniquely determined by the system* $(X_\alpha, \mathcal{O}_{X_\alpha})_\alpha$, *since* $\mathcal{O}_X = (\bigcup_\alpha \mathcal{O}_{X_\alpha})^+$.

Being a morphism is a local and sublocal property.

Proposition 3.1.11. *Let* $X = \overset{a}{\bigcup} \mathcal{U}$ *and* Y *be objects of* $\mathbf{GTS}(M)$. *For a mapping* $f : X \to Y$, *the following conditions are equivalent:*

(a) f *is a morphism in* $\mathbf{GTS}(M)$,
(b) $f|_U$ *is a morphism in* $\mathbf{GTS}(M)$ *for each* $U \in \mathcal{U}$.

Proof. By [P2, Proposition 2.2.45] the equivalence holds in the category $\mathbf{GTS}$. Notice that $\mathcal{O}_X$ is generated by all $\mathcal{O}_U$ with $U \in \mathcal{U}$ and a similar fact holds for the family $f^{-1}\mathcal{O}_Y$. The proposition follows. ∎

Proposition 3.1.12. *Let* $X = \overset{a}{\bigcup} \mathcal{U}$ *and* Y *be objects of* $\mathbf{GTS}(M)$. *For a mapping* $f : Y \to X$, *the following conditions are equivalent:*

(a) f *is a morphism in* $\mathbf{GTS}(M)$,
(b) *the family* $f^{-1}(\mathcal{U})$ *is admissible and* $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ *is a morphism in* $\mathbf{GTS}(M)$ *for each* $U \in \mathcal{U}$.

Proof. The equivalence holds on the level of strictly continuous mappings by [P2, Proposition 2.2.47]. If f is a morphism in $\mathbf{GTS}(M)$, then $f^{-1}\mathcal{O}_X|_{f^{-1}(U)} = f^{-1}(\mathcal{O}_X|_U)$ for each strict subspace $U \in \mathcal{U}$. If the family $f^{-1}(\mathcal{U})$ is admissible and each $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a morphism in $\mathbf{GTS}(M)$ for $U \in \mathcal{U}$, then $(f|_{f^{-1}(U)})^{-1}(\mathcal{O}_X|_U) = f^{-1}\mathcal{O}_X|_{f^{-1}(U)}$ for each U. ∎

Definition 3.1.13 (cf. [K, IV.1, Definitions 3 and 4]). For an object $(X, \mathcal{O}_X)$ of $\mathbf{GTS}(M)$ and a subset $Y \subseteq X$ we induce a gts over M on Y in the following way: take the generalized topology $\mathrm{Cov}_Y = \langle \mathrm{Cov}_X \cap_2 Y \rangle$ induced by (X, Cov_X) on Y, and take the function sheaf $\mathcal{O}_Y = (i_{YX}^{-1}\mathcal{O}_X)^+$. Such an object $(Y, \mathrm{Cov}_Y, \mathcal{O}_Y)$ of $\mathbf{GTS}(M)$ will be called a *subspace* in $\mathbf{GTS}(M)$ of $(X, \mathrm{Cov}_X, \mathcal{O}_X)$; it is a *strict subspace* in $\mathbf{GTS}$ if $\mathrm{Cov}_Y = \mathrm{Cov}_X \cap_2 Y$ and $\mathcal{O}_Y = i_{YX}^{-1}\mathcal{O}_X$. An *open* (*small*, respectively) *subspace* is a subspace $(Y, \mathrm{Cov}_Y, \mathcal{O}_Y)$ as above with $Y \in \mathrm{Op}_X$ ($Y \in \mathrm{Sm}_X$, respectively).

Fact 3.1.14. *Each open subspace is strict.*

Remark 3.1.15. All subspaces in our sense are initial subobjects and extremal subobjects in $\mathbf{GTS}(M)$.

Remark 3.1.16. The construct $\mathbf{GTS}(M)$ has discrete objects and indiscrete objects. The former are given by the discrete functor $D : \mathbf{Set} \to \mathbf{GTS}(M)$, and the latter by the indiscrete functor $I : \mathbf{Set} \to \mathbf{GTS}(M)$, where

$$D(X) = (X, \mathcal{P}^2(X), \mathrm{Full}(X)),$$
$$I(X) = (X, \mathrm{EssFin}(\{\emptyset, X\}), \mathrm{Empty}(X)).$$

Theorem 3.1.17. *The construct* $\mathbf{GTS}(M)$ *is topological.*

Proof. For a source (X, f_i) with mappings $f_i : X \to Y_i$ indexed by a class I, we may assume that I is a set. Assume each Y_i has a generalized topology Cov_i and a structure sheaf $\mathcal{O}_i$. Give X the initial generalized topology for the family $(f_i)_{i\in I}$ and $\mathcal{O}_X$ generated by all $f_i^{-1}(\mathcal{O}_i)$.

For any $(Z, \mathrm{Cov}_Z, \mathcal{O}_Z)$ and a mapping $h : Z \to X$, if all $f_i \circ h$ are morphisms in $\mathbf{GTS}(M)$, then h is a morphism in $\mathbf{GTS}$ by [P2, Theorem 2.2.60]. Also $\bigcup_i h^{-1}(f^{-1}(\mathcal{O}_i)) \subseteq \mathcal{O}_Z$. Since $\mathcal{O}_Z$ is a function sheaf, it also contains $(\bigcup_i h^{-1}(f^{-1}(\mathcal{O}_i)))^+$ and, by Fact 3.1.8, $h^{-1}((\bigcup_i f^{-1}(\mathcal{O}_i))^+)$. We have proved that h is a morphism in $\mathbf{GTS}(M)$. By [AHS, Theorem 21.5], only one object of $\mathbf{GTS}(M)$ in the fibre of the set X can do this job (the pair $(\mathbf{GTS}(M), U_M)$ is amnestic, where $U_M : \mathbf{GTS}(M) \to \mathbf{Set}$ is the forgetful functor). ∎

Corollary 3.1.18 ([AHS, Theorem 21.17]). *The category* $\mathbf{GTS}(M)$ *is complete, co-complete, wellpowered, co-wellpowered,* (*Epi, Extremal Mono-Source*)*-category, has regular factorizations and has separators and coseparators.*

Remark 3.1.19. For any non-empty set M, the category $\mathbf{GTS}$ may be fully embedded into $\mathbf{GTS}(M)$. Choose, for example, an element $m_0 \in M$ and for each open U of X define $\mathcal{C}_{m_0}(U)$ to be the singleton of the constant function with value m_0. Then $X \to (X, \mathcal{C}_{m_0})$ defines a faithful, injective on objects, and full functor from $\mathbf{GTS}$ to $\mathbf{GTS}(M)$.

