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Uniformity of holomorphic families of non-homeomorphic planar Riemann surfaces

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Abstract. We show the variation formula for the Schiffer span $s(t)$ for moving Riemann surfaces $R(t)$ with $t \in B = \{t \in \mathbb{C} \mid |t| < \rho\}$, and apply it to show the simultaneous uniformization of moving planar Riemann surfaces of class O_{AD}.

1. Introduction and main results. Let B be a disk in $\mathbb{C}_t$. Let $\pi : \mathcal{R} \to B$ be a holomorphic family of Riemann surfaces $R(t) = \pi^{-1}(t)$, $t \in B$. Assume that

(1) $\mathcal{R}$ is a 2-dimensional Stein manifold and $R(t)$ is irreducible and non-singular in $\mathcal{R}$;
(2) each $R(t)$, $t \in B$, is planar.

Nishino [9, Théorème II] showed that if $R(t)$, $t \in B$, is conformally equivalent to $\mathbb{C}$, then $\mathcal{R}$ is biholomorphic to $B \times \mathbb{C}$. Yamaguchi [13, Théorème 2] extended the above result by giving the variation formulas for the Robin constants $\lambda(t)$ for $R(t)$: if $R(t)$, $t \in B$, is of class O_G (i.e., parabolic), then $\mathcal{R}$ is biholomorphic to a univalent domain in $B \times \mathbb{P}$.

The purpose of this paper is to give a new uniformization.

In this paper we identify a holomorphic family $\pi : \mathcal{R} \to B$ with the variation $\mathcal{R}$ of Riemann surfaces $R(t)$, $t \in B$:

$$\mathcal{R} : t \in B \ \to \ R(t),$$

and write $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$. Let $\Gamma(B, \mathcal{R})$ denote the set of all holomorphic sections of $\mathcal{R}$ over B. Assume that $\mathcal{R}$ satisfies (1) and

(3) there exists a section $\mathbf{a} := \{a(t) \in R(t) \mid t \in B\} \in \Gamma(B, \mathcal{R})$.

In general, $R(t)$ might have infinitely many ideal boundary components and $\mathcal{R} : t \in B \to R(t)$ might not be topologically trivial. By Oka–Grauert

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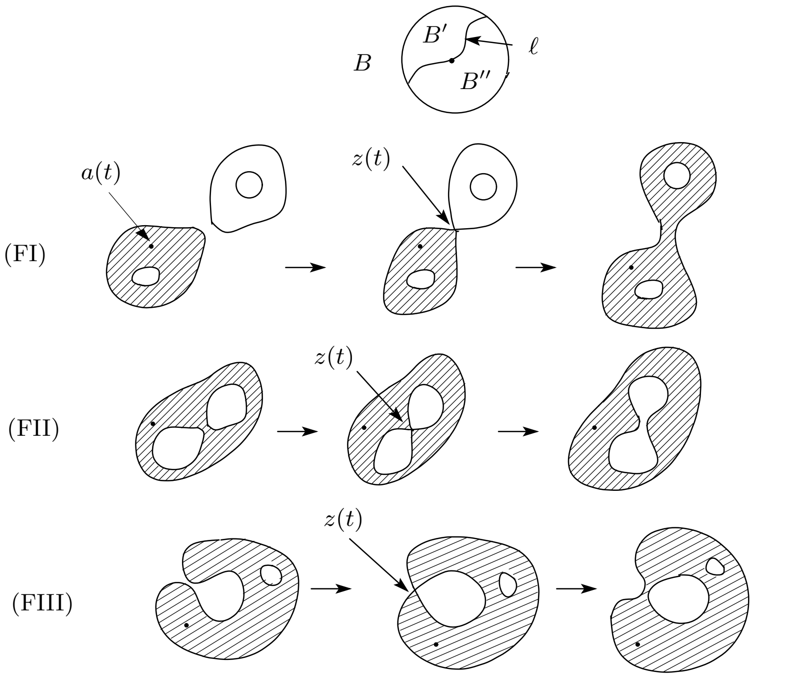
(see [10, Theorem 8.22]), $\mathcal{R}$ admits a C^ω strictly plurisubharmonic exhaustion function $\varphi(t,z)$. Fix a disk $B_0 \Subset B$ and consider $\mathcal{R}|_{B_0} = \pi^{-1}(B_0) = \bigcup_{t\in B_0}(t,R(t))$ and $\mathbf{a}|_{B_0} = \mathbf{a}\cap\mathcal{R}|_{B_0}$. Then, for $c \gg 1$, we have $\mathcal{R}(c) := \{(t,z)\in\mathcal{R}|_{B_0} \mid \varphi(t,z) < c\} \supset \mathbf{a}|_{B_0}$. There is an increasing sequence $\{c_n\}_n$ with $c_n > c$ and $\lim_{c\to\infty} c_n = \infty$ such that the connected component $\hat{\mathcal{R}}_n$ of $\mathcal{R}(c_n)$ containing $\mathbf{a}|_{B_0}$ satisfies the following:

(i) Each $\hat{\mathcal{R}}_n$ is a connected domain with real 3-dimensional C^ω surfaces $\partial\hat{\mathcal{R}}_n := \bigcup_{t\in B_0}(t,\partial\hat{R}_n(t))$ in $\mathcal{R}|_{B_0}$, where $\hat{R}_n(t) = R(t)\cap\hat{\mathcal{R}}_n$. Since $\hat{R}_n(t)$, $t\in B_0$, is not always connected, we denote by $R_n(t)$ the connected component of $\hat{R}_n(t)$ with $R_n(t) \ni a(t)$. If $\partial R_n(t)$, $t\in B_0$, consists of a finite number of C^ω smooth contours in $R(t)$, we call $\mathcal{R}_n := \bigcup_{t\in B_0}(t,R_n(t))$ a *smooth variation* in $\mathcal{R}|_{B_0}$.

(ii) If there exists a point $(t,z(t))\in\partial\hat{\mathcal{R}}_n$ with $\frac{\partial\varphi}{\partial z}(t,z(t)) = 0$ (so that $z(t)$ is a singular point of $\partial\hat{R}_n(t)$) and $\partial R_n(t)$ is not smooth at $z(t)$, then the variation $\mathcal{R}_n : t\in B_0 \to R_n(t)$ at $(t,z(t))$ is separated into two types as follows:

- if there is only one boundary point of $R_n(t)$ over the singular point $z(t)$, we say that the variation $\mathcal{R}_n$ is of (C1) *type* at $(t,z(t))$;
- if there are two or more boundary points of $R_n(t)$ over the singular point $z(t)$, we say that the variation $\mathcal{R}_n$ is of (C2) *type* at $(t,z(t))$.



$R_n(t')$, $t'\in B'$ $\quad$ $R_n(t)$, $t\in\ell$ $\quad$ $R_n(t'')$, $t''\in B''$

For example, if the shadowed part above is $R_n(t)$, then (FI) is of (C1) type, and both (FII) and (FIII) are of (C2) type at $(t, z(t))$. If at each point $(t, z(t)) \in \partial\hat{\mathcal{R}}_n$ with $\frac{\partial\varphi}{\partial z}(t, z(t)) = 0$ the variation $\mathcal{R}_n$ is of (C1) type, we say that $\mathcal{R}_n$ is of (C1) type. Moreover, if there exists an increasing sequence $\{c_n\}$ with $\lim_{n\to\infty} c_n = \infty$ such that each $\mathcal{R}_n$ is of (C1) type, we say that $\mathcal{R}|_{B_0}$ is of (C1) type.

