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On the stability of compressible Navier–Stokes–Korteweg equations

by Tong Tang and Hongjun Gao (Nanjing)

Abstract. We consider the compressible Navier–Stokes–Korteweg (N-S-K) equations. Through a remarkable identity, we reveal a relationship between the quantum hydrodynamic system and capillary fluids. Using some interesting inequalities from quantum fluids theory, we prove the stability of weak solutions for the N-S-K equations in the periodic domain $\Omega = \mathbb{T}^N$, when $N = 2, 3$.

1. Introduction. Due to its importance in science, the Navier–Stokes–Korteweg (N-S-K) system modeling viscous compressible fluids has been studied recently both from the theoretical and numerical point of view. In fluid mechanics, the existence, uniqueness and stability of weak solutions have been the object of active research. The present paper addresses the problem of stability of weak solutions for the following N-S-K equations in the periodic domain $\Omega = \mathbb{T}^N$ $(N = 2, 3)$:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} T + \operatorname{div} K, \end{cases}$$

where the unknown functions $\rho = \rho(x,t)$, $u = (u_1, \ldots, u_N)$ and P are the density, velocity and pressure respectively. The function ρ and u are periodic in Ω. Moreover, the viscous stress tensor T and the Korteweg stress tensor K are defined by

$$\begin{aligned} T &= 2\mu D(u) + (\lambda \operatorname{div} u)I, \\ K &= \big(\rho\kappa\Delta\rho + \tfrac{1}{2}(\kappa + \rho\kappa_\rho)|\nabla\rho|^2\big)I - \kappa\nabla\rho \otimes \nabla\rho, \end{aligned}$$

where $D(u) = \frac{1}{2}(\nabla u + \nabla^\top u)$ is the strain tensor, I is the identity matrix and $\mu = \mu(\rho)$, $\lambda = \lambda(\rho)$, $\kappa = \kappa(\rho)$ denote the shear coefficient viscosity of the fluid, the second viscosity coefficient and the capillary coefficient. As the

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fluid is assumed to be Newtonian, the two Lamé viscosity coefficients satisfy

$$\mu > 0, \qquad 2\mu + N\lambda \geq 0.$$

Generally speaking, the pressure P depends on the density and temperature. In that case, the system is not closed and should be complemented by the energy equation. However, there are physically relevant situations in which we assume that the fluid flow is barotropic, i.e., the pressure depends only on the density. This is the case when either the temperature or the entropy is supposed to be constant. The typical expression is $P(\rho) = \rho^\gamma$ ($\gamma > 1$ denotes the adiabatic exponent).

It was Van der Waals and Korteweg who first considered the compressible fluids model endowed with internal capillarity. Later, the model was developed by Dunn and Serrin [DS], who could describe the variation of density at the interfaces between the two phases, generally a liquid-vapor mixture. The N-S-K model can be reduced to many classical models by specifying various coefficients, such as the compressible Euler equations if we take $\mu = \lambda = \kappa = 0$, and the compressible Navier–Stokes equations if we take $\mu > 0$, $2\mu + N\lambda \geq 0$ and $\kappa = 0$.

As to the compressible Navier–Stokes equations, there is much literature and results depending on whether the viscosity coefficients are constant or not. If μ and λ are both constant, Kazhikhov [KS], Serre [S], Hoff [H], Matsumura and Nishida [MN], and Valli and Zajączkowski [VZ] had done much pioneering work. The first general result on the weak solutions was obtained by Lions [L]. Later, Feireisl et al. [FNP] and Jiang and Zhang [JZ] extended Lions' result. The case where the viscosity coefficients depend on the density has received much attention lately. Mellet and Vasseur [MV] proved the L^1 stability of weak solutions to the compressible Navier–Stokes equations. Li et al. [LLX] and Guo et al. [GJX] proved the existence of weak solutions respectively in the one-dimensional case and the three-dimensional spherical symmetric case.

The compressible N-S-K system has recently been extensively studied in fluid mechanics and applied mathematics due to its physical importance, complexity, rich phenomena and mathematical challenges. When the viscosity coefficients μ, λ and the capillarity coefficient κ are constant, a lot of mathematical results for that system have been obtained. Hattori and Li [HL1, HL2] proved the local and global existence of smooth solutions in Sobolev spaces. Danchin and Desjardins [DD] studied the existence of smooth solutions in Besov spaces.

In contrast to a system with constant viscosity coefficients, it is inevitable that the N-S-K system becomes complicated when the coefficients depend on the density. On the one hand, the viscosity coefficient is density-dependent, and degenerates at vacuum. Moreover, the strongly nonlinear third-order differential operator and dispersive structure of the momentum equation in the N-S-K system need hard analysis. Luckily, Bresch et al. [BD1]–[BD3], [BDL] introduced a new entropy estimate, which provided more information about the density. More precisely in [BDL], they dealt with a special Korteweg model: $\mu = \rho$, $\lambda = 0$, $\kappa = 1$ and obtained the stability of weak solutions in $\mathbb{T}^N$, $N = 2, 3$. Recently, Haspot [H2] obtained a similar stability result, assuming $\mu = \rho$, $\lambda = 0$ and the Korteweg stress tensor with $\kappa = 1/\rho$.