Fact 3.1.20 ([AHS, Proposition 21.15]). *The forgetful functor U_M from* $\mathbf{GTS}(M)$ *to* $\mathbf{Set}$ *preserves and uniquely lifts* (*small*) *limits and colimits. In particular, all* (*small*) *limits and colimits in* $\mathbf{GTS}(M)$ *are concrete.*

3.2. Weakly topological structures

Definition 3.2.1. A *structure with a topology* is a pair $(\mathcal{M}, \sigma)$ composed of a (first order, one-sorted) structure $\mathcal{M} = (M, \dots)$ and a topology σ given on the underlying set M of $\mathcal{M}$. This means the product topologies σ^n are defined on Cartesian powers M^n and the induced topologies σ_D exist on each definable (with parameters) set D in any M^n. The system $(\mathcal{M}, (\sigma_D)_D)$ will be called the corresponding *weakly topological structure* and briefly denoted also by $(\mathcal{M},\sigma)$.

Then all the projections $\pi_{n,i} : M^n \to M^{n-1}$ (forgetting the ith coordinate) are continuous and open.

REMARK 3.2.2. This setting seems to coincide with the case (i) in the introduction of [Pil]. We do not explore a special language L_t for topological structures considered in the case (ii) of the introduction of [Pil] or in [FZ].

Remember (after [Pil]) that a structure $\mathcal{M}$ is a *first order topological structure* (called in [FZ] a *topological structure with explicitly definable topology*) if the basis of the topology on M is uniformly definable in $\mathcal{M}$. (There is a formula $\Phi(x,\bar{y})$ of the (first order) language of $\mathcal{M}$ such that the family $\{\Phi(x,\bar{a})^M \mid \bar{a} \subseteq M\}$ is the basis of the topology of M.) We will write $\mathcal{M}$ instead of $(\mathcal{M},\sigma)$ in this situation.

EXAMPLE 3.2.3. Any o-minimal structure $(M,<,\dots)$ with $<$ being a dense order without endpoints together with the natural order topology yields a topological structure.

EXAMPLE 3.2.4. The field of complex numbers $(\mathbb{C},+,\cdot)$ considered with the Euclidean topology (but not with the Zariski topology) on every $\mathbb{C}^n$ (as in [Pil]) is a weakly topological structure.

EXAMPLE 3.2.5. The fields $\mathbb{Q}_p$ of p-adic numbers considered with their natural topologies (coming from valuations) and the product topologies on Cartesian powers $\mathbb{Q}_p^n$ (as in [Pil]) are topological structures.

REMARK 3.2.6. We will deal with sets (such as definable sets of affine spaces M^n or of definable spaces over $(\mathcal{M},\sigma)$) equipped with topologies coming from σ (as opposed to other topologies). For simplicity, we will speak about *σ-open* sets and *σ-continuous* functions or mappings in such situations.

DEFINITION 3.2.7. For each definable (with parameters) set $D \subseteq M^n$ of a weakly topological structure $(\mathcal{M},\sigma)$, we set (as in [P1, Fundamental Example 1]):

(a) an *open subset* of a gts D means a relatively σ-open, definable subset;
(b) an *admissible family* of a gts D means an essentially finite open family.

Each such D becomes a small gts, and $\pi_{n,i} : M^n \to M^{n-1}$ become strictly continuous and open. We define the *structure sheaf* $\mathcal{DC}_D$ as the function sheaf of all definable σ-continuous functions from (gts-)open subsets $U \subseteq D$ into M. Thus $(D,\mathcal{DC}_D)$ becomes a small (generalized topological) space over M, respecting the pair $(\mathcal{M},\sigma)$.

Fact 3.2.8. *Always* $\tau(\mathrm{Op}_{M^n}) \subseteq \sigma^n$, *and equality holds when* Op_{M^n} *is a basis of* σ^n.

Remark 3.2.9. In general σ has nothing to do with the structure $\mathcal{M}$. If we assume $\sigma = \tau(\mathrm{Op}_M)$, then $\sigma^n = \tau(\mathrm{Op}_M)^n = \tau(\mathrm{Op}_{M^n})$. For first order topological structures the natural topology on $\mathcal{M}$ satisfies $\sigma = \tau(\mathrm{Op}_X)$.

Example 3.2.10. All partially topological small spaces $(X, \tau, \mathrm{EssFin}(\tau))$ may be seen as definable sets M^1. Just put $M = X$, $\sigma = \tau$ and take all members of τ as relations of the language of $\mathcal{M}$.

Remark 3.2.11. For each definable $D \subseteq M^n$ the function sheaf $\mathcal{DC}_D$ contains the projections $\pi_i : D \to M$ $(i = 1, \ldots, n)$. The identity id_D is their diagonal product. More generally, all diagonal products of members of $\mathcal{DC}_D$ are definable σ-continuous.

Remark 3.2.12. Notice that each function in $\mathcal{DC}_D$ is also strictly continuous with respect to the small space $(D, \mathrm{EssFin}(\mathrm{Op}_D))$ induced on D.