For example, let $\mathcal{D}$ be a polynomially convex domain in $\mathbb{C}^2 = \mathbb{C}_z \times \mathbb{C}_w$. Let $B \subset \mathbb{C}_z$ and put $\mathcal{D}|_B = \mathcal{D} \cap (B \times \mathbb{C}_w)$. Assume that there exists a section $\mathbf{a} \in \Gamma(B, \mathcal{D})$. Then $\mathcal{D}|_B$ is of (C1) type.

The main result of this paper is the following:

MAIN THEOREM 1.1. *Let B be a disk in $\mathbb{C}_t$. Let $\pi : \mathcal{R} \to B$ be a holomorphic family of Riemann surfaces $R(t) = \pi^{-1}(t)$, $t \in B$, satisfying the following conditions:*

(1) *$\mathcal{R}$ is a 2-dimensional Stein manifold and $R(t)$ is irreducible and non-singular in $\mathcal{R}$;*
(2) *each $R(t)$, $t \in B$, is a planar Riemann surface;*
(3) *there exists a section $\mathbf{a} := \{a(t) \in R(t) \mid t \in B\} \in \Gamma(B, \mathcal{R})$;*
(4) *for any $t_0 \in B$, there exists a disk $B_0 = \{|t - t_0| < \rho_0\} \Subset B$ such that $\mathcal{R}|_{B_0}$ is of* (C1) *type.*

Assume that $E = \{t \in B \mid R(t)$ is of class $O_{\mathrm{AD}}\}$ is of positive logarithmic capacity in $\mathbb{C}_t$. Then

(i) *each $R(t)$, $t \in B$, is of class O_{AD};*
(ii) *$\mathcal{R}$ is biholomorphic to a univalent domain $\mathcal{D}$ in $B \times \mathbb{P}_w$ by the holomorphic transformation*

$$T : (t, z) \in \mathcal{R} \mapsto (t, w) = (t, M(t, z)) \in \mathcal{D}.$$

Here $M(t, z)$ is the maximizing function of the Schiffer span for $(R(t), a(t))$ (which is necessarily holomorphic as a function of the two complex variables (t, z) in $\mathcal{R} \setminus \mathbf{a}$).

REMARK 1.2. We denote by $\mathrm{AD}(R)$ the family of analytic functions which have a finite Dirichlet integral on a Riemann surface R. The class of Riemann surfaces on which $\mathrm{AD}(R)$ consists entirely of constants is denoted by O_{AD}. In the classification theory of Riemann surfaces we have the table (66) in [11, p. 390] of strict inclusion relations. The class O_{AD} is characterized as the largest class of Riemann surfaces in the table. As compared with [13, Théorème 2], condition (4) of (C1) type is not necessary for [13], but the condition of class O_{AD} is weaker than of class O_G.

REMARK 1.3. We showed in [3, Theorem 1.5] the following simultaneous uniformization. Let $\pi : \mathcal{S} \to B$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$, $t \in B$, of genus $g \geq 2$. Let $\widetilde{S}(t)$ denote the Schottky covering of $S(t)$ for each $t \in B$. Then the total space $\widetilde{\mathcal{S}} := \bigcup_{y\in B}(t, \widetilde{S}(t))$ is biholomorphic to a univalent domain in $B\times\mathbb{P}$. Our proof is based on the fact that $\widetilde{\mathcal{S}}$ is a Stein manifold. This is a new approach to the Bers simultaneous uniformization [2] for the Schottky group.

This result becomes a simple corollary of the above Main Theorem 1.1 since $\widetilde{S}(t)$ is planar and of class O_{AD} and since we can take an exhaustion $\widetilde{\mathcal{S}}_n = \bigcup_{t\in B}(t, \widetilde{S}_n(t))$ of $\widetilde{\mathcal{S}}$ which is a smooth variation and pseudoconvex in $\widetilde{\mathcal{S}}$. Moreover, our proof, which uses the variation of *Schiffer spans*, is more elementary than [3] which uses the variation of harmonic spans (cf. Remark 4.2). We note that, since Main Theorem 1.1 contains the case when the fibers are non-homeomorphic, the methods in the theory of the Teichmüller space are not available to prove it.

Now we recall the notion of Schiffer span in the potential theory of one complex variable. Let R be a planar bordered Riemann surface. Let $a \in R$ and let $V = \{|z| < r\}$ be a local coordinate of a neighborhood U of a in R such that a corresponds to $z = 0$. We denote by $\mathcal{P}(R)$ the set of all univalent functions P on R such that $P(z)-1/z$ is regular at 0. For $w = P(z) \in \mathcal{P}(R)$ let $\mathcal{E}_P$ denote the Euclidean area of $\mathbb{P}_w \setminus P(R)$, and set $\mathcal{E}(R) = \sup\{\mathcal{E}_P \mid P \in \mathcal{P}(R)\}$. Let P_1 be the vertical slit map and P_0 be the horizontal slit map on R. Following the studies of Koebe and Grunsky, in 1943 Schiffer [12, p. 209] introduced the quantity $s := \mathrm{Re}\{A_{01} - A_{11}\}$, called the *Schiffer span* for (R, a), where A_{i1} $(i = 0, 1)$ is the coefficient of z of the Taylor expansions of $P_i(z) - 1/z$ at 0, and showed $M(z) := (P_1 + P_0)/2 \in \mathcal{P}(R)$ and $\mathcal{E}(R) = \pi s/2$ (for details, see §3 and [1, §12, Chap. III]). In this paper we call $M(z)$ the *maximizing function of the Schiffer span*. By the standard approximation argument we define the Schiffer span and the maximizing function of the Schiffer span for any planar Riemann surface.

The key tool for the proof of the Main Theorem 1.1 is:

THEOREM 1.4. *Let the notation be as in Main Theorem* 1.1 *and let* $\mathcal{R}$ *satisfy* (1)–(4). *Then the Schiffer span* $s(t)$ *for* $(R(t), a(t))$ *is logarithmically subharmonic on* B (*i.e.*, $\log s(t)$ *is subharmonic on* B).

To the best of our knowledge, the above theorems are the first to show that the Schiffer span has some significant properties not only in one complex variable but in several complex variables.

2. Variation formulas for principal functions

2.1. Variation formulas for principal functions. Let B be a disk in $\mathbb{C}_t$. Let $\pi : \widetilde{\mathcal{R}} \to B$ be a holomorphic family of Riemann surfaces $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in B$, such that $\widetilde{R}(t)$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$, and set

$\widetilde{\mathcal{R}} = \bigcup_{t\in B}(t, \widetilde{R}(t))$. If a subdomain $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ satisfies the following conditions:

(I) $\widetilde{R}(t) \Supset R(t) \neq \emptyset$ for $t \in B$, and $R(t)$ is a connected Riemann surface of genus $g \geq 0$ such that $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$;

