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Solutions for the p-order Feigenbaum's functional equation $h(g(x)) = g^p(h(x))$

by Min Zhang (Qingdao) and Jianguo Si (Jinan)

Abstract. This work deals with Feigenbaum's functional equation

$$\begin{cases} h(g(x)) = g^p(h(x)), \\ g(0) = 1, \quad -1 \le g(x) \le 1, \quad x \in [-1,1], \end{cases}$$

where $p \ge 2$ is an integer, g^p is the p-fold iteration of g, and h is a strictly monotone odd continuous function on $[-1,1]$ with $h(0) = 0$ and $|h(x)| < |x|$ ($x \in [-1,1]$, $x \ne 0$). Using a constructive method, we discuss the existence of continuous unimodal even solutions of the above equation.

1. Introduction. In 1978, Feigenbaum [F1], [F2] and independently Coullet and Tresser [CT] introduced the notion of renormalization for real dynamical systems. In 1992, Sullivan [S] proved the uniqueness of the fixed point for the period-doubling renormalization operator. This fixed point of renormalization satisfies a functional equation known as the Cvitanović–Feigenbaum equation:

$$(1.1) \qquad \begin{cases} g(x) = -\frac{1}{\lambda} g(g(-\lambda x)), & 0 < \lambda < 1, \\ g(0) = 1, \quad -1 \le g(x) \le 1, & x \in [-1,1]. \end{cases}$$

This equation and its solution play an important role in the theory initiated by Feigenbaum [F1], [F2]. However, in general, finding an exact solution of the above equation is not an easy task. This problem can be studied in classes of smooth functions or of continuous functions. The existence of smooth solutions for (1.1) has been established in [EW], [E1], [E2], [S] and references therein. As far as we know, continuous solutions of (1.1) have been relatively little researched. In this direction, we refer to [YZ] and [M].

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In the last years, a number of authors considered the more general equation

$$(1.2)\qquad \begin{cases} g(x) = -\frac{1}{\lambda} g^p(-\lambda x), & -1 < \lambda < 1, \\ g(0) = 1, \quad -1 \le g(x) \le 1, & x \in [-1,1], \end{cases}$$

where $p \ge 2$ is an integer and g^p is the p-fold iteration of g. It is easy to see that (1.1) is a special case of (1.2). When $p = 3$, Chen [C] gave a method of constructing even C^1 solutions of (1.2). For p large enough, Eckmann, Epstein and Wittwer [EEW] showed that there exists a solution of (1.2) similar to the function $f(x) = |1 - 2x^2|$. For any $p \ge 2$, Liao [L] proved that (1.2) has single-valley continuous solutions.

In the present paper, we will consider Feigenbaum's functional equations

$$(1.3)\qquad \begin{cases} h(g(x)) = g^p(h(x)), \\ g(0) = 1, \quad -1 \le g(x) \le 1, \quad x \in [-1,1]. \end{cases}$$

where h is a strictly monotone odd continuous function on $[-1,1]$ that satisfies $h(0) = 0$ and $|h(x)| < |x|$ ($x \in [-1,1]$, $x \neq 0$). We will deal with solutions to (1.3) in classes of continuous unimodal endomorphisms of the interval $[-1,1]$ by the constructive method.

It is worth pointing out the vast difference between the problem considered in this paper and the well-developed theory of fixed points of the same equation in classes of smooth functions, at least C^{1+BV} with a critical point of polynomial type. Solutions in the smooth theory are unique for large sets of functions, typically depending only on the degree of the critical point. Our main result (Main Theorem 3.1) shows that solutions in the continuous theory are far from unique. Actually, in Main Theorem 3.1 any initial map φ_0 can be extended to a fixed point. This implies that for continuous functions the Feigenbaum functional equation (1.3) is underdetermined, while in smooth classes for deep reasons it has unique solutions.

We replace (1.3) by the equation

$$(1.4)\qquad \begin{cases} f(\varphi(x)) = \varphi^p(f(x)), \\ \varphi(0) = 1, \quad 0 \le \varphi(x) \le 1, \quad x \in [0,1], \end{cases}$$

to study the existence of a single-valley continuous solution of (1.4), where f is a strictly increasing continuous function on $[0,1]$ with $f(0) = 0$ and $f(x) < x$ ($x \in (0,1]$). Obviously, if $h(x) = -\lambda x$, then (1.3) becomes (1.2).

The following result states relations between equations (1.3) and (1.4); it will be proved in Appendix A.

Theorem 1.1. *For any* $p \ge 2$, *the solutions of* (1.3) *and* (1.4) *have the following relations:*

(i) *If g is a continuous unimodal even solution of* (1.3), *then*

$$\varphi(x) = |g(x)|, \qquad x \in [0,1], \tag{1.5}$$

is a single-valley continuous solution of (1.4), *where* $f(x) = |h(x)|$ *for* $x \in [0,1]$.