Proposition 3.2.13. *For a mapping* $f : D \to E$ *with definable* $D \subseteq M^m$ *and* $E \subseteq M^n$, *the following conditions are equivalent:*

(a) f *is a morphism of* $\mathbf{GTS}(M)$,
(b) f *is definable and* σ-*continuous,*
(c) f *is a diagonal product of functions from* $\mathcal{DC}_D$.

Proof. Left to the reader. ∎

Definition 3.2.14. We say that $(\mathcal{M}, \sigma)$ *satisfies* (DCD) if $\mathcal{M}$ admits a (finite) decomposition into definably connected definable sets: for each finite family of definable sets in some M^n there is a finite decomposition of M^n into definably connected definable sets compatible with the original family. (For stronger (cell) decompositions see [M] or [S].)

Remark 3.2.15. All o-minimal structures $\mathcal{M}$ (with their order dense without endpoints) satisfy (DCD), but t-minimality in the sense of [S] does not guarantee (DCD).

3.3. Definable spaces over structures. From now on assume that some weakly topological structure $(\mathcal{M}, \sigma)$ is given.

Definition 3.3.1 (cf. [D, p. 157]). An *affine definable space* over $(\mathcal{M}, \sigma)$ is an object of $\mathbf{GTS}(M)$ isomorphic to a definable subset of some M^n considered with its structure (from Definition 3.2.7) of a small space over M.

Definition 3.3.2 (cf. [D, p. 156]). A *definable space* over $(\mathcal{M}, \sigma)$ is an object of $\mathbf{GTS}(M)$ that has a finite open covering by affine definable subspaces. Since the open affine subspaces are strict, the structure sheaf is determined in an obvious way.

Definition 3.3.3. *Morphisms* of affine definable spaces and definable spaces over $(\mathcal{M}, \sigma)$ are their morphisms in $\mathbf{GTS}(M)$. We get full subcategories $\mathbf{ADS}(\mathcal{M}, \sigma)$, $\mathbf{DS}(\mathcal{M}, \sigma)$ of $\mathbf{GTS}(M)$.

Proposition 3.3.4. *The open subsets in a definable space are exactly the σ-open definable subsets. The morphisms of definable spaces are exactly the σ-continuous definable mappings.*

Proof. Follows from Definition 3.2.7 and Proposition 3.2.13. ∎

Fact 3.3.5. *Definable sets are stable under images and preimages of morphisms of* $\mathbf{DS}(\mathcal{M}, \sigma)$.

Remark 3.3.6. In general $\mathbf{ADS}(\mathcal{M}, \sigma)$ and further categories do not have indiscrete objects. That is why the class of epimorphisms may be different from the class of surjective morphisms.

Definition 3.3.7. A *subspace of a definable space* $(X, \mathcal{O}_X)$ given by a finite open covering $(X_i, \mathcal{O}_{X_i})$ by affine definable spaces is a definable subset $Y \subseteq X$ together with the function sheaf $\mathcal{O}_Y$ determined by the finite open covering $(Y_i, \mathcal{O}_{Y_i})$ by affine definable spaces with $Y_i = X_i \cap Y$.

Proposition 3.3.8 (cf. [D, p. 158]). *Each definable subspace* $(Y, \mathcal{O}_Y)$ *of a definable space* $(X, \mathcal{O}_X)$ *may be identified with an initial subobject of* $(X, \mathcal{O}_X)$ *in* $\mathbf{DS}(\mathcal{M}, \sigma)$.

Proof. First notice that the inclusion $i_{YX} : Y \to X$ is a definable continuous mapping, thus a monomorphism in $\mathbf{DS}(\mathcal{M}, \sigma)$. Assume $(Z, \mathcal{O}_Z)$ is any object of $\mathbf{DS}(\mathcal{M}, \sigma)$ and $h : Z \to Y$ is a mapping such that $i_{YX} \circ h : Z \to X$ is a morphism in $\mathbf{DS}(\mathcal{M}, \sigma)$. We will prove that h is a morphism. The graph of h is a definable subset of $Z \times X$, so also of $Z \times Y$. But h is also σ-continuous, since Y forms an initial subobject of X in $\mathbf{Top}$. ∎

Remark 3.3.9. The above does not mean that a subspace of a definable space in our sense is an initial subobject or an extremal subobject in $\mathbf{GTS}(M)$.

Remark 3.3.10. Each constructible set in a definable space is definable. By cell decomposition into locally closed cells (see [D, Chapter 3]), in definable spaces over o-minimal structures all definable sets are constructible.

Proposition 3.3.11. *Every weakly open* (*weakly closed, respectively*) *subspace of an object of* $\mathbf{DS}(\mathcal{M}, \sigma)$ *is an open* (*closed, respectively*) *subspace.*

Proof. This is clear for $\mathbf{ADS}(\mathcal{M}, \sigma)$ from Definition 3.2.7 and follows for $\mathbf{DS}(\mathcal{M}, \sigma)$ by applying finite unions. ∎

Proposition 3.3.12. *Finite products exist in the categories* $\mathbf{ADS}(\mathcal{M}, \sigma)$ *and* $\mathbf{DS}(\mathcal{M}, \sigma)$.

Proof. For definable sets $D \subseteq M^n$ and $E \subseteq M^m$, the product is the definable set $D \times E \subseteq M^{n+m}$ (with its natural space structure). Indeed, the projections $D \times E \to D$, $D \times E \to E$ are σ-continuous definable mappings, and for σ-continuous definable $f : Z \to D$, $g : Z \to E$, with Z definable in some M^k, the induced mapping $(f, g) : Z \to D \times E$ is σ-continuous definable, since $\sigma_{D\times E}$ is the product topology. For a σ-continuous definable function $h \in \mathcal{O}_{D\times E}(U)$, the function $h \circ (f, g) : (f, g)^{-1}(U) \to M$ is σ-continuous definable. This gives the proof for the affine definable case. The general definable case is similar. ∎

FACT 3.3.13. *The canonical projections from a finite product in* $\mathbf{ADS}(\mathcal{M}, \sigma)$ *as well as in* $\mathbf{DS}(\mathcal{M}, \sigma)$ *to its factors are open morphisms, but not in general closed morphisms.*

THEOREM 3.3.14. *The categories* $\mathbf{ADS}(\mathcal{M}, \sigma)$ *and* $\mathbf{DS}(\mathcal{M}, \sigma)$ *are finitely complete.*

Proof. Follows from Propositions 3.3.12 and 3.3.8. ∎

REMARK 3.3.15. Unfortunately, even finite products of affine definable spaces in $\mathbf{GTS}(M)$ usually are not affine definable spaces.