(II) the boundary $\partial\mathcal{R} = \bigcup_{t\in B}(t, \partial R(t))$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth and $\partial\mathcal{R}$ is transverse to each fiber $\widetilde{R}(t)$, $t \in B$,

then we say that $\mathcal{R}$ is a *smooth variation* in $\widetilde{\mathcal{R}}$ by regarding $\mathcal{R}$ as the variation $\mathcal{R} : t \in B \to R(t)$. We note that g and ν are independent of $t \in B$. Each $C_j(t)$ is oriented by $\partial R(t) = C_1(t) + \cdots + C_\nu(t)$. Assume that there exists a section $\mathbf{a} := \{a(t) \in R(t) \mid t \in B\} \in \Gamma(B, \mathcal{R})$. Let $\mathcal{V} := B \times \{|z| < r\}$ be a π-local coordinate of a neighborhood $\mathcal{U}$ of $\mathbf{a}$ in $\mathcal{R}$ such that $\mathbf{a}$ corresponds to $B \times \{0\}$. Let $t \in B$ be fixed. Then among all harmonic functions u on $R(t) \setminus \{a(t)\}$ with singularity $\mathrm{Re}(1/z)$ at $a(t)$ normalized so that $\lim_{z\to 0}(u(t, z) - \mathrm{Re}(1/z)) = 0$, we have two uniquely determined functions p_i $(i = 1, 0)$ with the following boundary conditions (L_i): for $j = 1, \ldots, \nu$,

(L_1) $p_1(t, z) = c_j(t)$ (constant) on $C_j(t)$ and $\displaystyle\int_{C_j(t)} \frac{\partial p_1(t, z)}{\partial n_z}\, ds_z = 0$;

(L_0) $\displaystyle\frac{\partial p_0(t, z)}{\partial n_z} = 0$ on $C_j(t)$.

Here $\partial/\partial n_z$ is the outer normal derivative and ds_z is the arc length element of $C_j(t)$ at z. We write

$$(2.1)\qquad p_i(t, z) = \mathrm{Re}\left\{\frac{1}{z} + \sum_{n=1}^{\infty} A_{in}(t) z^n\right\} \quad \text{at } z = 0 \quad (i = 0, 1).$$

We call $p_i(t, z)$ the *L_i-principal function* and $\alpha_i(t) := \mathrm{Re}\{A_{i1}(t)\}$ the *L_i-constant* for $(R(t), a(t))$ with respect to the local coordinate $\{|z| < r\}$.

We show the following variation formulas of the second order for $\alpha_i(t)$, which are the key tools for the proof of Theorem 1.4.

Lemma 2.1. *Let the notation be as above. Assume that $R(t)$, $t \in B$, is planar. Then*

$$(2.2)\qquad \frac{\partial^2 \alpha_1(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left|\frac{\partial p_1(t, z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left|\frac{\partial^2 p_1(t, z)}{\partial \bar{t} \partial z}\right|^2 dx\, dy,$$

$$(2.3)\qquad \frac{\partial^2 \alpha_0(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left|\frac{\partial p_0(t, z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left|\frac{\partial^2 p_0(t, z)}{\partial \bar{t} \partial z}\right|^2 dx\, dy.$$

Here

$$k_2(t,z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}}\right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)\left|\frac{\partial\varphi}{\partial z}\right|^{-3}$$

on $\partial\mathcal{R}$, where $\varphi(t,z)$ is a C^2 defining function of $\partial\mathcal{R}$.

Note that the *Levi curvature* $k_2(t,z)$ for $\partial\mathcal{R}$ (see [6, (1.3)] and [7, (7)]) does not depend on the choice of the defining function $\varphi(t,z)$ of $\partial\mathcal{R}$. We remark that the formulas (2.2) and (2.3) depend on the choice of the π-local coordinate $\mathcal{V} = B\times\{|z|<r\}$.

As noted in [3], since $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ if and only if $k_2(t,z)\geq 0$ on $\partial\mathcal{R}$, Lemma 2.1 implies:

Lemma 2.2. *Let $\mathcal{R}: t\in B\to R(t)$ be a smooth variation in $\widetilde{\mathcal{R}}$ and $\mathbf{a} := \{a(t)\in R(t)\mid t\in B\}\in\Gamma(B,\mathcal{R})$. If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t)$, $t\in B$, is planar, then*

(1) *the L_1-constant $\alpha_1(t)$ for $(R(t),a(t))$ is superharmonic on B;*
(2) *the L_0-constant $\alpha_0(t)$ for $(R(t),a(t))$ is subharmonic on B.*

2.2. Proof of Lemma 2.1

Lemma 2.3. *Let D be a domain in $\mathbb{C}_z$ bounded by C^ω smooth contours C_j $(j=1,\ldots,\nu)$ with $D\ni 0$, and $p_i(z)$ $(i=0,1)$ be an L_i-principal function for $(D,0)$. Let $u(z)$ be a harmonic function on $\overline{D}$ such that*

$$u(z) = u(0) + \mathrm{Re}\Big\{\sum_{n=1}^{\infty} a_n z^n\Big\} \qquad \text{at } z=0.$$

Then

$$\mathrm{Re}\{a_1\} = \frac{1}{2\pi}\int_{\partial D} p_0(z)\frac{\partial u(z)}{\partial n_z}\,ds_z. \tag{2.4}$$

If $u(z)$ satisfies the boundary conditions

$$\int_{C_j}\frac{\partial u(z)}{\partial n_z}\,ds_z = 0 \qquad (j=1,\ldots,\nu), \tag{2.5}$$

then

$$\mathrm{Re}\{a_1\} = -\frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial p_1(z)}{\partial n_z}\,ds_z. \tag{2.6}$$

Similarly, if $\widetilde{p}(z)$ is a harmonic function on $\overline{D}\setminus\{0\}$ such that

$$\widetilde{p}(z) = \mathrm{Im}\,\frac{1}{z} + 0 + \mathrm{Re}\Big\{\sum_{n=1}^{\infty}\widetilde{a}_n z^n\Big\} \qquad \text{at } z=0$$

and satisfies the condition (L_1) *on* ∂D*, then every* $u(z)$ *with* (2.5) *satisfies*

$$\mathrm{Im}\{a_1\} = -\frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial \widetilde{p}(z)}{\partial n_z}\,ds_z. \tag{2.7}$$

Proof of (2.4). Let $V_0 = \{|z| < \epsilon\}$ be a neighborhood of $z = 0$ in D. We see from Green's formula that

$$\int_{\partial D - \partial V_0} p_0(z)\frac{\partial u(z)}{\partial n_z}\,ds_z = \int_{\partial D - \partial V_0} u(z)\frac{\partial p_0(z)}{\partial n_z}\,ds_z.$$

It follows from the condition for the L_0-principal function $p_0(z)$ that

$$\int_{\partial D} p_0\frac{\partial u}{\partial n_z}\,ds_z = \int_{\partial V_0}(p_0 du^* - u dp_0^*) = \int_{\partial V_0}(\mathrm{Im}\{(p_0 + ip_0^*)\,d(u + iu^*)\} - d(p_0^* u)).$$

Since p_0^* is single-valued on $\overline{V_0}$ and $\partial V_0 = \{|z| = \epsilon\}$ is a closed curve, we have $\int_{\partial V_0} d(p_0^* u) = 0$. It follows that

$$\begin{aligned}\int_{\partial D} p_0\frac{\partial u}{\partial n_z}\,ds_z &= \mathrm{Im}\int_{\partial V_0}\left(\frac{1}{z} + A_{01}z + A_{02}z^2 + \cdots\right) d(u(0) + iu^*(0) + a_1 z + \cdots) \\ &= \mathrm{Im}\{2\pi i \cdot a_1\} = 2\pi\,\mathrm{Re}\{a_1\},\end{aligned}$$

which is (2.4). ∎

Proof of (2.6). From the condition for the L_1-principal function $p_1(z)$ on C_j and (2.5), we see that

$$\sum_{j=1}^{\nu}\int_{C_j} p_1\frac{\partial u}{\partial n_z}\,ds_z = \sum_{j=1}^{\nu} c_j\int_{C_j}\frac{\partial u}{\partial n_z}\,ds_z = 0.$$