In this paper, we will extend the previous results with the same Korteweg stress tensor, but for more general viscosity coefficients. More precisely, we take $\kappa = 1/\rho$, so the system under study becomes

$$
(1.1)\qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = 2\operatorname{div}(\mu(\rho)D(u)) + \nabla(\lambda(\rho)\operatorname{div} u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \operatorname{div}(\rho \nabla^2 \ln \rho), \end{cases}
$$

with the initial conditions

$$
(1.2)\qquad \rho|_{t=0} = \rho_0 \geq 0, \qquad \rho u|_{t=0} = m_0.
$$

By a direct computation, we find the following remarkable identity:

$$
(1.3)\qquad \operatorname{div}(\rho \nabla^2 \ln \rho) = 2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right),
$$

which reveals a relationship between the quantum hydrodynamic system and capillary fluids. The term $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ is the Bohm potential which can be interpreted as a quantum potential in a quantum fluid. Jüngel [J1] proved the global existence of weak solutions to the compressible quantum Navier–Stokes equations with $\mu = \rho$, $\lambda = 0$.

Our aim is to show the stability of weak solutions to the compressible N-S-K equations with viscosity coefficients vanishing at the vacuum in both two-dimensional and three-dimensional periodic domains. Although the periodic case does not correspond to a physical configuration, it retains the main mathematical difficulties of the problem. Comparing with works on the compressible Navier–Stokes equations, we will encounter extra difficulties in studying the compressible N-S-K equations. More precisely, the appearance of the nonlinear third-order derivatives of the density makes it difficult to establish entropy estimates and pass to the limit. However, utilizing some interesting inequalities, we overcome the above difficulties and obtain stability. The stability of weak solutions, in other words compactness, is a remarkable property which allows us to develop a rigorous existence theory, without any restriction on the size of the initial data and the length of the time interval.

The strategy of the proof is as follows. In Section 2, we define precisely the notion of weak solutions and state our main result. In order to deal with the convection term, we have chosen test functions of the form $\rho\varphi$ in the weak formulation, where φ is some smooth function. In Section 3, we give the energy equality and an entropy estimate, and use some interesting inequalities to obtain an H^2 estimate for $\sqrt{\rho}$, which is the key to stability analysis. Finally, in Section 4, we give the proof of Theorem 2.2. The most delicate task here is to control possible concentrations rather than oscillations of weakly converging fields.

2. Notation and main result

2.1. Conditions on $\mu(\rho), \lambda(\rho)$. To the best of our knowledge, a rigorous mathematical analysis for compressible flows is beyond the available mathematical framework. Hence, we need to add some additional hypotheses on the viscosity coefficients μ and λ.

First we assume that $\mu(\rho), \lambda(\rho)$ are two $C^2(0,\infty) \cap C[0,\infty)$ functions satisfying

$$\lambda(\rho) = 2\rho\mu'(\rho) - 2\mu(\rho). \tag{2.1}$$

This relation is fundamental to get more regularity of the density. Moreover, we assume that there exists a positive constant $\nu > 0$ such that

$$\mu'(\rho) \geq \nu, \quad \mu(0) \geq 0, \qquad \forall \rho > 0, \tag{2.2}$$

$$|\lambda'(\rho)| \leq C\mu'(\rho), \qquad \forall \rho > 0, \tag{2.3}$$

$$M_1\mu(\rho) \leq 2\mu(\rho) + N\lambda(\rho) \leq M_2\mu(\rho), \quad \forall \rho \geq 0, \tag{2.4}$$

where M_1, M_2 are two positive constants.

In addition, for some small $\varepsilon > 0$, we assume that

$$|\mu''(\rho)| \leq \varepsilon/\rho, \quad \forall \rho > 0. \tag{2.5}$$

Remark. The functions $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ satisfy assumptions (2.1)–(2.5). So do the functions: $\mu(\rho) = \rho + \varepsilon(\rho+1)\ln(\rho+1)$, $\lambda(\rho) = 2\varepsilon[\rho - \ln(\rho+1)]$; $\mu(\rho) = \rho + \varepsilon\rho\ln(\rho+1)$, $\lambda(\rho) = 2\varepsilon\frac{\rho^2}{\rho+1}$; $\mu(\rho) = \rho + \frac{\varepsilon}{2}\rho\ln(\rho^2+1)$, $\lambda(\rho) = \varepsilon\frac{\rho^2}{\rho^2+1}$. We will verify this in the appendix.

Remark. Assumptions (2.3) and (2.4) will be used to pass to the limit in the term $\nabla(\lambda(\rho_n)\operatorname{div} u_n)$. For more details refer to [MV].

2.2. Weak solutions. Before stating the compactness result, we need to specify the definition of weak solutions which we will apply. It is necessary to require that the weak solutions should satisfy the natural energy estimates and from the viewpoint of physics, the mass and momentum conservation laws should also be satisfied, at least in the sense of distributions. Based on those considerations, the definition of reasonable global-in-time weak solutions goes as follows.