FACT 3.3.16. *Finite coproducts exist in* $\mathbf{DS}(\mathcal{M}, \sigma)$.

REMARK 3.3.17. Often also finite coproducts exist in $\mathbf{ADS}(\mathcal{M}, \sigma)$, but not always (as in the case of M a singleton).

EXAMPLE 3.3.18. Consider the field of real numbers $\mathbb{R}$ (with its natural topology) as a structure $\mathcal{M}$ and two continuous semialgebraic functions of one real variable $f_1(x) = x$, $f_2(x) = x + 1$. The quotient set $\mathbb{R}/\mathbb{Z}$ together with the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a coequalizer of this pair in **Set**. But q is not semialgebraic. A coequalizer in $\mathbf{ADS}(\mathbb{R})$ or $\mathbf{DS}(\mathbb{R})$ exists, but is a one-point space. We get an example when a coequalizer in $\mathbf{ADS}(\mathcal{M})$ and $\mathbf{DS}(\mathcal{M})$ is not concrete.

FACT 3.3.19. *If* $\mathcal{M}$ *is a first order topological structure, then the* (*weak*) *closure of any definable set in a definable space is definable* (*see* [Pil]), *thus is the closure of this set* (*the smallest closed subspace containing the set*). *Hence each definable space over* $\mathcal{M}$ *has the closure property* (CPG).

PROPOSITION 3.3.20. *Assume* $(\mathcal{M}, \sigma)$ *satisfies* (DCD). *Then each object of* $\mathbf{DS}(\mathcal{M}, \sigma)$ *has a finite number of clopen definably connected components, and is a finite direct sum of them.*

Proof. Follows from [P2, Remark 2.2.90] and Proposition 3.3.11. ∎

In the particular case of o-minimal expansions of fields, definable spaces were extensively studied in [D, Chapter 10].

3.4. Locally definable spaces over structures

Definition 3.4.1 (cf. [P1]). A *locally definable space* over $(\mathcal{M},\sigma)$ is an object of $\mathbf{GTS}(M)$ that has an admissible covering by affine definable open (hence strict) subspaces. (Sections of the structure sheaf of a locally definable space are admissible unions of sections of the structure sheaves of these affine open subspaces. This property uniquely determines an object of $\mathbf{GTS}(M)$ by Fact 3.1.10.)

Definition 3.4.2. *Morphisms* of locally definable spaces over $(\mathcal{M},\sigma)$ are their morphisms in $\mathbf{GTS}(M)$. We thus get a full subcategory $\mathbf{LDS}(\mathcal{M},\sigma)$ of $\mathbf{GTS}(M)$.

Fact 3.4.3 (cf. [DK, I.1.3]). *A mapping $f: X \to Y$ between objects of $\mathbf{LDS}(\mathcal{M},\sigma)$ is a morphism iff the image of each open definable subspace of the domain is contained in an open definable subspace of the range and when restricted to open definable subspaces in the domain and in the range it is a mapping of definable spaces, hence is σ-continuous definable.*

Locally definable spaces in particular cases were extensively studied in [DK] and [P1].

Definition 3.4.4 (cf. [P1]). Locally definable subsets (i.e. subsets having definable traces on each definable open subspace from Definition 3.4.1) of locally definable spaces are called *subspaces* in $\mathbf{LDS}(\mathcal{M},\sigma)$.

Remark 3.4.5. Locally definable subsets with their induced admissible coverings by open affine definable subspaces may be identified with certain initial subobjects of an object in $\mathbf{LDS}(\mathcal{M},\sigma)$. This does not mean they are initial subobjects in $\mathbf{GTS}(M)$.

Fact 3.4.6. *Every small subspace of a locally definable space is a definable space.*

Fact 3.4.7. *Every weakly open (weakly closed, respectively) subspace of an object of $\mathbf{LDS}(\mathcal{M},\sigma)$ is an open (closed, respectively) subspace.*

Fact 3.4.8. *The preimage of a subspace by a morphism of $\mathbf{LDS}(\mathcal{M},\sigma)$ is always a subspace of the domain.*

Remark 3.4.9. The image of a morphism is usually not a subspace, since any subset of the target space is the image of a discrete object of $\mathbf{LDS}(\mathcal{M},\sigma)$.

Fact 3.4.10. *In $\mathbf{LDS}(\mathcal{M},\sigma)$ direct sums exist.*

Theorem 3.4.11 (cf. [DK, I.2.5]). *The category $\mathbf{LDS}(\mathcal{M},\sigma)$ is finitely complete.*

Proof. Equalizers are easy. We need to check only the existence of binary products. For locally definable spaces $(X,\mathcal{O}_X) = \overset{a}{\bigcup}_{\alpha\in A}(X_\alpha,\mathcal{O}_{X_\alpha})$,

$(Y,\mathcal{O}_Y)=\overset{a}{\bigcup}_{\beta\in B}(Y_\beta,\mathcal{O}_{Y_\beta})$, where $(X_\alpha,\mathcal{O}_{X_\alpha})$, $(Y_\beta,\mathcal{O}_{Y_\beta})$ are affine definable (strict) subspaces over $(\mathcal{M},\sigma)$, consider the space

$$\overset{a}{\bigcup_{(\alpha,\beta)\in A\times B}}(X_\alpha,\mathcal{O}_{X_\alpha})\times(Y_\beta,\mathcal{O}_{Y_\beta})=\overset{a}{\bigcup_{(\alpha,\beta)\in A\times B}}(X_\alpha\times Y_\beta,\mathcal{O}_{X_\alpha\times Y_\beta}).$$