It follows from Green's formula that

$$\begin{aligned}\int_{\partial D} u\frac{\partial p_1}{\partial n_z}\,ds_z &= \int_{\partial V_0}(u\,dp_1^* - p_1\,du^*) \\ &= \mathrm{Im}\int_{\partial V_0}(u(0) + iu^*(0) + a_1 z + \cdots)\,d\left(\frac{1}{z} + A_{11}z + A_{12}z^2 + \cdots\right) \\ &= \mathrm{Im}\{-2\pi i \cdot a_1\} = -2\pi\,\mathrm{Re}\{a_1\},\end{aligned}$$

which is (2.6). ∎

Proof of (2.7). Similarly, it follows from the condition (L_1) for $\widetilde{p}$ and (2.5) that

$$\begin{aligned}\int_{\partial D} u\frac{\partial\widetilde{p}}{\partial n_z}\,ds_z &= \int_{\partial V_0}(ud\widetilde{p}^* - \widetilde{p}du^*)\\ &= \mathrm{Im}\int_{\partial V_0}(u(0)+iu^*(0)+a_1z+\cdots)\Big(\frac{i}{z^2}+\widetilde{a}_1+2\widetilde{a}_2z+\cdots\Big)\\ &= \mathrm{Im}\{2\pi i\cdot a_1 i\} = -2\pi\,\mathrm{Im}\{a_1\},\end{aligned}$$

which is (2.7). ∎

It suffices to prove Lemma 2.1 at $t=0$. If necessary, take a smaller disk B of center $t=0$. Then by the standard use of the immersion theorem for open Riemann surfaces [8], we have a π-biholomorphic mapping T from $\widetilde{\mathcal{R}}$ to an unramified Riemann domain $\widetilde{\mathcal{D}}$ over $B\times\mathbb{C}_w$ such that the holomorphic section $\mathbf{a}$ of $\mathcal{R}$ over B corresponds to the constant section $B\times\{w=0\}$ of $\mathcal{D}:=T(\mathcal{R})$ over B. Then $\mathcal{D}$ is a smooth variation in $\widetilde{\mathcal{D}}$. It suffices to show the lemma for the unramified domain $\mathcal{D}$ over $B\times\mathbb{C}_w$ with the section $B\times\{0\}$. For convenience we use anew the notations $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ for $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. We find a neighborhood $V=\bigcup_{j=1}^{\nu}V_j$ (disjoint union) of $\partial R(0)=\bigcup_{j=1}^{\nu}C_j(0)$ such that $(B\times V)\cap\mathbf{a}=\emptyset$, V_j is a thin tubular neighborhood of $C_j(0)$ with $V_j\supset C_j(t)$ for $t\in B$, and $p_i(t,z)$ is harmonic on $(R(0)\cup V)\setminus\{0\}$. Then $p_i(t,z)$ is defined in the product $B\times\widehat{R}(0)$, where $\widehat{R}(0):=R(0)\cup V$.

Proof of (2.2). We see from (2.1) that $\frac{\partial^2 p_1(t,z)}{\partial t\partial\bar{t}}$ is harmonic for z on $\overline{R(t)}$ by setting $\frac{\partial^2 p_1}{\partial t\partial\bar{t}}(t,0)=0$. For all t sufficiently close to $t=0$, we may assume $\frac{\partial^2 p_1(t,z)}{\partial t\partial\bar{t}}$ is harmonic on $\overline{R(0)}$. Since $\int_{C_j(0)}\frac{\partial}{\partial n_z}\big(\big(\frac{\partial^2 p_1}{\partial t\partial\bar{t}}\big)(0,z)\big)\,ds_z=0$ $(j=1,\dots,\nu)$ by (L_1), it follows from (2.6) that

$$\frac{\partial^2\alpha_1(t)}{\partial t\partial\bar{t}}(0) = -\frac{1}{2\pi}\int_{\partial R(0)}\frac{\partial^2 p_1}{\partial t\partial\bar{t}}(0,z)\frac{\partial p_1(0,z)}{\partial n_z}\,ds_z.$$

Under the boundary condition (L_1), we proved in [3, Lemma (A)] that along each $C_j(0)$ $(j=1,\dots,\nu)$,

$$\begin{aligned}\frac{\partial^2 p_1}{\partial t\partial\bar{t}}(0,z)\frac{\partial p_1(0,z)}{\partial n_z}\,ds_z &= 2k_2(0,z)\left|\frac{\partial p_1(0,z)}{\partial z}\right|^2 ds_z + \frac{\partial^2 c_j}{\partial t\partial\bar{t}}(0)\frac{\partial p_1(0,z)}{\partial n_z}\,ds_z\\ &\quad + 4\,\mathrm{Im}\Big\{\frac{\partial p_1}{\partial t}(0,z)\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\,dz\Big\} - 4\,\mathrm{Im}\Big\{\frac{\partial c_j}{\partial t}(0)\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\,dz\Big\}.\end{aligned}$$

We note that

$$\int_{C_j(0)}\frac{\partial^2 c_j}{\partial t\partial\bar{t}}(0)\frac{\partial p_1(0,z)}{\partial n_z}\,ds_z = \frac{\partial^2 c_j}{\partial t\partial\bar{t}}(0)\int_{C_j(0)}\frac{\partial p_1(0,z)}{\partial n_z}\,ds_z = 0 \quad \text{by } (L_1).$$

Since $p_1(t,z)$ is of class C^ω for (t,z) in a neighborhood of $(0,\partial R(0))$ and harmonic for z on $\overline{R(0)}\setminus\{0\}$ with (L_1), by Green's formula we have

$$\int_{C_j(0)} \frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\,dz = \frac{\partial}{\partial\bar{t}}\bigg(\int_{C_j(0)} \frac{\partial p_1(t,z)}{\partial z}(t,z)\,dz\bigg)\bigg|_{t=0} = 0.$$

Thus we have

$$\begin{aligned}\text{(2.8)}\qquad \frac{\partial^2\alpha_1}{\partial t\partial\bar{t}}(0) = &-\frac{1}{\pi}\int_{\partial R(0)} k_2(0,z)\left|\frac{\partial p_1(0,z)}{\partial z}\right|^2 ds_z\\ &-\frac{2}{\pi}\,\mathrm{Im}\bigg\{\int_{\partial R(0)} \frac{\partial p_1}{\partial t}(0,z)\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\,dz\bigg\}.\end{aligned}$$

Since $\frac{\partial p_1}{\partial t}(0,z)$ and $\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)$ are harmonic on $\overline{R(0)}$, it follows from Green's formula that

$$\begin{aligned}&\int_{\partial R(0)} \frac{\partial p_1}{\partial t}(0,z)\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\,dz\\ &\quad= \iint_{R(0)}\bigg(\frac{\partial^2 p_1}{\partial t\partial\bar{z}}(0,z)\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z) + \frac{\partial p_1}{\partial t}(0,z)\frac{\partial^3 p_1}{\partial\bar{t}\partial z\partial\bar{z}}\bigg)(0,z)\,d\bar{z}\wedge dz\\ &\quad= 2i\iint_{R(0)}\left|\frac{\partial^2 p_1}{\partial\bar{t}\partial z}(0,z)\right|^2 dx\,dy.\end{aligned}$$