Definition 2.1. We say that (ρ, u) is a *weak solution* of (1.1)–(1.2) on $\Omega \times [0,T]$, with initial conditions, if

$$\begin{aligned}
&\sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\
&\rho \in L^\infty(0,T;L^\gamma(\Omega)) \cap L^2(0,T;W^{1,3}(\Omega)),\\
&\sqrt{\rho}\, u \in L^\infty(0,T;L^2(\Omega)), \rho u \in L^2(0,T;W^{1,3/2}(\Omega)),
\end{aligned}$$

with $\rho \geq 0$, and $(\rho, \rho u)$ satisfying

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \mathcal{D}'((0,T)\times\Omega),$$

and moreover the following equality holds for all smooth test functions $\varphi(t,x)$ with compact support such that $\varphi(T,\cdot) = 0$:

$$\begin{aligned}
&-\int_\Omega \rho_0^2 u_0 \cdot \varphi(\cdot,0)\,dx\\
&\quad = \int_0^T\int_\Omega \Big(\rho^2 u\cdot\varphi_t - \rho^2 u \operatorname{div} u\cdot\varphi + \rho u\otimes\rho u : \nabla\varphi + \frac{\gamma}{\gamma+1}\rho^\gamma \operatorname{div}\varphi\\
&\qquad - 2\Delta\sqrt{\rho}(2\sqrt{\rho}\nabla\rho\cdot\varphi + \rho^{3/2}\operatorname{div}\varphi) - \mu(\rho)D(u) : (\nabla\rho\otimes\varphi + \rho\nabla\varphi)\\
&\qquad - \lambda(\rho)\operatorname{div} u(\nabla\rho\cdot\varphi + \rho\operatorname{div}\varphi)\Big)\,dx\,dt,
\end{aligned}$$

where the product "$A : B$" means summation over both indices of the matrices A and B.

We now give the main result of this paper.

Theorem 2.2. *Assume that $\gamma > 1$ if $N = 2$ and $\gamma > 3$ if $N = 3$. Assume further $\mu(\rho), \lambda(\rho)$ are two $C^2(0,\infty)\cap C[0,\infty)$ functions of ρ satisfying conditions* (2.1)–(2.5). *Let (ρ_n, u_n) be a sequence of weak solutions of* (1.1) *which satisfy the entropy equalities* (3.1) *and* (3.2), *with initial data*

$$\rho_n|_{t=0} = \rho_0^n(x) \geq 0, \qquad \rho_n u_n|_{t=0} = m_0^n(x) = \rho_0^n(x)u_0^n(x),$$

where ρ_0^n, u_0^n are such that

$$\rho_0^n \to \rho_0 \quad \text{in } L^1(\Omega), \qquad \rho_0^n u_0^n \to \rho_0 u_0 \quad \text{in } L^1(\Omega),$$

and satisfy (*with C being a constant independent of n*)

$$\int_\Omega \bigg[\rho_0^n\frac{|u_0^n|^2}{2} + \frac{1}{\gamma-1}(\rho_0^n)^\gamma + |\nabla\sqrt{\rho_0^n}|^2\bigg]\,dx \leq C, \qquad \int_\Omega \frac{1}{\rho_0^n}|\nabla\mu(\rho_0^n)|^2\,dx \leq C.$$

Then, up to a subsequence, $(\rho_n, \rho_n u_n)$ converges strongly to a weak solution of (1.1)–(1.2) *satisfying the entropy equalities* (3.1) *and* (3.2).

3. The energy equality and entropy estimate. In this section, we will give an energy equality and state an entropy estimate, both important in the proof of Theorem 2.2. When deriving a priori estimates it is customary to assume that all quantities appearing in the equations are as smooth as is necessary.

By multiplying the momentum equation by u, using the mass equation and integrating by parts, we get the following energy equality:

$$\frac{d}{dt}\int_{\Omega}\left[\rho\frac{u^2}{2}+\frac{1}{\gamma-1}\rho^{\gamma}+2|\nabla\sqrt{\rho}|^2\right]dx+\int_{\Omega}2\mu(\rho)|D(u)|^2dx+\int_{\Omega}\lambda(\rho)(\operatorname{div}u)^2\,dx=0, \tag{3.1}$$

where we use the identity (1.3), since

$$\begin{aligned}\int_{\Omega}\operatorname{div}(\rho\nabla^2\ln\rho)u\,dx&=\int_{\Omega}2\rho\nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big)u\,dx=-2\int_{\Omega}\operatorname{div}(\rho u)\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big)\,dx\\&=2\int_{\Omega}\partial_t\rho\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big)\,dx=-2\int_{\Omega}\partial_t|\nabla\sqrt{\rho}|^2\,dx.\end{aligned}$$

It is well-known that equality (3.1) alone is not sufficient to build up a reasonable stability theory for weak solutions to the compressible N-S-K equations in the sense of distributions, since we cannot obtain any estimates on the dissipations. Therefore, we need to investigate further estimates. Inspired by [BDL] and [MV], we get the following lemma which offers the crucial estimate.

LEMMA 3.1. *Assume that* $\mu(\rho),\lambda(\rho)$ *are two* $C^2(0,\infty)\cap C[0,\infty)$ *functions satisfying* (2.1)–(2.5). *Then the following equality holds for smooth solutions of* (1.1)*:*

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\left[\frac{1}{2}\rho|u+\nabla\psi(\rho)|^2+\frac{1}{\gamma-1}\rho^{\gamma}+2|\nabla\sqrt{\rho}|^2\right]dx+\int_{\Omega}\nabla\psi(\rho)\cdot\nabla\rho^{\gamma}\,dx\\+2\int_{\Omega}\mu(\rho)|A(u)|^2\,dx-\int_{\Omega}\operatorname{div}(\rho\nabla^2\ln\rho)\cdot\nabla\psi(\rho)\,dx=0,\end{aligned} \tag{3.2}$$

with $\psi'=2\mu'/\rho$ *and* $A(u)=(\nabla u-\nabla^t u)/2$.