This space is the product of the given locally definable spaces, since the projections are clearly morphisms in $\mathbf{LDS}(\mathcal{M},\sigma)$, and for given morphisms $f: Z\to X$, $g:Z\to Y$ in $\mathbf{LDS}(\mathcal{M},\sigma)$ from the space $(Z,\mathcal{O}_Z)=\overset{a}{\bigcup}_{\gamma\in\Gamma}(Z_\gamma,\mathcal{O}_{Z_\gamma})$, the induced mapping $(f,g):Z\to X\times Y$ is a morphism in $\mathbf{LDS}(\mathcal{M},\sigma)$. Indeed, we may assume that $\{Z_\gamma\}_{\gamma\in\Gamma}$ is an admissible covering by affine definable spaces that is a refinement of the preimage of the covering $\{X_\alpha\times Y_\beta\}_{(\alpha,\beta)\in A\times B}$ and thus each Z_γ is mapped into some $X_{\alpha_0}\times Y_{\beta_0}$ by some σ-continuous definable mapping, and the whole (f,g) is an admissible union of such partial mappings. ■

FACT 3.4.12. *Finite limits in* $\mathbf{LDS}(\mathcal{M},\sigma)$ *are concrete.*

DEFINITION 3.4.13. A locally definable space is *paracompact* or *Lindelöf* if so is its underlying locally small space.

Paracompactness and Lindelöfness were extensively studied in [DK] in the special case of locally semialgebraic spaces.

DEFINITION 3.4.14 (cf. [P1]). We can also generalize the following concept from [P1]: a subset Y of a locally definable space X over $(\mathcal{M},\sigma)$ may be called *local* if for each $y\in Y$ there is an open small neighbourhood U of y such that $Y\cap U$ is definable in U. On such a Y, we could define a locally definable space by the formula

$$Y=\overset{a}{\bigcup}\{Y\cap U\mid U\text{ small, open}, Y\cap U\text{ definable in }U\}.$$

This definition is canonical in the sense that it does not depend on the choice of open neighbourhoods. However, often when we consider local subsets, this locally definable space structure is not important (see Examples 11 and 12 in [P1]).

EXAMPLE 3.4.15. Any weakly open and any weakly discrete set in a locally definable space is local.

PROPOSITION 3.4.16. *If $\mathcal{M}$ is a first order topological structure, then each locally definable space over $\mathcal{M}$ satisfies* (CPL), *and any subspace has closure* (*i.e. the smallest ambient closed subspace*).

Proof. Follows from Fact 3.3.19. ■

FACT 3.4.17. *Assume* $(\mathcal{M},\sigma)$ *satisfies* (DCD). *Then each object of* $\mathbf{LDS}(\mathcal{M},\sigma)$ *is a locally finite direct sum of its connected components.*

Question 3.4.18. Which objects of **LSS** can be realized as underlying gts's of objects of categories $\mathbf{LDS}(\mathcal{M},\sigma)$?

3.5. Weakly definable spaces over structures

Definition 3.5.1. A *weakly definable space* over $(\mathcal{M},\sigma)$ is an object $(X,\mathcal{O}_X)$ of $\mathbf{GTS}(M)$ that has an exhaustion $(X_\alpha)_{\alpha\in A}$ of X composed of definable (so small) subspaces such that a function $h: V\to M$ $(V\in \mathrm{Op}_X)$ belongs to $\mathcal{O}_X$ iff each restriction $h|_{V\cap X_\alpha}$ belongs to $\mathcal{O}_{X_\alpha}$.

Definition 3.5.2. *Morphisms* of weakly definable spaces over $(\mathcal{M},\sigma)$ are their morphisms in $\mathbf{GTS}(M)$. We thus get a full subcategory $\mathbf{WDS}(\mathcal{M},\sigma)$ of $\mathbf{GTS}(M)$.

Remark 3.5.3. The image of a morphism may be any subset of the target space, since discrete objects exist in $\mathbf{WDS}(\mathcal{M},\sigma)$.

Definition 3.5.4 (cf. [P1]). Piecewise definable subsets (i.e. subsets having definable traces on any piece) of weakly definable spaces with their induced structure of weakly small spaces (given by the induced exhaustions) are called *subspaces* in $\mathbf{WDS}(\mathcal{M},\sigma)$.

Remark 3.5.5. Piecewise definable subsets may be identified with initial subobjects in $\mathbf{WDS}(\mathcal{M},\sigma)$. This does not mean that they are initial subobjects in $\mathbf{GTS}(M)$.

Proposition 3.5.6. *Every weakly open* (*weakly closed, respectively*) *subspace of an object of* $\mathbf{WDS}(\mathcal{M},\sigma)$ *is an open* (*closed, respectively*) *subspace.*

Proof. Follows from Fact 3.3.11. ∎

Weakly definable spaces in particular cases were extensively studied in [K] and [P1].

Corollary 3.5.7. *Each object of* $\mathbf{WDS}(\mathcal{M},\sigma)$ *is a direct limit of the diagram in* $\mathbf{GTS}(M)$ *of:*

(a) *all its pieces,*
(b) *all of its closed definable subspaces,*
(c) *all its definable subspaces with inclusions as morphisms.*

Proposition 3.5.8. *If* $\mathcal{M}$ *is a first order topological structure, then each weakly definable space has the closure property* (CPW), *and any subspace of a piece has closure* (*i.e. the smallest ambient closed subspace*).