By substituting this into (2.8), we obtain (2.2). ∎

Proof of (2.3). For each fixed $t\in B$, we consider the harmonic conjugate function $p_0^*(t,z)$ of $p_0(t,z)$ in $R(t)$ such that

$$p_0(t,z) + ip_0^*(t,z) = \frac{1}{z} + 0 + \sum_{n=1}^{\infty} A_{0n}(t)z^n \qquad \text{at } z=0.$$

Then $p_0^*(t,z)$ *is a single-valued function on* $R(t)$ *such that*

$$p_0^*(t,z) = \mathrm{Im}\,\frac{1}{z} + \mathrm{Im}\bigg\{\sum_{n=1}^{\infty} A_{0n}(t)z^n\bigg\} \qquad \textit{at } z=0$$

and $p_0^*(t,z)$ *satisfies* (L_1) *on* $\partial R(t) = \sum_{j=1}^{\nu} C_j(t)$. Since $\frac{\partial^2}{\partial t\partial\bar{t}}$ is a real operator, we have $\frac{\partial^2 p_0^*(t,z)}{\partial t\partial\bar{t}} = \mathrm{Im}\{\sum_{n=1}^{\infty}\frac{\partial^2 A_{0n}(t)}{\partial t\partial\bar{t}}z^n\}$ at $z=0$. Note that $\frac{\partial^2 p_0^*(t,z)}{\partial t\partial\bar{t}}$ is harmonic on $\overline{R(0)}$ by setting $\frac{\partial^2 p_0^*}{\partial t\partial\bar{t}}(t,0)=0$. For every t sufficiently close to $t=0$, we may assume that $\frac{\partial^2 p_0^*(t,z)}{\partial t\partial\bar{t}}$ is harmonic on $\overline{R(0)}$. Since p_0

satisfies (L_0) on $\partial R(t)$, by Green's formula we have

$$\int\limits_{C_j(0)} \frac{\partial}{\partial n_z}\Big(\frac{\partial^2 p_0^*}{\partial t\partial\bar t}(0,z)\Big)\,ds_z = \frac{\partial^2}{\partial t\partial\bar t}\Big(\int\limits_{C_j(0)} \frac{\partial p_0^*(t,z)}{\partial n_z}\,ds_z\Big)\Big|_{t=0} = 0.$$

Since $p_0^*(0,z)$ satisfies (L_1) on $\partial R(0)$, and $\frac{\partial^2 p_0^*}{\partial t\partial\bar t}(0,z)$ is harmonic on $\overline{R(0)}$ and satisfies (2.5) for $C_j(0)$, it follows from (2.7) and $\mathrm{Im}\{\frac{1}{i}\frac{\partial^2 A_{01}(t)}{\partial t\partial\bar t}\} = -\mathrm{Re}\{\frac{\partial^2 A_{01}(t)}{\partial t\partial\bar t}\}$ that

$$\mathrm{Re}\Big\{\frac{\partial^2 A_{01}}{\partial t\partial\bar t}(0)\Big\} = \frac{1}{2\pi}\int\limits_{\partial R(0)} \frac{\partial^2 p_0^*}{\partial t\partial\bar t}(0,z)\frac{\partial p_0^*(0,z)}{\partial n_z}\,ds_z.$$

By the same calculation in the proof of (2.2), we see that the right-hand side above becomes

$$\frac{1}{\pi}\int\limits_{\partial R(0)} k_2(0,z)\Big|\frac{\partial p_0^*(0,z)}{\partial z}\Big|^2 ds_z + \frac{4}{\pi}\iint\limits_{R(0)}\Big|\frac{\partial^2 p_0^*}{\partial\bar t\partial z}(0,z)\Big|^2 dx\,dy.$$

Since

$$\frac{\partial p_0(t,z)}{\partial z} = \frac{1}{i}\,\frac{\partial p_0^*(t,z)}{\partial z},\qquad \frac{\partial^2 p_0^*(t,z)}{\partial\bar t\partial z} = \frac{1}{i}\,\frac{\partial^2 p_0(t,z)}{\partial\bar t\partial z},$$

and $\frac{\partial^2}{\partial t\partial\bar t}$ is real, the assertion (2.3) is shown. ∎

3. Schiffer span and its geometric meaning. We now recall the slit mapping theory in one complex variable. Let R be a planar bordered Riemann surface with a finite number of smooth contours C_j $(j = 1,\dots,\nu)$. Let $a \in R$ and let $\{|z| < r\}$ be a local coordinate of a neighborhood U of a in R. We denote by $\mathcal{P}(R)$ the set of all univalent functions P on R such that $P(z) - 1/z$ is regular at 0. Koebe constructed the vertical slit mapping P_1 and the horizontal slit mapping P_0 for (R,a) with $\mathrm{Re}\,P_1(z) = c_j$ (constant) and $\mathrm{Im}\,P_0(z) = \tilde c_j$ (constant) on C_j $(j = 1,\dots,\nu)$, respectively. The *Schiffer span* (or analytic span) s for (R,a) is defined to be $s := \mathrm{Re}\{A_{01} - A_{11}\}$, where A_{i1} $(i = 0,1)$ is the coefficient of z of the Taylor expansion of $P_i(z) - 1/z$ at 0, and s is positive (see [11, pp. 45–46]).

PROPOSITION 3.1 (Schiffer [12]). *For $w = P(z) \in \mathcal{P}(R)$ let $\mathcal{E}_P$ denote the Euclidean area of $\mathbb{P}_w \setminus P(R)$, and set $\mathcal{E}(R) = \sup\{\mathcal{E}_P \mid P \in \mathcal{P}(R)\}$. Let $M(z) := (P_1 + P_0)/2$. Then*

(i) $M(z) \in \mathcal{P}(R)$ *and* (ii) $\mathcal{E}(R) = \mathcal{E}_M = \pi s/2$.

We call $M(z)$ the *maximizing function of the Schiffer span* s for (R,a) with respect to the local coordinate $\{|z| < r\}$.

To extend our argument to open planar Riemann surfaces which may have infinitely many ideal boundary components, we recall the definition of the vertical slit and the horizontal slit mappings and the Schiffer span: Let R be such an open planar Riemann surface. Let $a \in R$ and let $\{|z| < r\}$ be a local coordinate of a neighborhood U of a in R such that a corresponds to $z = 0$. We choose a canonical exhaustion $\{R_n\}_{n=1}^{\infty}$ of R such that $a \in R_1$, $R_n \Subset R_{n+1}$, $R = \bigcup_{n=1}^{\infty} R_n$, and $R \setminus R_n$ has no relatively compact connected component in R. Then each R_n $(n = 1, 2, \dots)$ carries the vertical slit mapping $P_1^n(z)$ and the horizontal slit mapping $P_0^n(z)$ such that

$$P_i^n(z) = \frac{1}{z} + A_{i1}^n z + A_{i2}^n z^2 + \cdots \qquad \text{at } z = 0 \ (i = 0, 1).$$