Proof. The proof is similar to one in [MV], with minor changes. ∎

Next, we need to control all terms in (3.2). By a straightforward calculation, we have

$$\begin{aligned}\nabla^2\ln\rho&=-\frac{1}{\rho^2}\nabla\rho\otimes\nabla\rho+\frac{1}{\rho}\nabla^2\rho,\\ \nabla\mu'\otimes\nabla\ln\rho&=\frac{\mu''}{\rho}\nabla\rho\otimes\nabla\rho.\end{aligned} \tag{3.3}$$

Combining the above identity (3.3) and condition (2.5), we deduce

$$|\nabla\mu'\otimes\nabla\ln\rho| \le \varepsilon\Big|\nabla^2\ln\rho - \frac{1}{\rho}\nabla^2\rho\Big|. \tag{3.4}$$

Thus

$$\begin{aligned}
-\int_\Omega \operatorname{div}(\rho\nabla^2\ln\rho)\cdot\nabla\psi(\rho)\,dx &= -\int_\Omega \operatorname{div}(\rho\nabla^2\ln\rho)\cdot\frac{2\mu'}{\rho}\nabla\rho\,dx \\
&= -2\int_\Omega \operatorname{div}(\rho\nabla^2\ln\rho)\cdot\mu'\nabla\ln\rho\,dx \\
&= 2\int_\Omega \rho\nabla^2\ln\rho : (\nabla\mu'\otimes\nabla\ln\rho + \mu'\nabla^2\ln\rho)\,dx.
\end{aligned} \tag{3.5}$$

In the following, we utilize (2.5) and the Cauchy inequality to handle the term $\int_\Omega \nabla^2\ln\rho : \nabla\mu'\otimes\nabla\ln\rho$:

$$\begin{aligned}
\Big|\int_\Omega \rho\nabla^2\ln\rho : \nabla\mu'\otimes\nabla\ln\rho\,dx\Big| &\le \int_\Omega |\rho\nabla^2\ln\rho|\varepsilon\Big|\nabla^2\ln\rho - \frac{1}{\rho}\nabla^2\rho\Big|\,dx \\
&\le \varepsilon\int_\Omega \rho|\nabla^2\ln\rho|^2 dx + \varepsilon\int_\Omega \rho|\nabla^2\ln\rho|\,\Big|\frac{1}{\rho}\nabla^2\rho\Big|\,dx.
\end{aligned}$$

Using the Cauchy inequality, we deduce

$$\begin{aligned}
\int_\Omega \rho|\nabla^2\ln\rho|\Big|\frac{1}{\rho}\nabla^2\rho\Big|\,dx &= \int_\Omega |\sqrt{\rho}\nabla^2\ln\rho|\Big|\frac{\nabla^2\rho}{\sqrt{\rho}}\Big|\,dx \\
&\le \Big(\int_\Omega \rho|\nabla^2\ln\rho|^2\,dx\Big)^{1/2}\Big(\int_\Omega \Big|\frac{\nabla^2\rho}{\sqrt{\rho}}\Big|^2\,dx\Big)^{1/2}.
\end{aligned}$$

By direct calculation, we have

$$\begin{aligned}
\nabla^2\sqrt{\rho} &= -\frac{1}{4}\,\frac{1}{\rho^{3/2}}\nabla\rho\otimes\nabla\rho + \frac{1}{2\sqrt{\rho}}\nabla^2\rho, \\
\frac{\nabla^2\rho}{\sqrt{\rho}} &= 2\nabla^2\sqrt{\rho} + \frac{1}{2}\,\frac{1}{\rho^{3/2}}\nabla\rho\otimes\nabla\rho, \\
|\nabla\sqrt[4]{\rho}|^4 &= \Big|\frac{1}{4}\,\frac{\nabla\rho}{\rho^{3/4}}\Big|^4 = \frac{1}{4^4}\,\frac{|\nabla\rho|^4}{\rho^3}.
\end{aligned}$$

Hence, we deduce that

$$\Big|\frac{\nabla^2\rho}{\sqrt{\rho}}\Big|^2 \le C_1|\nabla^2\sqrt{\rho}|^2 + C_2\frac{|\nabla\rho|^4}{\rho^3} \le C_1|\nabla^2\sqrt{\rho}|^2 + C_2|\nabla\sqrt[4]{\rho}|^4.$$

Now, we use some interesting inequalities to obtain the H^2 estimate for $\sqrt{\rho}$

(which is derived from [J1]):

$$\int_\Omega \rho|\nabla^2 \ln\rho|^2\,dx \ge \kappa_N \int_\Omega |\nabla^2\sqrt{\rho}|^2\,dx,$$

with $\kappa_2 = 7/8$, $\kappa_3 = 11/15$, and the inequality

$$\int_\Omega \rho|\nabla^2 \ln\rho|^2\,dx \ge \kappa \int_\Omega |\nabla \sqrt[4]{\rho}|^4\,dx, \qquad \kappa > 0.$$

Recalling condition (2.2), we have

$$\int_\Omega \mu'\rho|\nabla^2 \ln\rho|^2\,dx \ge \nu \int_\Omega \rho|\nabla^2 \ln\rho|^2\,dx.$$