Proof. Follows from Fact 3.3.19. ∎

Fact 3.5.9. *Assume* $(\mathcal{M},\sigma)$ *satisfies* (DCD). *Then each object of* $\mathbf{WDS}(\mathcal{M},\sigma)$ *is a piecewise finite union of its clopen connected components, so also a direct sum of its connected components.*

Let us denote by $\mathbf{WDS}_1(\mathcal{M},\sigma)$ the full subcategory of strongly (but see Proposition 3.6.1) T_1 objects of $\mathbf{WDS}(\mathcal{M},\sigma)$. From Theorem 2.2.30, we get

FACT 3.5.10. *In an object of* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *each small subspace is contained in a piece, hence it is a definable* (*sub*)*space.*

FACT 3.5.11 (cf. [K, IV.2.3]). *A mapping* $f: X \to Y$ *between objects of* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *is a morphism iff the image of a piece is contained in a piece and when restricted to pieces in the domain and in the range, the mapping is* σ*-continuous definable.* (*Here a piece may be understood both as a member of some chosen exhaustion and as any closed definable subspace.*)

FACT 3.5.12. *The category* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *has equalizers for pairs of parallel morphisms and direct sums.*

THEOREM 3.5.13 (cf. [K, IV.3, p. 32]). *The category* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *is finitely complete.*

Proof. We need to check only the existence of binary products. Notice that by Theorem 3.4.11, finite products exist in the category $\mathbf{DS}(\mathcal{M},\sigma)$, and the product of T_1 spaces is T_1. We will drop the structure sheaves in notation. If $(X_\alpha)_{\alpha\in A}$, $(Y_\beta)_{\beta\in B}$ are exhaustions of objects X, Y, then define $X\times Y = \overset{e}{\bigcup}_{(\alpha,\beta)\in A\times B}(X_\alpha\times Y_\beta)_{(\alpha,\beta)\in A\times B}$. This is an exhaustion determining the structure sheaf. The resulting space is clearly T_1, and the canonical projections $X\times Y\to X$, $X\times Y\to Y$ are clearly morphisms of $\mathbf{WDS}_1(\mathcal{M},\sigma)$. If $Z=\overset{e}{\bigcup}_{\gamma\in\Gamma} Z_\gamma$ is any object of $\mathbf{WDS}_1(\mathcal{M},\sigma)$ and $f: Z\to X$, $g: Z\to Y$ are morphisms, then the mapping $(f,g): Z\to X\times Y$ is a morphism. Indeed, we may assume that for each Z_{γ_0} there are X_{α_0} and Y_{β_0} such that $f(Z_{\gamma_0})\subseteq X_{\alpha_0}$ and $g(Z_{\gamma_0})\subseteq Y_{\beta_0}$, the generalized topology of the product of definable spaces contains the generalized topology of their generalized topological product, and the definable σ-continuous mappings are stable under composition and diagonal product. Thus $X\times Y$ is the product of X and Y in $\mathbf{WDS}_1(\mathcal{M},\sigma)$. ∎

FACT 3.5.14. *Finite limits in* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *are concrete.*

FACT 3.5.15. *The preimage of a subspace under a morphism of* $\mathbf{WDS}_1(\mathcal{M},\sigma)$ *is always a subspace of the domain.*

QUESTION 3.5.16. Is Fact 3.5.11 extendable to $\mathbf{WDS}(\mathcal{M},\sigma)$?

3.6. Separation axioms

PROPOSITION 3.6.1.

(a) *Each weakly* T_1 *object of* $\mathbf{LDS}(\mathcal{M},\sigma)$ *or* $\mathbf{WDS}(\mathcal{M},\sigma)$ *is strongly* T_1.
(b) *If* (M,σ) *is* T_1, *then all objects of* $\mathbf{LDS}(\mathcal{M},\sigma)$ *and* $\mathbf{WDS}(\mathcal{M},\sigma)$ *are strongly* T_1.

Proof. (a) Any singleton is always a subspace. For a weakly T_1 space, the generated topology is T_1 by [P2, Proposition 2.2.82]. Hence each point is closed.

(b) Each point in M is closed in σ, hence each point in M^n is closed in σ^n. Each point is a closed subspace in an object of $\mathbf{ADS}(\mathcal{M},\sigma)$. The statement for locally definable and weakly definable spaces follows. ∎

Thus we speak just about T_1 *objects* of $\mathbf{LDS}(\mathcal{M},\sigma)$ or of $\mathbf{WDS}(\mathcal{M},\sigma)$.

PROPOSITION 3.6.2. *For an object Z of $\mathbf{LDS}(\mathcal{M},\sigma)$ or of $\mathbf{WDS}_1(\mathcal{M},\sigma)$, the following conditions are equivalent:*

(a) *Z is weakly Hausdorff;*
(b) *Z is strongly Hausdorff;*
(c) *Z has its diagonal Δ_Z closed.*

Proof. If a space X is weakly Hausdorff, then its generated topology is Hausdorff, thus the diagonal Δ_X is closed in the generated topology. But Δ_X is always a subspace, hence, by Fact 3.4.7 or Proposition 3.5.6, a closed subspace.

If the diagonal Δ_X is a closed subspace, then the generated topology of X is Hausdorff, so X is weakly Hausdorff. But it is also strongly T_1, since for any $x_0 \in X$, the set $\{(x_0,x_0)\}$ is closed in $\{x_0\}\times X$ and the projection $\{x_0\}\times X \to X$ is closed. Thus X is strongly Hausdorff. ∎

Because of Proposition 3.6.2, we speak about T_2 *objects* of $\mathbf{LDS}(\mathcal{M},\sigma)$ or of $\mathbf{WDS}(\mathcal{M},\sigma)$. The following example shows that even if σ is Hausdorff, an object of $\mathbf{ADS}(\mathcal{M},\sigma)$ may not be Hausdorff.

EXAMPLE 3.6.3. Consider as $\mathcal{M}$ the pure set $\mathbb{R}$ of real numbers, and as σ the natural topology on $\mathbb{R}$. Then only finite or cofinite sets are definable. The sets in Op_M are the empty set and the cofinite sets, so M as an affine definable space is not weakly Hausdorff.