(ii) *If φ is a single-valley continuous solution of* (1.4), *then φ has the minimum* 0 *at a point α in* $(0,1)$ *and*

$$g(x) = \operatorname{sgn}(\alpha - |x|)\varphi(|x|), \qquad x \in [-1,1], \tag{1.6}$$

is a continuous unimodal even solution of (1.3), *where*

$$h(x) = \begin{cases} \operatorname{sgn}(\alpha - \varphi^{p-2}(1))f(x), & x \in [0,1], \\ \operatorname{sgn}(\varphi^{p-2}(1) - \alpha)f(-x), & x \in [-1,0). \end{cases}$$

2. Basic definitions and lemmas. In this section, we will give some characterizations of single-valley continuous solutions of (1.4); they will be proved in Appendix A.

Definition 2.1. We call φ a *single-valley-extended continuous solution* of (1.4) if (1) φ is a continuous solution of (1.4); (2) there exists $\alpha \in (f(1),1)$ such that φ is strictly decreasing on $[f(1),\alpha]$ and strictly increasing on $[\alpha,1]$.

Definition 2.2. We call φ a *single-valley continuous solution* of (1.4) if (1) φ is a single-valley-extended continuous solution of (1.4); (2) there exists $\alpha \in (f(1),1)$ such that φ is strictly decreasing on $[0,\alpha]$ and strictly increasing on $[\alpha,1]$.

Lemma 2.3. *Suppose that φ is a single-valley continuous solution of* (1.4) *and* $\varphi(\alpha) = 0$. *Then:*

(i) 0 *is a recurrent but not periodic point of* φ,

(ii) $\varphi(x)$ *has a unique fixed point* $\beta = \varphi(\beta)$ *in* $[0,1]$, *and*

$$\varphi^{p-1}(1) = f(1) = \lambda < \beta < \alpha. \tag{2.1}$$

(iii) *If* $x \in [0,\lambda]$ *and* $0 \le i \le p-1$, *then* $\varphi^i(x) = \alpha$ *if and only if* $x = f(\alpha)$ *and* $i = p-1$.

(iv) *If* $1 \le i \le p-1$, *then* $\varphi^i(x) > \lambda$ *for all* $x \in [0, f(\alpha)]$, *and* $\varphi^i(x) > f(\alpha)$ *for all* $x \in (f(\alpha), \lambda]$.

(v) *If* $1 \le i \le p-1$, *then φ has no periodic point of period i on* $[0,\lambda]$.

Lemma 2.4. *Suppose that φ is a single-valley continuous solution of* (1.4). *Let* $J = [0,\lambda]$, $J_0 = \varphi(J)$ *and* $J_i = \varphi^i(J_0)$. *Then:*

(i) $\varphi^i : J_0 \to J_i$ *is a homeomorphism for all* $i = 0, 1, \ldots, p-2$.

(ii) $J_0, J_1, \ldots, J_{p-2} \subset (\lambda, 1]$ *are pairwise disjoint.*

LEMMA 2.5. *Suppose that φ is a single-valley continuous solution of* (1.4). *Then the equation $\varphi^{p-1}(x) = f(x)$ has only one solution $x = 1$ in* $(\varphi(f(\alpha)), 1]$.

LEMMA 2.6. *Let φ_1, φ_2 be two single-valley continuous solutions of* (1.4). *If*

$$\varphi_1(x) = \varphi_2(x), \qquad x \in [\lambda, 1],$$

then $\varphi_1(x) = \varphi_2(x)$ on $[0, 1]$.

3. Constructive method of solution. In this section, we will prove constructively the existence of single-valley continuous solutions of (1.4).

MAIN THEOREM 3.1. *Fix a strictly increasing continuous function f on* $[0, 1]$ *with $f(0) = 0$ and $f(x) < x$ ($x \in (0, 1]$). Denote $f(1) = \lambda$. Suppose that φ_0 is a continuous function on $[\lambda, 1]$ and satisfies the following conditions:*

(i) *there exists an $\alpha \in (\lambda, 1)$ such that $\varphi_0(\alpha) = 0$ and φ_0 is strictly decreasing on $[\lambda, \alpha]$ and strictly increasing on $[\alpha, 1]$,*
(ii) *$\varphi_0^{p-1}(1) = f(1) = \lambda$ and $\varphi_0^p(\lambda) = f(\varphi_0(1))$,*
(iii) *if $J_0 = [\varphi_0(\lambda), 1]$ and $J_i = \varphi_0^i(J_0)$, then*
   (1) *$J_0, J_1, \ldots, J_{p-2} \subset (\lambda, 1]$ are pairwise disjoint,*
   (2) *$\varphi_0^i : J_0 \to J_i$ is a homeomorphism for all $i = 0, 1, \ldots, p-2$,*
   (3) *α is in the interior of J_{p-2},*
(iv) *the equation $\varphi_0^{p-1}(x) = f(x)$ has only one solution $x = 1$ in $(\alpha_0, 1]$, where $\alpha_0 \in J_0$ with $\varphi_0^{p-1}(\alpha_0) = 0$.*