Choosing ε small enough (so that it can be controlled by ν), we get

$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\rho|u+\nabla\psi(\rho)|^2 + \frac{1}{\gamma-1}\rho^\gamma + 2|\nabla\sqrt{\rho}|^2\right]dx + \int_\Omega \nabla\psi(\rho)\cdot\nabla\rho^\gamma\,dx$$
$$+2\int_\Omega \mu(\rho)|A(u)|^2\,dx + C\int_\Omega |\nabla^2\sqrt{\rho}|^2\,dx \le 0.$$

Combining (3.1)–(3.2) and the finite initial energy, we get the following estimates:

$$\begin{aligned}
&\|\sqrt{\rho}\,u\|_{L^\infty(0,T;L^2(\Omega))} \le C,\\
&\|\sqrt{\rho}\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} \le C,\\
&\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))} \le C,\\
&\|\sqrt{\rho}\,\nabla\psi\|_{L^\infty(0,T;L^2(\Omega))} = \|\mu'(\rho)\,\nabla\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))} \le C,\\
&\|\sqrt{\mu(\rho)}\,\nabla u\|_{L^2(0,T;L^2(\Omega))} \le C,\\
&\|\sqrt{\mu'(\rho)\rho^{\gamma-2}}\,\nabla\rho\|_{L^2(0,T;L^2(\Omega))} \le C,
\end{aligned}$$

where $C > 0$ is a constant.

4. The proof of Theorem 2.2. With the a priori estimates obtained in the previous sections, we now study the stability of sequences of weak solutions (ρ_n, u_n) and pass to the limit in the nonlinear terms. To begin with, we recall the following facts:

$$\|\sqrt{\rho_n}\,u_n\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{4.1}$$

$$\|\sqrt{\rho_n}\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} \le C, \tag{4.2}$$

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} \le C, \tag{4.3}$$

$$\|\mu'(\rho_n)\nabla\sqrt{\rho_n}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{4.4}$$

$$\|\sqrt{\mu(\rho_n)}\,\nabla u_n\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{4.5}$$

$$\Big\|\sqrt{\mu'(\rho_n)\rho_n^{\gamma-2}}\,\nabla\rho_n\Big\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{4.6}$$

The hypothesis on the viscosity coefficient (2.2) yields

$$\|\sqrt{\rho_n}\,\nabla u_n\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{4.7}$$

$$\|\nabla\rho_n^{\gamma/2}\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{4.8}$$

where $C > 0$ is (here and in the following) a generic constant independent of n.

The proof of Theorem 2.2 will be given in a sequence of seven lemmas. In the first two steps, we show the convergence of the density and pressure (note that the convergence of the pressure is straightforward). The key fact is the strong convergence of $\rho_n u_n$, and of the diffusion terms.

Lemma 4.1. *For every ρ_n satisfying the mass equation of system* (1.1), *we have*

$$\begin{aligned}&\sqrt{\rho_n} \textit{ is bounded in } L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ &\partial_t\sqrt{\rho_n} \textit{ is bounded in } L^2(0,T;H^{-1}(\Omega)).\end{aligned}$$

Then, up to a subsequence, $\sqrt{\rho_n}$ converges a.e. and

$$\sqrt{\rho_n}\to\sqrt{\rho} \quad \textit{in } L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^r(\Omega))\ (1\le r<6).$$

Moreover,

$$\rho_n\to\rho \quad \textit{in } L^2(0,T;W^{1,p}(\Omega))\ \big(3<p<\tfrac{6\gamma}{\gamma+3}\big).$$

Proof. From the above estimate (4.2), we easily get the bound of $\sqrt{\rho_n}$.

Using the continuity equation of $(1.1)_1$, we write

$$\partial_t\sqrt{\rho_n} = -\tfrac12\sqrt{\rho_n}\operatorname{div} u_n - u_n\cdot\nabla\rho_n = \tfrac12\sqrt{\rho_n}\operatorname{div} u_n - \operatorname{div}(u_n\sqrt{\rho_n}).$$

Then we deduce from (4.1) and (4.5) that

$$\partial_t\sqrt{\rho_n} \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)).$$

We have the compactness of $\sqrt{\rho_n}$ by Aubin's lemma, i.e.,

$$\sqrt{\rho_n}\to\sqrt{\rho} \quad \text{in } L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^r(\Omega))\ (1\le r<6).$$

Next we will discuss the compactness of ρ_n.

Combining the estimates (4.2)–(4.3) with the Gagliardo–Nirenberg inequality and Sobolev embeddings, we easily get the following facts (for more details see [J1, Lemma 4.3]):

$$\begin{aligned}&\rho_n \text{ is bounded in } L^2(0,T;W^{2,p}(\Omega)),\\ &\rho_n u_n \text{ is bounded in } L^2(0,T;W^{1,3/2}(\Omega)),\end{aligned}$$

where $p = 2\gamma/(\gamma+1)$ if $N = 3$ and $p < 2$ if $N = 2$. It follows that $\partial_t\rho_n$ is bounded in $L^2(0,T;L^{3/2}(\Omega))$, since $\partial_t\rho_n = -\operatorname{div}(\rho_n u_n)$. Thus, again using Aubin's lemma, we obtain the compactness of ρ_n,

$$\rho_n \to \rho \quad \text{in } L^2(0,T;W^{1,p}(\Omega)) \ \big(3 < p < \tfrac{6\gamma}{\gamma+3}\big). \ \blacksquare$$

The next lemma concerns the convergence of the pressure.