Since the Dirichlet integral $\iint_{R_n} \big[\big(\frac{\partial(P_i^n - P_i^m)}{\partial x}\big)^2 + \big(\frac{\partial(P_i^n - P_i^m)}{\partial y}\big)^2\big]\, dx\, dy$ tends to 0 as $m \geq n \to \infty$, $P_i^n(z)$ converges to a certain univalent function $P_i(z)$ on R uniformly on any compact set in R, so that

$$P_i(z) = \frac{1}{z} + A_{i1} z + A_{i2} z^2 + \cdots \qquad \text{at } z = 0 \ (i = 0, 1). \tag{3.1}$$

Further, $P_i(z)$ does not depend on the choice of the canonical exhaustion $\{R_n\}_n$ of R, so that $P_i(z)$ is uniquely determined by R, a and the local coordinate $\{|z| < r\}$. We call $P_1(z)$ (resp. $P_0(z)$) the *vertical* (resp. *horizontal*) *slit mapping* and $\operatorname{Re} A_{11}$ (resp. $\operatorname{Re} A_{01}$) the L_1-(resp. L_0-)*constant* for (R, a) with respect to the local coordinate $\{|z| < r\}$. We call $s := \operatorname{Re} A_{01} - \operatorname{Re} A_{11}$ the *Schiffer span* for (R, a), and $M(z) := (P_1 + P_0)/2$ the *maximizing function of the Schiffer span* s for (R, a). We note $s_n \searrow s$ as $n \to \infty$.

The following is well-known in one complex variable (cf. [11, p. 46]):

PROPOSITION 3.2. *R is of class O_{AD} if and only if* (1) $s = 0$ (*i.e.*, $\operatorname{Re} A_{11} = \operatorname{Re} A_{01}$), *or* (2) $P_1(z) = P_0(z) = M(z) := (P_1(z) + P_0(z))/2$ *on* R, *or* (3) $\mathcal{P}(R)$ *consists of only one function* $P_1(z)$.

We prepare the following for the proof of Main Theorem 1.1.

LEMMA 3.3. *Let R be of class O_{AD} and $P_1(z)$ be as in* (3.1). *For a point b $(\neq 0)$ in $\{|z| < r\}$ let $P_1^b(z)$ denote the vertical slit map for (R, b):*

$$P_1^b(z) = \frac{1}{z - b} + A_{11}^b (z - b) + A_{12}^b (z - b)^2 + \cdots \qquad \text{at } z = b. \tag{3.2}$$

Then

$$A_{11}^b = \frac{1}{4}\left(\frac{P_1''(b)}{P_1'(b)}\right)^2 - \frac{1}{6}\left(\frac{P_1'''(b)}{P_1'(b)}\right).$$

Proof. From the Taylor expansion about $z = b$, we have

$$P_1(z) - P_1(b) = P_1'(b)(z - b)\left(1 + \sum_{n=2}^{\infty} \frac{P_1^{(n)}(b)}{n!\,P_1'(b)}(z - b)^{n-1}\right).$$

Hence

$$\frac{P_1'(b)}{P_1(z)-P_1(b)}=\frac{1}{z-b}-\frac{P_1''(b)}{2P_1'(b)}+\left[\left(\frac{P_1''(b)}{2P_1'(b)}\right)^2-\frac{P_1'''(b)}{3!P_1'(b)}\right](z-b)+\cdots.$$

We set $\widetilde{P}_1(z):=\frac{P_1'(b)}{P_1(z)-P_1(b)}+\frac{P_1''(b)}{2P_1'(b)}$, which is a univalent function on R. By (3) in Proposition 3.2 (using b instead of 0) we have $\widetilde{P}_1(z)=P_1^b(z)$ on R, which proves the lemma. ∎

By simple calculations we have $\lim_{b\to 0}A_{11}^b=A_{11}$.

4. Variation formula for the Schiffer span. We return to a smooth variation $\mathcal{R}: t\in B\to R(t)\Subset \widetilde{R}(t)$ in $\widetilde{\mathcal{R}}:=\bigcup_{t\in B}(t,\widetilde{R}(t))$. Let $\mathbf{a}:=\{a(t)\in R(t)\mid t\in B\}\in\Gamma(B,\mathcal{R})$. If $R(t)$, $t\in B$, is planar, then Lemma 2.1 immediately implies the following variation formula for the Schiffer span $s(t)$:

$$\begin{aligned}\text{(4.1)}\qquad \frac{\partial^2 s(t)}{\partial t\partial\bar t}&=\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\left(\left|\frac{\partial p_1(t,z)}{\partial z}\right|^2+\left|\frac{\partial p_0(t,z)}{\partial z}\right|^2\right)ds_z\\&\quad+\frac{4}{\pi}\iint_{R(t)}\left(\left|\frac{\partial^2 p_1(t,z)}{\partial\bar t\partial z}\right|^2+\left|\frac{\partial^2 p_0(t,z)}{\partial\bar t\partial z}\right|^2\right)dx\,dy.\end{aligned}$$

Lemma 4.1. *Under the same assumptions as in Lemma 2.2, the Schiffer span $s(t)$ for $(R(t),a(t))$ with respect to the local coordinate $\{|z|<r\}$ is logarithmically subharmonic on B.*

Proof. We divide the proof into two steps.

Step 1. *Let $\varphi(t)$ be any non-vanishing holomorphic function on B and consider the holomorphic transformation*

$$T:(t,z)\in\mathcal{V}=B\times\{|z|<r\}\mapsto(t,\zeta)=(t,\varphi(t)z)\in B\times\mathbb{C}_\zeta.$$

Take a bidisk $\hat{\mathcal{V}}=B\times\{|\zeta|<\hat r\}\subset T(\mathcal{V})$ and consider the Schiffer span $\hat s(t)$ for $(R(t),a(t))$ with respect to the local coordinate $\{|\zeta|<\hat r\}$. Then

$$s(t)=|\varphi(t)|^2\hat s(t),\qquad t\in B.$$

Indeed, let $\hat P_1(t,\zeta)$ and $\hat P_0(t,\zeta)$ be the vertical slit and the horizontal slit mappings for $(R(t),a(t))$ with respect to the local coordinate $\{|\zeta|<\hat r\}$. Then, for $\zeta=\varphi(t)z$,

$$\frac{\hat P_1(t,\zeta)+\hat P_0(t,\zeta)}{2}=\frac{1}{\zeta}+\hat c_1(t)\zeta+\cdots=\frac{1}{\varphi(t)z}+\hat c_1(t)\varphi(t)z+\cdots$$

at $z=0$, so that $\left(\frac{\hat P_1+\hat P_0}{2}\right)\cdot\varphi(t)\in\mathcal{P}(R(t))$ by Proposition 3.1(i). It follows from the definition of $s(t)$ and Proposition 3.1(ii) that $\hat s(t)|\varphi(t)|^2\le s(t)$. Similarly, we have $s(t)\frac{1}{|\varphi(t)|^2}\le\hat s(t)$, which proves Step 1.