Then there exists a unique single-valley continuous function φ satisfying the equation

$$\begin{cases} f(\varphi(x)) = \varphi^p(f(x)), & x \in [0, 1], \\ \varphi(x) = \varphi_0(x), & x \in [\lambda, 1]. \end{cases} \tag{3.1}$$

Conversely, if φ_0 is the restriction to $[\lambda, 1]$ of a single-valley continuous solution to (1.4), *then the above conditions* (i)–(iv) *must hold.*

Proof. Suppose that φ_0 satisfies (i)–(iv). Define

$$\psi_+ = \varphi_0^{p-1}|_{[\alpha_0, 1]}, \qquad \psi_- = \varphi_0^{p-1}|_{[\varphi_0(\lambda), \alpha_0]}.$$

By (iii) and (iv), ψ_+ and ψ_- are both homeomorphisms. Since

$$\psi_+(\alpha_0) = \varphi_0^{p-1}(\alpha_0) = 0 \le \psi_+(1), \qquad \psi_-(\alpha_0) = \varphi_0^{p-1}(\alpha_0) = 0 \le \psi_-(\varphi_0(\lambda)),$$

ψ_+ is strictly increasing and ψ_- is strictly decreasing. It is trivial that $\{f^k(1)\}$ is decreasing and $\lim_{k\to\infty} f^k(1) = 0$. Let

$$\Delta_k = [f^{k+1}(1), f^k(1)], \qquad k = 0, 1, 2, \ldots. \tag{3.2}$$

Then

$$[0,1] = \bigcup_{k=0}^{\infty} \Delta_k.$$

We define φ on Δ_k by induction as follows:

Obviously, $\varphi = \varphi_0$ is well defined on Δ_0. For $x \in \Delta_1 = [f^2(1), f(1)]$, we set

$$\varphi_1(x) = \begin{cases} \psi_+^{-1}(f(\varphi_0(f^{-1}(x)))), & x \in [f^2(1), f(\alpha)], \\ \psi_-^{-1}(f(\varphi_0(f^{-1}(x)))), & x \in [f(\alpha), f(1)]. \end{cases} \tag{3.3}$$

Trivially, φ_1 is a strictly decreasing continuous function on Δ_1. For $x \in \Delta_0$ we have

$$\varphi_0^{p-1}(\varphi_1(f(x))) = f(\varphi_0(x)). \tag{3.4}$$

By (ii), we get

$$\psi_-(\varphi_0(\lambda)) = \varphi_0^{p-1}(\varphi_0(\lambda)) = \varphi_0^p(\lambda) = f(\varphi_0(1)).$$

Letting $x = f(1)$ in (3.3) yields

$$\varphi_1(f(1)) = \psi_-^{-1}(f(\varphi_0(1))) = \varphi_0(\lambda) = \varphi_0(f(1)), \tag{3.5}$$

i.e. φ_0 and φ_1 have the same value at the common endpoint of Δ_0 and Δ_1. Suppose that φ is well defined as a strictly decreasing continuous function φ_k on Δ_k for all $k \le m$, where $m \ge 1$ is a certain integer. Let

$$\varphi_{m+1}(x) = \psi_+^{-1}(f(\varphi_m(f^{-1}(x)))), \quad x \in \Delta_{m+1}. \tag{3.6}$$

Then φ is well defined as a strictly decreasing continuous function φ_k on Δ_k for all $k \ge 1$, and for $x \in \Delta_k$ we have

$$\psi_+(\varphi_{k+1}(f(x))) = f(\varphi_k(x)). \tag{3.7}$$

For $k = 1, \ldots, m$, where $m \ge 1$ is a certain integer, we suppose that

$$\varphi_k(f^k(1)) = \varphi_{k-1}(f^k(1)). \tag{3.8}$$

Let $x = f^{m+1}(1)$ in (3.6). Then we have

$$\begin{aligned} \varphi_{m+1}(f^{m+1}(1)) &= \psi_+^{-1}(f(\varphi_m(f^m(1)))) = \psi_+^{-1}(f(\varphi_{m-1}(f^m(1)))) \\ &= \varphi_m(f^{m+1}(1)), \end{aligned} \tag{3.9}$$

i.e. φ_k and φ_{k+1} have the same value at the common endpoint of Δ_k and Δ_{k+1} $(k = 1, 2, \ldots)$. Thus, we can let

$$\varphi(x) = \begin{cases} 1, & x = 0, \\ \varphi_k(x), & x \in \Delta_k. \end{cases} \tag{3.10}$$

Since φ_k is strictly decreasing continuous on Δ_k $(k \ge 1)$ and (3.5), (3.8) and (3.9) hold, we see that φ is a single-valley continuous function on $(0, 1]$.