Lemma 4.2. *The pressure ρ_n^γ is bounded in $L^{5/3}((0,T)\times\Omega)$ when $N = 3$, and $L^r((0,T)\times\Omega)$ for all $r \in [1,2)$ when $N = 2$. In particular, ρ_n^γ converges to ρ^γ strongly in $L^1((0,T)\times\Omega)$.*

Proof. See [MV, Lemma 4.2] for more details. ■

Lemma 4.3. *$\mu(\rho_n)/\sqrt{\rho_n}$ and $\lambda(\rho_n)/\sqrt{\rho_n}$ are bounded in $L^\infty(0,T,L^6(\Omega))$.*

Proof. See [MV, Lemma 4.5], with a little modification. ■

Lemma 4.4. *Let $m_n = \rho_n u_n$ be a sequence satisfying the momentum equation* $(1.1)_2$. *Then*

$$\begin{aligned} \rho_n u_n &\to m \quad \text{in } L^2(0,T;L^q(\Omega)) \ (1 \le q < 3),\\ \rho_n u_n &\to m \quad \text{for a.e . } (x,t) \text{ in } \Omega\times(0,T). \end{aligned}$$

Proof. Firstly, we can prove directly from Lemma 4.1 that $\rho_n u_n$ is bounded in $L^2(0,T;W^{1,3/2}(\Omega))$.

Next, we rewrite the momentum equation and deal with $\partial_t(\rho_n u_n)$:

$$\begin{aligned} \partial_t(\rho_n u_n) = &-\operatorname{div}(\rho_n u_n\otimes u_n) - \nabla\rho_n^\gamma + 2\operatorname{div}(\mu(\rho_n)D(u_n))\\ &+\nabla(\lambda(\rho_n)\operatorname{div} u_n) + \operatorname{div}(\rho_n\nabla^2\ln\rho). \end{aligned}$$

Since the sequence $(\rho_n u_n\otimes u_n)$ is bounded in $L^\infty(0,T;L^1(\Omega))$, it follows that $\operatorname{div}(\rho_n u_n\otimes u_n)$ is bounded in $L^\infty(0,T;(H^s(\Omega))^*)$ for $s > N/2+1$. It is easy to check that $\nabla\rho_n^\gamma$ is also bounded in $L^\infty(0,T;(H^s(\Omega))^*)$.

From [J1, Lemma 4.4] and [MV, Lemma 4.4], we have:

$$\begin{aligned} &\operatorname{div}(\rho_n\nabla^2\ln\rho_n) \text{ is bounded in } L^4(0,T;(W^{1,3}(\Omega))^*)\hookrightarrow L^{4/3}(0,T;(H^s(\Omega))^*),\\ &\operatorname{div}(\mu(\rho_n)\nabla u_n) \text{ is bounded in } L^\infty(0,T;W^{-2,4/3}(\Omega)). \end{aligned}$$

Therefore, $\partial_t(\rho_n u_n)$ is uniformly bounded in $L^{4/3}(0,T;(H^s(\Omega))^*)$. Using Aubin's lemma, we get the compactness of $\rho_n u_n$ in $L^2(0,T;L^q(\Omega))$ $(1 \le q < 3)$. ■

Lemma 4.5. *There exists a function u such that $m = \rho u$ and*

$$\rho_n u_n \to \rho u \quad \text{in } L^2(0,T;L^q(\Omega)) \ (1 \le q < 3).$$

In particular, $m = 0$ a.e. on the set $\{\rho = 0\}$.

Proof. We set $m_n = \rho_n u_n$. Since $m_n/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Fatou's lemma yields

$$\int_\Omega \liminf \frac{m_n^2}{\rho_n}\,dx < \infty.$$

In particular, $m = 0$ a.e. on $\{\rho = 0\}$. So if we define the limit velocity $u := m/\rho$ in $\{\rho \neq 0\}$ and $u := 0$ in $\{\rho = 0\}$, we obtain $m = \rho u$. ■

Considering the previous results, we show that we can pass to the limit in all terms of the weak solution of (1.1).

First, using the above lemmas, we can obtain the following convergences (for more details see [J1]):

$$\begin{aligned}
&\rho_n^2 u_n \to \rho^2 u \quad \text{strongly in } L^1(0,T;L^q(\Omega)) \ (q<3),\\
&\rho_n u_n \otimes \rho_n u_n \to \rho u \otimes \rho u \quad \text{strongly in } L^1(0,T;L^{q/2}(\Omega)) \ (q<3),\\
&\Delta\sqrt{\rho_n}\sqrt{\rho_n}\,\nabla\rho_n \rightharpoonup \Delta\sqrt{\rho}\sqrt{\rho}\,\nabla\rho \quad \text{weakly in } L^1(0,T;L^1(\Omega)).
\end{aligned}$$

Here, we need the assumption $\gamma > 3$ if $N = 3$, which allows us to obtain compactness of ρ in $W^{1,p}$ with $p > 3$. Then it remains to pass to the limit in the nonlinear terms $\rho_n^2 \operatorname{div}(u_n)u_n$, $\mu(\rho_n)D(u_n)\rho_n$, $\mu(\rho_n)D(u_n)\nabla\rho_n$, $\lambda(\rho_n)\operatorname{div} u_n\nabla\rho_n$ and $\rho_n\lambda(\rho_n)\operatorname{div} u_n$.