That is why we should consider the topologies $\tau(\mathrm{Op}_{M^n})$ in addition to σ. Remember that $\tau(\mathrm{Op}_M)^n \subseteq \tau(\mathrm{Op}_{M^n}) \subseteq \sigma^n$. For a first order topological structure (considered with its natural topology) also $\sigma^n \subseteq \tau(\mathrm{Op}_M)^n$.

FACT 3.6.4. *If $(M,\tau(\mathrm{Op}_M))$ is Hausdorff (T_2), then all M^n as well as all objects of $\mathbf{ADS}(\mathcal{M},\sigma)$ are (strongly) Hausdorff.*

EXAMPLE 3.6.5. An object of $\mathbf{DS}(\mathcal{M})$ may not be (weakly) Hausdorff even if $(M,\tau(\mathrm{Op}_M))$ is Hausdorff: look at the real unit interval $[0,1]$ and consider the connected definable space formed by "doubling" (by a pair of "charts") only a finite number of points of the interval. No pair of points formed by "doubling" can be separated by open sets.

Corollary 3.6.6. *Each weakly regular* (*weakly normal, respectively*) *locally definable or weakly definable space over* $(\mathcal{M},\sigma)$ *is strongly regular* (*strongly normal, respectively*).

That is why we speak about T_3 *objects* and T_4 *objects* of $\mathbf{LDS}(\mathcal{M},\sigma)$ and $\mathbf{WDS}(\mathcal{M},\sigma)$. As Example 3.6.5 shows, a definable space over $\mathcal{M}$ may not inherit regularity or normality from $(M,\tau(\mathrm{Op}_M))$. For a first order topological structure, at least affine definable spaces inherit regularity.

Proposition 3.6.7. *If* $\mathcal{M}$ *is a first order topological structure with its natural topology regular* (*Hausdorff*), *then all objects of* $\mathbf{ADS}(\mathcal{M})$ *are regular.*

Proof. The equality $\sigma=\tau(\mathrm{Op}_M)$ holds for $\mathcal{M}$. If the topological space (M,σ) is regular (Hausdorff), then so are all (M^n,σ^n) and all affine definable spaces. By Proposition 3.3.19, the property (CPG) is satisfied for each definable subset of M^n, and each affine definable space. By Fact 2.1.10, each affine definable space is weakly regular. By Corollary 3.6.6, this space is strongly regular. ■

Question 3.6.8. Assume $(M,\tau(\mathrm{Op}_M))$ is Hausdorff, and X is an object of $\mathbf{WDS}(\mathcal{M},\sigma)$ having an exhaustion composed of affine definable spaces. For which $(\mathcal{M},\sigma)$ must X be Hausdorff?

Question 3.6.9. Assume $\mathcal{M}$ is a first order topological structure with its natural topology regular (Hausdorff), and X is an object of $\mathbf{WDS}(\mathcal{M})$ having an exhaustion composed of affine definable subspaces of M^n. For which $\mathcal{M}$ must X be regular?

Example 3.6.10. Take the full structure on $M=\mathbb{R}$ (i.e. all n-ary relations on $\mathbb{R}$ are in the language of $\mathcal{M}$) with the lower limit topology. Then $\mathcal{M}$ is a normal (Hausdorff) first order topological structure, but the Sorgenfrey plane $\mathbb{R}^2$ is not normal as a topological space or a definable set.

4. Completeness

4.1. Locally small and weakly small cases

Definition 4.1.1 (cf. [DK, I.5, Definition 2] and [K, IV.5, Definition 2]). Let $\mathcal{C}$ be one of the (considered earlier) categories of spaces having binary products. We will say that an object Z of $\mathcal{C}$ is $\mathcal{C}$-*complete* if the mapping $Z\to\{*\}$ is universally closed in $\mathcal{C}$, which means that for each object Y of $\mathcal{C}$ the canonical projection $Z\times Y\to Y$ (which is the base extension of $Z\to\{*\}$) is a closed mapping.

Remark 4.1.2. The quasi-compact spaces are exactly the **Top**-complete spaces (see [B]). All objects of **SS** are **SS**-complete (by [P2, Fact 2.3.15]), **LSS**-complete (by Proposition 2.1.40), and (if strongly T_1) $\mathbf{WSS}_1$-complete (by Proposition 2.2.39). On the other hand, even the one-dimensional semi-algebraic affine space $\mathbb{R}$ is not $\mathbf{ADS}(\mathbb{R})$-complete.

QUESTION 4.1.3. Which objects are $\mathbf{GTS}$-complete?

QUESTION 4.1.4. Which objects are $\mathbf{GTS}(M)$-complete?

PROPOSITION 4.1.5. *Any closed subspace of a $\mathcal{C}$-complete space is $\mathcal{C}$-complete.*

Proof. Let C be a closed subspace of a $\mathcal{C}$-complete space Z. For any Y in $\mathcal{C}$, the space $C \times Y$ is a closed subset of $Z \times Y$. Thus the image under the projection along C of any closed subset of $C \times Y$ is closed in Y. ∎

PROPOSITION 4.1.6. *The image $g(C)$ of a $\mathcal{C}$-complete subspace C of Y under a morphism $g : Y \to Z$ in $\mathcal{C}$ is a $\mathcal{C}$-complete subspace.*

Proof. Consider an object W of $\mathcal{C}$. For a closed subset A of $g(C) \times W$, we have $\pi_{g(C)}(A) = \pi_C((g|_C \times \mathrm{id}_W)^{-1}(A))$ is a closed set. ∎

PROPOSITION 4.1.7. *If Y is an object of $\mathcal{C}$ having its diagonal $\Delta_Y \subseteq Y \times Y$ closed, then:*

(a) *each of its $\mathcal{C}$-complete subspaces is closed;*
(b) *the graph of each morphism $f : X \to Y$ in $\mathcal{C}$ is closed.*

Proof. (a) If C is a $\mathcal{C}$-complete subspace of Y, then $\Delta_Y \cap (C \times Y)$ is relatively closed in $C \times Y$, and its projection on Y, equal to C, is closed.