Now, we prove that φ is continuous at $x = 0$. Trivially, $\{f^k(\alpha)\}$ is strictly decreasing and $\lim_{k\to\infty} f^k(\alpha) = 0$. Since φ is strictly decreasing in $(0, \alpha]$, the sequence $\{\varphi_k(f^k(\alpha))\}_{k=2}^{\infty}$ is strictly increasing in $[\alpha, 1]$. Let

$$\lim_{k\to\infty} \varphi_k(f^k(\alpha)) = \gamma.$$

Then $\gamma \in [\alpha, 1]$. From (3.7), we have $\psi_+(\varphi_{k+1}(f^{k+1}(\alpha))) = f(\varphi_k(f^k(\alpha)))$, i.e., $\varphi_0^{p-1}(\varphi_{k+1}(f^{k+1}(\alpha))) = f(\varphi_k(f^k(\alpha)))$. Letting $k \to \infty$, we get $\varphi_0^{p-1}(\gamma) = f(\gamma)$. By condition (iv), we know $\gamma = 1 = \varphi(0)$. This proves that φ is continuous at $x = 0$. Thus, φ is a single-valley continuous function on $[0, 1]$. We see that φ satisfies (3.1) by (3.4) and (3.7), and φ is unique by Lemma 2.6.

Obviously, if φ_0 is the restriction to $[\lambda, 1]$ of a single-valley continuous solution to (1.4), then conditions (i)–(iv) must hold by the lemmas in Section 2. ■

Example 3.2. Let $\varphi_0 : [1/4, 1] \to [0, 1]$ be defined by

$$\varphi_0(x) = \begin{cases} -\frac{3+\sqrt{7}}{4}x + \frac{9+3\sqrt{7}}{16}, & 1/4 \le x \le 3/4, \\ x - 3/4, & 3/4 \le x \le 1. \end{cases}$$

Obviously, φ_0 satisfies the conditions of Main Theorem 3.1 with $f(x) = x/4$ and $\lambda = f(1) = 1/4$, $\alpha = 3/4$. Hence it is the restriction to $[1/4, 1]$ of a single-valley-extended continuous solution φ to (1.4). Since φ_0 has the

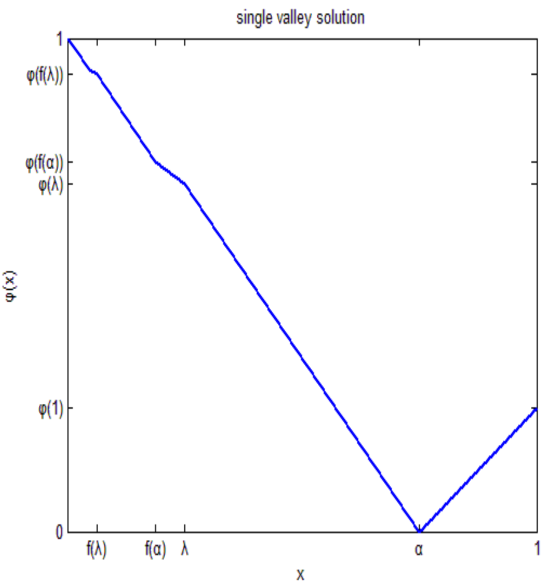

Fig. 1. The graph of single-valley solution

minimum point $\alpha = 3/4$ and $1/4 < \varphi_0(1/4) = (3+\sqrt{7})/8 < 3/4$, φ is a single-valley continuous solution. Its graph is depicted in Figure 1.

EXAMPLE 3.3. Let $\varphi_0 : [1/4, 1] \to [0, 1]$ be defined by

$$\varphi_0(x) = \begin{cases} -\frac{6+\sqrt{34}}{8}x + \frac{18+3\sqrt{34}}{32}, & 1/4 \le x \le 3/4, \\ x - 3/4, & 3/4 \le x \le 1, \end{cases}$$

Obviously, φ_0 satisfies the conditions of Main Theorem 3.1 with $f(x) = x^2/4$ and $\lambda = f(1) = 1/4$, $\alpha = 3/4$. Hence it is the restriction to $[1/4, 1]$ of a single-valley-extended continuous solution φ to (1.4). Since φ_0 has the minimum point $\alpha = 3/4$ and $1/4 < \varphi_0(1/4) = (6+\sqrt{34})/16 < 3/4$, φ is a single-valley continuous solution. Its graph is similar to Figure 1.

Appendix A

Proof of Theorem 1.1. (i) Obviously, φ is single-valley continuous. Firstly, we will prove that

$$\varphi^i(f(x)) = |g^i(h(x))|, \quad i = 1, \dots, p. \tag{A.1}$$

For $i = 1$, by (1.5) and since g is even, we get

$$\varphi(f(x)) = |g(f(x))| = |g(|h(x)|)| = |g(h(x))|,$$

i.e., (A.1) holds for $i = 1$. Suppose that it holds for $i = k$, where k is a certain integer. Since (1.5) holds and g is even, we have

$$\varphi^{k+1}(f(x)) = \varphi(\varphi^k(f(x))) = \varphi(|g^k(h(x))|) = |g(|g^k(h(x))|)| = |g^{k+1}(h(x))|,$$

i.e., (A.1) holds for $i = k+1$. Thus, (A.1) is proved by induction.

By (1.5) and since $f(x) = |h(x)|$ and h is odd, we have

$$f(\varphi(x)) = f(|g(x)|) = |h(|g(x)|)| = |h(g(x))|.$$

Since $g(x)$ satisfies (1.3), we get

$$|h(g(x))| = |g^p(h(x))|.$$

From (A.1) ($i = p$),

$$f(\varphi(x)) = \varphi^p(f(x)),$$

i.e., φ satisfies (1.4).