We consider the term $\mu(\rho_n)D(u_n)\rho_n$; others are handled in a similar way. We introduce functions $\beta \in C^\infty(\mathbb{R})$ (similarly to [BDL, J1]) such that $\beta(s) = 1$ for $s \ge 2$, $\beta(s) = 0$ for $s \le 1$, and $0 \le \beta(s) \le 1$. For any $\alpha > 0$, β_α is defined by $\beta_\alpha(\cdot) = \beta(\cdot/\alpha)$. This function allows us to deal with the density close to zero, so we can estimate the low-density part of $\mu(\rho_n)D(u_n)\rho_n$ by

$$\begin{aligned}
&\|(1-\beta_\alpha(\rho_n))\rho_n\mu(\rho_n)D(u_n)\|_{L^1(0,T;L^1)}\\
&\quad = \left\|(1-\beta_\alpha(\rho_n))\rho_n \cdot \frac{\mu(\rho_n)}{\sqrt{\rho_n}} \cdot \sqrt{\rho_n}\,D(u_n)\right\|_{L^1(0,T;L^1)}\\
&\quad \le \|(1-\beta_\alpha(\rho_n))\rho_n\|_{L^2(0,T;L^3)}\|\sqrt{\rho_n}\,D(u_n)\|_{L^2(0,T;L^2)}\left\|\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right\|_{L^\infty(0,T;L^6)}\\
&\quad \le C\alpha,
\end{aligned}$$

$$\begin{aligned}
&\|(1-\beta_\alpha(\rho_n))\nabla\rho_n\mu(\rho_n)D(u_n)\|_{L^1(0,T;L^1)}\\
&\qquad\qquad \le \|(1-\beta_\alpha(\rho_n))\sqrt{\rho_n}\|_{L^\infty(0,T;L^\infty)}\|\nabla\sqrt{\rho_n}\|_{L^2(0,T;L^3)},
\end{aligned}$$

$$\left\|\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right\|_{L^\infty(0,T;L^6)}\|\sqrt{\rho_n}D(u_n)\|_{L^2(0,T;L^2)} \le C\sqrt{\alpha},$$

$$\begin{aligned}
&\|(1-\beta_\alpha(\rho_n))\rho_n^2 u_n \operatorname{div} u_n\|_{L^1(0,T;L^1)} \\
&\quad \le \|(1-\beta_\alpha(\rho_n))\sqrt{\rho_n}\|_{L^\infty(0,T;L^\infty)} \|\sqrt{\rho_n}\operatorname{div} u_n\|_{L^2(0,T;L^2)} \|\rho_n u_n\|_{L^2(0,T;L^2)} \\
&\quad \le C\sqrt{\alpha}.
\end{aligned}$$

Therefore, we are reduced to study $\beta_\alpha(\rho_n)\rho_n\mu(\rho_n)D(u_n)$ for a given positive α. We split it into two terms $\beta_\alpha(\rho_n)\rho_n^{3/2}D(u_n)$ and $\mu(\rho_n)/\sqrt{\rho_n}$.

First, we observe that $\mu(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^6)$. This term converges a.e. to $\mu(\rho)/\sqrt{\rho}$ (defined to be zero on the vacuum set), so converges strongly in $L^2((0,T)\times\Omega)$. We write

$$\begin{aligned}
\beta_\alpha(\rho_n)\rho_n^{3/2}D(u_n) &= D(\beta_\alpha(\rho_n)\rho_n^{3/2}u_n) \\
&\quad - \rho_n u_n \otimes \nabla\rho_n \cdot \sqrt{\rho_n}\left[\beta_\alpha'(\rho_n) + \frac{3}{2}\,\frac{\beta_\alpha(\rho_n)}{\rho_n}\right].
\end{aligned}$$

Using a similar method to [BDL, J1], we have (where $p > 2$)

$$\beta_\alpha(\rho_n)\rho_n^{3/2}D(u_n) \rightharpoonup \beta_\alpha(\rho)\rho^{3/2}D(u) \quad \text{weakly in } L^p(0,T;L^p(\Omega)).$$

Moreover, from strong convergence of $\mu(\rho_n)/\sqrt{\rho_n}$ in $L^2((0,T)\times\Omega)$, we infer that

$$\beta_\alpha(\rho_n)\rho_n^{3/2}D(u_n)\frac{\mu(\rho_n)}{\sqrt{\rho_n}} \rightharpoonup \beta_\alpha(\rho)\rho^{3/2}D(u)\frac{\mu(\rho)}{\sqrt{\rho}} \qquad \text{weakly in } L^{\frac{2p}{p+2}}(0,T;L^{\frac{2p}{p+2}}(\Omega)).$$

We write, for a test function φ,

$$\begin{aligned}
(4.9)\qquad &\int_\Omega \big(\mu(\rho_n)D(u_n)\rho_n - \mu(\rho)D(u)\rho\big)\nabla\varphi\,dx \\
&\quad = \int_\Omega \big(\beta_\alpha(\rho_n)\rho_n\mu(\rho_n)D(u_n) - \beta_\alpha(\rho)\rho\mu(\rho)D(u)\big)\nabla\varphi\,dx \\
&\qquad + \int_\Omega \big((\beta_\alpha(\rho) - \beta_\alpha(\rho_n)\big)\rho\mu(\rho)D(u)\nabla\varphi\,dx \\
&\qquad + \int_\Omega (1-\beta_\alpha(\rho_n))\big(\rho_n\mu(\rho_n)D(u_n) - \rho\mu(\rho)D(u)\big)\nabla\varphi\,dx.
\end{aligned}$$