Step 2. $\log s(t)$ *is subharmonic on B.*

Since $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, it follows from (4.1) that $s(t)$ is subharmonic on B. More precisely we have

$$\frac{\partial^2 s(t)}{\partial t\partial\overline{t}} \geq \frac{4}{\pi}\iint\limits_{R(t)}\left(\left|\frac{\partial^2 p_0(t,z)}{\partial\overline{t}\partial z}\right|^2+\left|\frac{\partial^2 p_1(t,z)}{\partial\overline{t}\partial z}\right|^2\right)dx\,dy \geq 0.$$

Similarly, $\hat{s}(t)$ defined in Step 1 is subharmonic on B. Thus $s(t)|\varphi(t)|^2$ *is subharmonic on B for any non-vanishing holomorphic function $\varphi(t)$ on B.* By the standard argument due to Hartogs, $\log s(t)$ is subharmonic on B. In fact, if it is not, we get a contradiction as follows: Assume that there exist $t_0 \in B$ and $B_0 = \{|t-t_0| < r\} \subset B$ such that $\frac{1}{2\pi}\int_0^{2\pi}\log s(t_0+re^{i\theta})\,d\theta < \log s(t_0)$. By using means of the Poisson integral, we construct a harmonic function $u(t)$ on B_0 with $u(t) = \log s(t)$ on ∂B_0. Then $u(t_0) < \log s(t_0)$. For the non-vanishing holomorphic function $\phi(t) := e^{-(u+iu^*)}$ on B_0, $|\phi(t)|s(t)$ is not subharmonic on B_0 because $|\phi(t)|s(t) = 1$ on ∂B_0 and $|\phi(t_0)|s(t_0) > 1$. This is a contradiction. ∎

REMARK 4.2. Let $\mathcal{R}$ be as above. Assume that there exists another section $\mathbf{b} := \{b(t) \in R(t) \mid t \in B\} \in \Gamma(B,\mathcal{R})$ with $\mathbf{a}\cap\mathbf{b} = \emptyset$. Then the L_i-principal function $q_i(t,z)$ $(i = 0,1)$ for $(R(t), a(t), b(t))$ with *two logarithmic poles* is determined such that $q_i = -\log|z-a(t)| + h_a(t,z)$ near $z = a(t)$ and $q_i = \log|z-b(t)| + \beta_i(t) + h_b(t,z)$ near $z = b(t)$, where h_a and h_b are harmonic with $h_a(t,a(t)) = h_b(t,b(t)) = 0$ and $\beta_i(t)$ is a real constant, and q_i satisfies the condition (L_i) on $\partial R(t)$. In [3, Lemma 1.3] and [5, Lemma 2.2], we showed variation formulas of the same type for $\beta_i(t)$ $(i = 0,1)$, respectively, and applied them to the variation of *harmonic spans* $h(t) := \beta_1(t) - \beta_0(t)$. In contrast to the formulas for L_i-constants $\alpha_i(t)$ and the Schiffer span $s(t)$ for $(R(t), a(t))$, those for $\beta_i(t)$ and $h(t)$ for $(R(t), a(t), b(t))$ do *not* depend on the choice of π-local coordinates. We note that $h(t)$ *is subharmonic on B* (*see* [5, Theorem 4.1]), *but $s(t)$ is moreover logarithmically subharmonic.*

As regards the harmonic span $h(t)$, we earlier studied certain *non-smooth variations* $\mathcal{R} : t \in B \to R(t)$ under pseudoconvexity. In [4, Theorem 1.3] we showed that for the variation $\mathcal{R}$ of (C2) type there exists a counterexample such that $h(t)$ is not subharmonic on B, while for the variation $\mathcal{R}$ of (C1) type, $h(t)$ is subharmonic on B:

THEOREM 4.3 ([5, Theorem 5.1]). *Let $\mathcal{R}$ satisfy* (1), (2) *and* (4) *of Main Theorem* 1.1. *Assume that there exist two sections $\mathbf{a}, \mathbf{b} \in \Gamma(B,\mathcal{R})$ with $\mathbf{a}\cap\mathbf{b} = \emptyset$. Let $h(t)$ be the harmonic span for $(R(t), a(t), b(t))$. Then $h(t)$ is subharmonic on B.*

By similar considerations to those for Theorem 4.3, using Lemma 4.1 we can show Theorem 1.4.

5. Proof of Main Theorem 1.1. From the assumption and Proposition 3.2(1), the Schiffer span $s(t)$ for $(R(t), a(t))$ is 0 on E. Since E is of positive logarithmic capacity, Theorem 1.4 and the suction principle of subharmonic functions imply $s(t) \equiv 0$ on B, and hence we obtain assertion (i).

For each $t \in B$, let $P_1(t, z)$ be a vertical slit mapping for $(R(t), a(t))$ with respect to the local coordinate $\{|z| < r\}$ such that

$$P_1(t, z) = \frac{1}{z} + A_{11}(t)z + A_{12}(t)z^2 + \cdots \quad \text{at } z = 0. \tag{5.1}$$

For the proof of (ii) it is enough to show that P_1 *is holomorphic as a function of the two complex variables* (t, z) *in* $\mathcal{R}$, by Proposition 3.2(2). Since P_1 is holomorphic for z in $R(t) \setminus \{0\}$ and each $R(t)$ is non-singular in $\mathcal{R}$, by analytic continuation it suffices to show that P_1 is holomorphic for (t, z) only in $\mathcal{V} = B \times \{|z| < r\}$, i.e., $A_{11}(t), A_{12}(t), \dots$ are holomorphic on B. We divide the proof into four steps.

Step 1. *The* L_1*-constant* $\mathrm{Re}\{A_{11}(t)\}$ *for* $(R(t), a(t))$ *is harmonic on* B.

By similar considerations to those for Theorem 1.4, using Lemma 2.2 we can show that $\mathrm{Re}\{A_{11}(t)\}$ is superharmonic and $\mathrm{Re}\{A_{01}(t)\}$ is subharmonic on B under the conditions of $\mathcal{R}$. It follows from Main Theorem 1.1(i) and Proposition 3.2(1) that $\mathrm{Re}\{A_{11}(t)\} = \mathrm{Re}\{A_{01}(t)\}$ on B, so that $\mathrm{Re}\{A_{11}(t)\}$ is harmonic on B.

Step 2. $A_{11}(t)$ *is holomorphic for* $t \in B$.

Let $\phi(t)$ be *any* non-vanishing holomorphic function on B and consider the transformation

$$T : (t, z) \in \mathcal{V} \mapsto (t, w) = (t, \phi(t)z) \in B \times \mathbb{C}_w.$$

We take a bidisk $\widehat{V} = B \times \{|w| < \hat{r}\} \subset T(\mathcal{V})$. Let $\widehat{P}_1(t, w)$ denote the vertical slit mapping for $(\widehat{R}(t), a(t))$ with respect to the local coordinate $\{|w| < \hat{r}\}$ such that

$$\widehat{P}_1(t, w) := \frac{1}{w} + \widehat{A}_{11}(t)w + \widehat{A}_{12}(t)w^2 + \cdots \quad \text{at } w = 0.$$

For the same reason as in Step 1, $\mathrm{Re}\{\widehat{A}_{11}(t)\}$ is harmonic on B. If we set $\check{P}_1(t, z) := \phi(t)\widehat{P}_1(t, \phi(t)z)$ on $R(t)$, $t \in B$, then

$$\check{P}_1(t, z) = \frac{1}{z} + (\widehat{A}_{11}(t)\phi(t)^2)z + (\widehat{A}_{12}(t)\phi(t)^3)z^2 + \cdots \quad \text{at } z = 0.$$

Since $P_1(t, z)$ is a univalent function on $R(t)$, and $R(t)$ is of class O_{AD}, Proposition 3.2(3) implies $\check{P}_1(t, z) = P_1(t, z)$, and hence $\widehat{A}_{11}(t) = A_{11}(t)\phi(t)^{-2}$. Thus $\mathrm{Re}\{A_{11}(t)\phi(t)^{-2}\}$ is harmonic on B. Since $\phi(t)$ is any non-vanishing holomorphic function, $A_{11}(t)$ is holomorphic for $t \in B$, as desired.