(ii) Suppose that φ has the minimum at a point α. By (1.4) we have

$$f(\varphi(\alpha)) = \varphi^p(f(\alpha)) \ge \varphi(\alpha).$$

Since $f(0) = 0$ and $f(x) < x$ ($x \in (0, 1]$), we get

$$\varphi(\alpha) = 0.$$

Trivially, g is a continuous unimodal even function. We now prove that

$$g^i(h(x)) = \operatorname{sgn}(\alpha - \varphi^{i-1}(f(|x|)))\varphi^i(f(|x|)), \quad i = 1, \dots, p. \tag{A.2}$$

For $i = 1$, in view of (1.6) and the definition of $h(x)$, we get

$$g(h(x)) = \mathrm{sgn}(\alpha - |h(x)|)\varphi(|h(x)|)| = \mathrm{sgn}(\alpha - f(|x|))\varphi(f(|x|)),$$

i.e., (A.2) holds for $i = 1$. Suppose that it holds for $i = k$, where k is a certain integer. From (1.6) and the definition of $h(x)$, we have

$$\begin{aligned} g^{k+1}(h(x)) = g(g^k(h(x))) &= g\big(\mathrm{sgn}(\alpha - \varphi^{k-1}(f(|x|)))\varphi^k(f(|x|))\big) \\ &= \mathrm{sgn}\big(\alpha - \big|\mathrm{sgn}(\alpha - \varphi^{k-1}(f(|x|)))\varphi^k(f(|x|))\big|\big) \\ &\quad \cdot \varphi\big(\big|\mathrm{sgn}(\alpha - \varphi^{k-1}(f(|x|)))\varphi^k(f(|x|))\big|\big) \\ &= \mathrm{sgn}\big(\alpha - \varphi^k(f(|x|))\big)\varphi^{k+1}(f(|x|)), \end{aligned}$$

i.e., (A.2) holds for $i = k + 1$. Thus, (A.2) is proved by induction.

Finally, we will prove that g satisfies (1.3) by considering the following two cases.

Case 1: $\alpha - \varphi^{p-2}(1) > 0$. Then by the definition of $h(x)$, we have

$$h(x) = \begin{cases} f(x), & x \in [0, 1], \\ -f(-x), & x \in [-1, 0). \end{cases}$$

By Lemma 2.4(i), φ^{p-1} is strictly monotone on $[0, f(1)]$. From $\varphi^{p-1}(0) = \varphi^{p-2}(1) < \alpha = \varphi^{p-1}(f(\alpha))$, it follows that φ^{p-1} is strictly increasing on $[0, f(1)]$.

Case 1.1: $|x| \le \alpha$. Since f is strictly increasing, we have $f(|x|) \le f(\alpha)$. Moreover, $\alpha = \varphi^{p-1}(f(\alpha)) \ge \varphi^{p-1}(f(|x|))$. From (A.2) ($i = p$), (1.4), (1.6) and the definition of $h(x)$, we have

$$g^p(h(x)) = \varphi^p(f(|x|)) = f(\varphi(|x|)) = f(g(x)) = h(g(x)).$$

Case 1.2: $|x| > \alpha$. Since f is strictly increasing, we have $f(|x|) > f(\alpha)$. Moreover, $\alpha = \varphi^{p-1}(f(\alpha)) < \varphi^{p-1}(f(|x|))$. From (A.2) ($i = p$), (1.4), (1.6) and the definition of $h(x)$, we have

$$g^p(h(x)) = -\varphi^p(f(|x|)) = -f(\varphi(|x|)) = -f(-g(x)) = h(g(x)),$$

i.e., $g(x)$ satisfies (1.3).

Case 2: $\alpha - \varphi^{p-2}(1) < 0$. Then by the definition of $h(x)$,

$$h(x) = \begin{cases} -f(x), & x \in [0, 1], \\ f(-x), & x \in [-1, 0). \end{cases}$$

The rest of the proof is similar to that in Case 1, and we omit it. ∎

Proof of Lemma 2.3. (i) We prove that for all $n \ge 0$ and each $x \in [0, 1]$,

$$f^n(\varphi(x)) = \varphi^{p^n}(f^n(x)). \tag{A.3}$$

Obviously, (A.3) holds for $n = 1$ by (1.4). Suppose that it holds for $n \le k$, where k is a certain integer. Then, by induction and (1.4), we have

$$\begin{aligned}\varphi^{p^{k+1}}(f^{k+1}(x)) &= (\varphi^{p^k})^p(f^{k+1}(x)) = (\varphi^{p^k})^{p-1} \circ \varphi^{p^k}(f^{k+1}(x))\\ &= (\varphi^{p^k})^{p-1}(f^k(\varphi(f(x)))) = (\varphi^{p^k})^{p-2} \circ \varphi^{p^k}(f^k(\varphi(f(x))))\\ &= (\varphi^{p^k})^{p-2}(f^k(\varphi^2(f(x)))) = \cdots = (\varphi^{p^k})^{p-i}(f^k(\varphi^i(f(x))))\\ &= \cdots = f^k(\varphi^p(f(x))) = f^k(f(\varphi(x))) = f^{k+1}(\varphi(x)),\end{aligned}$$

i.e., (A.3) holds for $n = k + 1$. Thus, (A.3) is proved by induction.