For fixed $\alpha > 0$, the first term on the right in (4.9) converges to zero as $n \to 0$. Furthermore, the last integral can be estimated by $C\alpha$ uniformly in n. For the second integral, we recall that $\beta_\alpha(\rho_n)$ converges strongly to $\beta_\alpha(\rho)$ in $L^p(0,T;L^p(\Omega))$ for any $p < \infty$. What is more, since $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ and $\mu(\rho)/\sqrt{\rho} \in L^\infty(0,T;L^6(\Omega))$ and $\sqrt{\rho}\,D(u) \in L^2(0,T;L^2(\Omega))$, we have

$$\rho\mu(\rho)D(u) = \rho\frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}\,D(u) \in L^r(0,T;L^r)\ \big(r = \tfrac{3\gamma}{2\gamma+3} > 1\big).$$

Letting $n \to \infty$ in (4.9), one gets

$$\mu(\rho_n)D(u_n)\rho_n \rightharpoonup \mu(\rho)D(u)\rho \quad \text{weakly in } L^1(0,T;L^1(\Omega)).$$

We can easily pass to the limit, so the proof of Theorem 2.2 is complete.

Remark. As shown in [MV], the motivation for considering the stability of weak solutions is to show their existence. The key ingredient is the construction of approximate sequences of solutions (ρ_n, u_n) which preserve physical bounds and the mathematical entropy, uniformly with respect to smoothing parameters. As to the compressible Navier–Stokes equations with viscosity coefficients depending on the density, to the best of our knowledge, there are two ways to built such approximate sequences. Bresch and Desjardins [BD2] constructed approximate solutions by adding a drag term and cold pressure in $\mathbb{T}^2$. Li et al. [LLX] and Guo et al. [GJX] used another way, adding an approximate density term respectively in the one-dimensional case and the three-dimensional spherical symmetric case. However, the above methods cannot be applied to our model directly, since the main problem is that the lower bound of ρ is difficult to obtain. In [J1], Jüngel used a clever effective velocity variable $v = u + \nabla \ln \rho$, changing the quantum Navier–Stokes equations into the viscous quantum Euler model. Then following the standard Faedo–Galerkin method, he constructed an approximation. Though a similar effective velocity can be taken as in [J2], $v = u + \frac{1}{\rho}\nabla\mu(\rho)$, we cannot obtain the crucial H^2 estimate for $\sqrt{\rho}$. Therefore, the construction of approximate solutions in this framework is still elusive, and left for further investigation.

5. Appendix. There are many functions that could satisfy the hypothesis on the viscosity. In the following, we want to verify these conditions. In particular, we take $\mu(\rho) = \rho + \varepsilon(\rho+1)\ln(\rho+1)$, $\lambda(\rho) = 2\varepsilon[\rho - \ln(\rho+1)]$ for example; others are similar.

First, $\mu''(\rho) = \varepsilon/(\rho+1)$ obviously satisfies (2.5).

Then, as $\mu'(\rho) = 1 + \varepsilon[\ln(\rho+1)+1]$, $\lambda'(\rho) = 2\varepsilon\big(1 - \frac{1}{\rho+1}\big)$, we easily get (2.2).

We have $|\lambda'(\rho)| \le 2\mu'(\rho)$; to see that, we just need to prove $\varepsilon\big|1 - \frac{1}{\rho+1}\big| \le \varepsilon\big(1+\frac{1}{\rho+1}\big) \le 1+\varepsilon(1+\ln(\rho+1))$. Let $F(\rho) = \varepsilon[\frac{1}{\rho+1} - \ln(\rho+1)] - 1$; obviously we have $F(\rho) \le F(0) < 0$, which proves (2.3).

Finally, we find $2\mu(\rho) + N\lambda(\rho) = 2[(1+N\varepsilon)\rho + \varepsilon\rho\ln(\rho+1) - (N-1)\varepsilon\ln(\rho+1)]$, which obviously yields $2\mu(\rho) + N\lambda(\rho) \le M_2\mu(\rho)$ if we take $M_2 \ge 2(1+N\varepsilon)$.

Let

$$F(\rho) = [2(1+N\varepsilon) - M_1]\rho + (2-M_1)\varepsilon\rho\ln(\rho+1) - (2N-2+M_1)\varepsilon\ln(\rho+1).$$

By a simple calculation, we have

$$F'(\rho) = 2(1+N\varepsilon) - M_1 + (2-M_1)\varepsilon\ln(\rho+1) + (2-M_1)\varepsilon\frac{\rho}{\rho+1} - (2N-2+M_1)\varepsilon\frac{1}{\rho+1}.$$

Using $F''(\rho) > 0$, we find that $F'(\rho) > F'(0)$. So if we want to have $F'(0) = 2(1+N\varepsilon) - M_1 - (2N-2+M_1)\varepsilon > 0$, we just need $M_1 < 2$, which implies $M_1\mu(\rho) \leq 2\mu(\rho) + N\lambda(\rho)$. This verifies (2.4).

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Tong Tang, Hongjun Gao
Jiangsu Key Laboratory for NSLSCS
and
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023, China
E-mail: tt0507010156@126.com
gaohj@njnu.edu.cn

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