STEP 3. *The function*

$$\mathcal{A}(t,z) := 3\Big(\frac{\partial^2 P_1/\partial z^2}{\partial P_1/\partial z}(t,z)\Big)^2 - 2\frac{\partial^3 P_1/\partial z^3}{\partial P_1/\partial z}(t,z)$$

is holomorphic as a function of the two complex variables $(t,z) \in \mathcal{V}$.

Since each $\mathcal{A}(t,z)$, $t \in B$, is holomorphic for z in $V = \{|z| < r\}$, it is enough to show that, for any $b \in V$, $\mathcal{A}(t,b)$ is holomorphic for $t \in B$. To show this, we consider the constant section $\mathbf{b} := \{z = b \mid t \in B\}$. Putting $w = z - b$, the bidisk $\mathcal{V}_b := B \times \{|w| < r - |b|\}$ is a local coordinate for the section $\mathbf{b}$ of $\mathcal{R}$ over B. We have the vertical slit mapping $\widetilde{P}_1(t,w)$ for $(\widetilde{R}(t), b)$ such that

$$\widetilde{P}_1(t,w) = \frac{1}{w} + \widetilde{A}_{11}(t)w + \widetilde{A}_{12}(t)w^2 + \cdots \qquad \text{at } w = 0.$$

Then Step 2 implies that $\widetilde{A}_{11}(t)$ is holomorphic on B. We set $\widehat{P}_1(t,z) := \widetilde{P}_1(t, z-b)$, so that $\widehat{P}_1(t,z)$ is univalent on $R(t)$ and

$$\widehat{P}_1(t,z) = \frac{1}{z-b} + \widetilde{A}_{11}(t)(z-b) + \widetilde{A}_{12}(t)(z-b)^2 + \cdots \qquad \text{at } z = b.$$

By Proposition 3.2(3), $\widehat{P}_1(t,z)$ is identical with the vertical slit mapping $P_1^b(t,z)$ for $(R(t), b)$ defined in (3.2). By Lemma 3.3 we have $\widetilde{A}_{11}(t) = \mathcal{A}(t,b)$, $t \in B$. It follows that $\mathcal{A}(t,b)$ is holomorphic for $t \in B$. Thus $\mathcal{A}(t,z)$ is holomorphic on B, and Step 3 is proved.

STEP 4. $P_1(t,z)$ *is holomorphic for* $(t,z) \in \mathcal{V}$.

We simply put $\mathcal{V} = B \times V$ where $V = \{|z| < r\}$. We set

$$g(t,z) := \frac{\partial^2 P_1(t,z)}{\partial z^2}\Big/\frac{\partial P_1(t,z)}{\partial z} = \frac{\partial}{\partial z}\Big(\log \frac{\partial P_1(t,z)}{\partial z}\Big) \qquad \text{on } B \times V.$$

Then

$$\mathcal{A}(t,z) = g(t,z)^2 - 2\frac{\partial g(t,z)}{\partial z} \qquad \text{on } B \times V.$$

By (5.1) and a simple calculation we have

$$g(t,z) = \frac{-2}{z}\Big\{\frac{1 + A_{12}(t)z^3 + 3A_{13}(t)z^4 + \cdots}{1 - A_{11}(t)z^2 - 2A_{12}(t)z^3 + \cdots}\Big\} = \frac{-2}{z}\Big\{1 + \sum_{n=2}^{\infty} B_n(t)z^n\Big\},$$

where $B_2(t) = A_{11}(t)$, and $B_k(t) = F_k(A_{11}(t), \ldots, A_{1k-1}(t))$ for $k \geq 3$. Here, $F_k(X_1, \ldots, X_{k-1})$ is a polynomial of $X_1, \ldots, X_{k-1}$. Thus by straightforward calculation,

$$\mathcal{A}(t,z) = 4[3B_2(t) + 4B_3(t)z + \{B_2(t)^2 + 5B_4(t)\}z^2 + \cdots].$$

Since $\mathcal{A}(t,z)$ is holomorphic as a function of the two complex variables (t,z) in $B \times V$ by Step 3, and since $B_2(t) = A_{11}(t)$ is holomorphic for $t \in B$ by Step 2, by induction each $B_j(t)$ $(j = 2, 3, \ldots)$ is holomorphic on B.

Thus $g(t,z)$ itself is holomorphic as a function of (t,z) in $B\times V$, and so is $\log \frac{\partial P_1(t,z)}{\partial z}$. Since

$$\begin{aligned}\log \frac{\partial P_1}{\partial z} &= \log\Big(-\frac{1}{z^2}\Big)+\log\Big(1-\sum_{n=1}^{\infty} nA_{1n}(t)z^{n+1}\Big)\\ &= \log\Big(-\frac{1}{z^2}\Big)-\sum_{k=1}^{\infty} G_k(t)z^{k+1},\end{aligned}$$

where $G_k(t)=kA_{1k}(t)$ $(k=1,2)$, $G_k(t)=kA_{1k}(t)+\widetilde{F}_k(G_1(t),\dots,G_{k-1}(t))$ $(k\ge 3)$, and $\widetilde{F}_k(X_1,\dots,X_{k-1})$ is a polynomial of $X_1,\dots,X_{k-1}$, it follows that each $G_k(t)$ $(k=1,2,\dots)$ is holomorphic for $t\in B$, and so is each $A_{1k}(t)$ on B. Thus Step 4 is proved.

Thus, the proof of Main Theorem 1.1 is complete. ∎

6. Example for Lemma 2.2 and Lemma 4.1. For $t\in B=\{|t|<\rho\}$, let $D(t)=\{z\in\mathbb{C}_z \mid |z|<r(t)\}$ such that $\log r(t)$ is superharmonic on B. Then $\mathcal{D}=\bigcup_{t\in B}(t,D(t))$ is pseudoconvex in $B\times\mathbb{C}_z$. By use of the Joukowski transformation we consider the vertical (resp. horizontal) slit mapping P_1 (resp. P_0) for $(D(t),0)$:

$$P_1(t,z)=\frac{1}{z}-\frac{z}{r(t)^2} \quad\text{and}\quad P_0(t,z)=\frac{1}{z}+\frac{z}{r(t)^2} \quad \text{on } D(t). \tag{6.1}$$

Then P_1 (resp. P_0) maps $D(t)$ onto the slit domain $\mathbb{P}\setminus\big[-\frac{2i}{r(t)},\frac{2i}{r(t)}\big]$ (resp. $\mathbb{P}\setminus\big[\frac{-2}{r(t)},\frac{2}{r(t)}\big]$). The L_1-constant $\alpha_1(t)=-1/r(t)^2<0$ is superharmonic on B, while the L_0-constant $\alpha_0(t)=1/r(t)^2$ is subharmonic on B. By (6.1) the Schiffer span $s(t)$ for $(D(t),0)$ is $s(t)=2/r(t)^2>0$ on B. Hence, $\log s(t)=\log 2-2\log r(t)$ is subharmonic on B. We consider the maximizing function

$$M(t,z):=\frac{1}{2}\{P_1(t,z)+P_0(t,z)\}=\frac{1}{z},$$

and see that the complement of $M(t,D(t))$ in $\mathbb{P}_w$ is equal to $\{|w|\le 1/r(t)\}$. It is convex and its Euclidean area is $\pi/r(t)^2$, which is equal to $\pi s(t)/2$.

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