Letting $x = 0$ in (A.3), we have

$$f^n(1) = f^n(\varphi(0)) = \varphi^{p^n}(f^n(0)) = \varphi^{p^n}(0). \tag{A.4}$$

Trivially $\{f^n(1)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(1) = 0$. Hence

$$\lim_{n\to\infty} \varphi^{p^n}(0) = \lim_{n\to\infty} f^n(1) = 0. \tag{A.5}$$

i.e., 0 is a recurrent but not periodic point of φ.

(ii) Let $x = 0$ in (1.4). Then

$$f(1) = f(\varphi(0)) = \varphi^p(f(0)) = \varphi^p(0) = \varphi^{p-1}(1).$$

Let $\varphi^{p-1}(1) = f(1) = \lambda$. Firstly, we prove that $\beta < \alpha$. Since $\varphi(0) = 1$, $\varphi(\alpha) = 0$ and φ is strictly decreasing in $[0, \alpha]$, it follows that φ has a unique fixed point in $[0, \alpha]$. Suppose that φ has another fixed point q; then $q \in (\alpha, 1]$. By (A.5), we have $q \ne 1$. Thus, $q \in (\alpha, 1)$. Since φ is strictly increasing in $[\alpha, 1]$, it follows that $q = \varphi(q) < \varphi(1)$. By induction, $q = \varphi^m(q) < \varphi^m(1)$ for all $m \ge 0$. In particular,

$$q = \varphi^{p^n-1}(q) < \varphi^{p^n-1}(1) = \varphi^{p^n-1}(\varphi(0)) = \varphi^{p^n}(0).$$

This contradicts (A.5). Thus, we have proved that φ has a unique fixed point β in $[0, 1]$ and $\beta < \alpha$.

Secondly, we prove that $\lambda < \beta$. Suppose that $\lambda \ge \beta$. Since $0 \le f(x) \le f(1) = \lambda$, there exists $\gamma \in [0, 1]$ such that $f(\gamma) = \beta$, i.e., $\gamma = f^{-1}(\beta)$. By (1.4), we have

$$\beta = \varphi^p(\beta) = \varphi^p(f(\gamma)) = f(\varphi(\gamma)) = f(\varphi(f^{-1}(\beta))).$$

Thus, $f^{-1}(\beta) = \varphi(f^{-1}(\beta))$. And since φ has a unique fixed point β in $[0, 1]$, we have $f^{-1}(\beta) = \beta$. Thus, $\beta = f(\beta)$. As $f(0) = 0$ and $f(x) < x$ ($x \in (0, 1]$), we deduce that $\beta = 0$. This contradicts $\varphi(0) = 1$. Thus, we have proved that $\lambda < \beta$.

(iii) Firstly, we prove the sufficiency. By (1.4), we have $0 = f(\varphi(\alpha)) = \varphi^p(f(\alpha))$. Since α is the unique minimum point of φ, it follows that $\varphi^{p-1}(f(\alpha)) = \alpha$. Thus the sufficiency is proved.

To prove the necessity, suppose that $\varphi^i(x) = \alpha$ for some $x \in [0, \lambda]$ and $0 \le i \le p - 1$. Then $x \ne 0$ and α is not a periodic point of φ by (i).

We claim that $x \neq \lambda$. Suppose that $x = \lambda$. Since

$$\varphi^{p+1}(\alpha) = \varphi^p(\varphi(\alpha)) = \varphi^p(0) = \varphi^p(f(0)) = f(\varphi(0)) = f(1) = \lambda,$$

we have

$$\alpha = \varphi^i(x) = \varphi^i(\lambda) = \varphi^{i+p+1}(\alpha).$$

This contradicts that α is not a periodic point. Thus $x \neq \lambda$.

We next prove that $x \notin (f(\alpha), \lambda)$. Suppose that $x \in (f(\alpha), \lambda)$. Then φ^{i+1} is not strictly monotone in $(f(\alpha), \lambda)$. This contradicts φ^p being strictly monotone in $(f(\alpha), \lambda)$ since (1.4), and $f(\varphi(x))$ is strictly monotone in $(\alpha, 1)$. Thus $x \notin (f(\alpha), \lambda)$. By a similar argument we find that $x \notin (0, f(\alpha))$. So $x = f(\alpha)$. Since α is not a periodic point, we know that $\varphi^j(x) \neq \alpha$ for all $j \neq i$. Hence, $x = f(\alpha)$ and $i = p - 1$. Thus the necessity is proved.