(b) The graph of f is a subspace of $X \times Y$, being the preimage of Δ_Y under a morphism $f \times \mathrm{id}_Y$. ∎

DEFINITION 4.1.8. A category $\mathcal{C}$ will be called *nice* if the following property holds: if $\mathcal{C}$ has an infinite topological discrete object, then $\mathcal{C}$ has a strongly T_1 object with a countable non-closed set.

LEMMA 4.1.9. *If $\mathcal{C}$ is nice, and has a topological discrete infinite object, then this object is not $\mathcal{C}$-complete.*

Proof. Let Z be an infinite topological discrete space. Take an injective sequence $\{z_n\}_{n\in\mathbb{N}}$ of elements of Z. Let W be a strongly T_1 object with a non-closed countable set $C = \{c_n\}_{n\in\mathbb{N}}$ (determined by an injective sequence). Then $\{(z_n, c_n) : n \in \mathbb{N}\}$ is closed in $Z \times W$, $C = \pi_Z(\{(z_n, c_n) : n \in \mathbb{N}\})$, so Z is not complete by Definition 4.1.1. ∎

EXAMPLE 4.1.10. Consider the discrete topology σ, and the full structure on a non-empty set M. Then all objects of $\mathbf{LDS}(\mathcal{M})$ and $\mathbf{WDS}(\mathcal{M})$ are (not necessarily topological) discrete. Neither $\mathbf{LDS}(\mathcal{M})$ nor $\mathbf{WDS}_1(\mathcal{M})$ is nice. All objects of $\mathbf{LDS}(\mathcal{M})$ and of $\mathbf{WDS}_1(\mathcal{M})$ are complete in their respective categories, while they may not be small.

Theorem 4.1.11. *Assume $\mathcal{C}$ is nice and C is a strongly T_1, $\mathcal{C}$-complete space. If C is a generalized topological direct sum, then the number of summands is finite.*

Proof. Take one point from each summand. The resulting subspace S is topological discrete and closed, so $\mathcal{C}$-complete by Proposition 4.1.5. By Lemma 4.1.9, S is finite. ∎

Corollary 4.1.12. *If $(\mathcal{M},\sigma)$ satisfies* (DCD)*, then strongly T_1 objects complete in $\mathbf{LDS}(\mathcal{M},\sigma)$ or in $\mathbf{WDS}_1(\mathcal{M},\sigma)$ have, if this category is nice, only finitely many connected components.*

Theorem 4.1.13 (cf. [DK, I.5.10]). *Assume $\mathcal{C}$ is nice, and is one of: $\mathbf{LSS}$, $\mathbf{LDS}(\mathcal{M},\sigma)$, $\mathbf{WSS}_1$, or $\mathbf{WDS}_1(\mathcal{M},\sigma)$. Let C be a strongly T_1 and Lindelöf (if $\mathcal{C}=\mathbf{LDS}(\mathcal{M},\sigma)$ or $\mathbf{LSS}$) $\mathcal{C}$-complete space. Then C is small.*

Proof. In the case of a locally definable (or locally small) space: if $C = \overset{a}{\bigcup}_{n\in\mathbb{N}} C_n$ is an admissible covering by affine definable (small) spaces and C is not small, then we may assume that for each $n\in\mathbb{N}$ we can choose $x_n \in C_n \setminus (C_0\cup\dots\cup C_{n-1}) \neq \emptyset$. The set $B=\{x_n \mid n\in\mathbb{N}\}$ is an infinite, not small, locally finite subspace, so, as a T_1 space, closed and topological discrete (by Proposition 2.1.34). Thus B is $\mathcal{C}$-complete (by Proposition 4.1.5).

In the case of a weakly definable (or weakly small) space: if C is not small, then the index function η of an exhaustion $(C_\alpha)_{\alpha\in A}$ has infinite image, and we can choose an element $x_\alpha \in C^0_\alpha$ for each $\alpha\in\eta(C)$. The set $B=\{x_\alpha \mid \alpha\in\eta(C)\}$ is an infinite (but piecewise finite) closed subspace all of whose subsets are also closed. Thus B is a $\mathcal{C}$-complete (by Proposition 4.1.5) and a topological discrete space (by Proposition 2.2.38).

In both cases we get a contradiction with Lemma 4.1.9. ∎

Definition 4.1.14. A *relatively complete* set in an object of $\mathcal{C}$ is a subset of a complete subspace. The collection of all relatively complete sets of X will be denoted by Rc_X.

Fact 4.1.15. *For any object of $\mathcal{C}$, the collection Rc_X is a bornology.*

Remark 4.1.16. We get further faithful functors $F_{\mathcal{C}}:\mathcal{C}\to\mathbf{UBor}$, with $F_{\mathcal{C}}(X)=(X,\tau(\mathrm{Op}_X),\mathrm{Rc}_X)$ and $F_{\mathcal{C}}(f)=f$.

The bornology Rc_X is in general different from the bornology Sm_X.

Example 4.1.17. Consider the situation from Example 4.1.10. Then Rc_X is a trivial bornology on each object of $\mathbf{LDS}(\mathcal{M})$ or $\mathbf{WDS}_1(\mathcal{M})$ (all subsets are bounded), while Sm_X is usually non-trivial.

Example 4.1.18. In the semialgebraic space $\mathbb{R}$ (with the underlying small space denoted by $\mathbb{R}_{\mathrm{salg}}$) we have a trivial bornology $\mathrm{Sm}_{\mathbb{R}}$ and a non-trivial bornology $\mathrm{Rc}_{\mathbb{R}}$ of bounded sets (in the traditional meaning). By the process of localization (see Example 2.1.15 or [P1]), we pass to the space $\mathbb{R}_{\mathrm{loc}}$, where the two bornologies coincide (a small set is exactly a relatively complete set).

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