(iv) Firstly, we claim that

$$\varphi^i(f(\alpha)) > \lambda, \quad \forall 1 \leq i \leq p-1. \tag{A.6}$$

Suppose that there exists $1 \leq j \leq p-1$ such that $\varphi^j(f(\alpha)) = x \leq \lambda$. Then

$$\varphi^{p-1-j}(x) = \varphi^{p-1-j}(\varphi^j(f(\alpha))) = \varphi^{p-1}(f(\alpha)) = \alpha$$

by (iii). This contradicts (iii). Thus (A.6) is proved.

Secondly, we prove that

$$\varphi^i(x) > \lambda, \quad \forall x \in [0, f(\alpha)],\ 1 \leq i \leq p-1. \tag{A.7}$$

Trivially, $\varphi^i : [0, f(\alpha)] \to \varphi^i([0, f(\alpha)])$ is a homeomorphism by (iii). From (A.6) it suffices to show that $\varphi^i(0) > \lambda$. Suppose that there exists $1 \leq j \leq p-1$ such that $\varphi^j(0) = x \leq \lambda$. Then

$$\varphi^{p-j}(x) = \varphi^{p-j}(\varphi^j(0)) = \varphi^p(0) = \varphi^p(f(0)) = f(\varphi(0)) = \lambda.$$

Thus

$$\varphi^j(\lambda) = \varphi^j(\varphi^{p-j}(x)) = \varphi^p(x) = f(\varphi(f^{-1}(x))) \leq f(1) = \lambda.$$

Since φ^j is also a homeomorphism on $[0, \lambda]$ by (iii), we have $\varphi^j(f(\alpha)) \leq \lambda$. This contradicts (A.6). Thus (A.7) is proved.

Thirdly, we prove that

$$\varphi^i(x) > f(\alpha), \quad \forall x \in (f(\alpha), \lambda],\ 1 \leq i \leq p-1. \tag{A.8}$$

Suppose that there exist $1 \leq j \leq p-1$ and $x \in (f(\alpha), \lambda]$ such that $\varphi^j(x) = y \leq f(\alpha)$. Then

$$\varphi^{p-j}(y) = \varphi^{p-j}(\varphi^j(x)) = \varphi^p(x) = f(\varphi((f^{-1}(x)))) \leq f(1) = \lambda.$$

This contradicts (A.7). Thus (A.8) is proved.

(v) Suppose that there exist $1 \leq j \leq p-1$ and $x \in [0, \lambda]$ such that x is a periodic point of φ with period j, i.e., $\varphi^j(x) = x$. Then $x \in (f(\alpha), \lambda]$ by (A.7). Let

$$y = \min\{x, \varphi(x), \dots, \varphi^{j-1}(x)\}.$$

Then $y \in (f(\alpha), \lambda]$ by (A.8). From $f^{-1}(y) \in (\alpha, 1]$, $\varphi(\alpha) = 0 < \alpha$ and (ii), we have $\varphi(f^{-1}(y)) < f^{-1}(y)$. Thus, by (1.4) and since f is strictly increasing, we know that

$$\varphi^p(y) = f(\varphi(f^{-1}(y))) < f(f^{-1}(y)) = y.$$

This contradicts the definition of y. Thus we have proved that φ has no periodic point of period i on $[0, \lambda]$.

Proof of Lemma 2.4. (i) For all $i = 0, 1, \ldots, p-2$, $\varphi^{i+1} : J \to J_i$ is a homeomorphism by Lemma 2.1(iii). Thus $\varphi^i : J_0 \to J_i$ is also a homeomorphism.

(ii) Firstly, we prove that for all $i = 0, 1, \ldots, p-2$, we have $J_i \subset (\lambda, 1]$, i.e., $J_i \cap J = \emptyset$. We claim that $\varphi^{i+1}(\lambda) > \lambda$. Suppose that there exists $1 \le j \le p-1$ such that $\varphi^j(\lambda) \le \lambda$. Then by (A.6),

$$\varphi^j(f(\alpha)) > \lambda > f(\alpha).$$

Thus φ^j has a fixed point in $[f(\alpha), \lambda]$. This contradicts Lemma 2.1(v). Hence $\varphi^{i+1}(\lambda) > \lambda$. Since $\varphi^{i+1} : J \to J_i$ is a homeomorphism and (A.7) holds, we conclude that $J_i \cap J = \emptyset$.

Secondly, we prove that the J_i are pairwise disjoint for all $0 \le i \le p-2$. Suppose that there exist $0 \le i < j \le p-2$ such that $J_i \cap J_j = J_{ij} \neq \emptyset$. Let $y \in J_{ij}$. Then there exist $x_i, x_j \in [0, \lambda]$ such that $\varphi^{i+1}(x_i) = y = \varphi^{j+1}(x_j)$. Thus we have

$$\begin{aligned}\varphi^{p-j+i}(x_i) &= \varphi^{p-1-j}(\varphi^{i+1}(x_i)) = \varphi^{p-1-j}(\varphi^{j+1}(x_j)) \\ &= \varphi^p(x_j) = f(\varphi(f^{-1}(x_j))) \le f(1) = \lambda.\end{aligned}$$

This contradicts $J_i \subset (\lambda, 1]$. Thus we have proved that $J_0, J_1, \ldots, J_{p-2}$ are pairwise disjoint.

Proof of Lemma 2.5. Obviously, $x = 1$ is a solution of the equation $\varphi^{p-1}(x) = f(x)$ by (1.4). Suppose that $x = x_0$ is an arbitrary solution of this equation, i.e., $\varphi^{p-1}(x_0) = f(x_0)$. Since $(\varphi(f(\alpha)), 1] \subset \varphi([0, f(\alpha)])$, there exists $y_0 \in [0, f(\alpha)]$ such that $\varphi(y_0) = x_0$. Thus, $\varphi^{p-1}(\varphi(y_0)) = f(x_0)$. By (1.4), we get $f(\varphi(f^{-1}(y_0))) = \varphi^{p-1}(\varphi(y_0)) = f(x_0)$. It follows that $\varphi(f^{-1}(y_0)) = x_0$ since f is strictly monotone. As $f^{-1}(y_0) \in [0, \alpha]$ and φ is strictly decreasing in $[0, \alpha]$, we have $y_0 = f^{-1}(y_0)$, i.e., $f(y_0) = y_0$. From $f(0) = 0$, $f(x) < x$ ($x \in (0, 1]$), we have $y_0 = 0$. Consequently, $x_0 = \varphi(y_0) = \varphi(0) = 1$.

Proof of Lemma 2.6. There exist $\alpha, \beta \in (\lambda, 1)$ such that $\varphi_i(\alpha) = 0$ and $\varphi_i(\beta) = \beta$ ($i = 1, 2$) by (2.1). Denote $\varphi_0(x) = \varphi_1(x) = \varphi_2(x)$ ($x \in [\lambda, 1]$), $\alpha_1 = \varphi_1(f(\alpha)), \alpha_2 = \varphi_2(f(\alpha))$.

We now prove that $\alpha_1 = \alpha_2$. Trivially,

$$\alpha_1 > \varphi_1(f(1)) = \varphi_0(\lambda) > \varphi_0(\beta) = \beta > \lambda,$$

by (2.1). Similarly, $\alpha_2 > \varphi_0(\lambda) > \lambda$. By Lemmas 2.4 and 2.3(iii), we have

$$\varphi_0^{p-1}(\alpha_1) = \varphi_1^{p-1}(\alpha_1) = \varphi_1^p(f(\alpha)) = 0.$$

Similarly, $\varphi_0^{p-1}(\alpha_2) = 0$. Since φ_0^{p-1} has a unique zero in $[\varphi_0(\lambda), 1]$, we conclude that $\alpha_1 = \alpha_2$.

Let $\alpha_0 = \alpha_1 = \alpha_2$. Then $\alpha_0 \in [\varphi_0(\lambda), 1]$ and $\varphi_0^{p-1}(\alpha_0) = 0$. Define

$$\psi_+ = \varphi_0^{p-1}|_{[\alpha_0,1]}, \qquad \psi_- = \varphi_0^{p-1}|_{[\varphi_0(\lambda),\alpha_0]}. \tag{A.9}$$

By Lemmas 2.3(iii) and 2.4, ψ_+ and ψ_- are both homeomorphisms. Since

$$\psi_+(\alpha_0) = 0 \le \psi_+(1), \qquad \psi_-(\alpha_0) = 0 \le \psi_-(\varphi_0(\lambda)),$$

ψ_+ is strictly increasing and ψ_- is strictly decreasing. Trivially, $\{f^k(1)\}$ is decreasing and $\lim_{k\to\infty} f^k(1) = 0$. Let

$$\Delta_k = [f^{k+1}(1), f^k(1)], \qquad k = 0, 1, 2, \ldots.$$

Then $[0,1] = \bigcup_{k=0}^\infty \Delta_k$.

We now prove $\varphi_1(x) = \varphi_2(x)$ on Δ_k by induction.

Obviously, $\varphi_1(x) = \varphi_2(x)$ on Δ_0. Suppose that $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \le m$, where $m \ge 0$ is a certain integer. Let

$$\varphi(x) = \varphi_1(x) = \varphi_2(x), \qquad x \in [f^{m+1}(1), 1].$$

If $x \le f(1) = \lambda$, then $\varphi_i(x) \ge \varphi_i(\lambda) = \varphi_0(\lambda) > \varphi_0(\beta) = \beta > \lambda$. Thus by (1.4) we have

$$f(\varphi(f^{-1}(x))) = f(\varphi_i(f^{-1}(x))) = \varphi_i^{p-1}(\varphi_i(x)) = \varphi_0^{p-1}(\varphi_i(x)) \qquad (i = 1, 2,\ x \in \Delta_{m+1}). \tag{A.10}$$

By (A.9), if $\varphi_i(x) \in [\alpha_0, 1]$, then (A.10) is equivalent to

$$\varphi_i(x) = \psi_+^{-1}(f(\varphi(f^{-1}(x)))), \qquad x \in \Delta_{m+1}. \tag{A.11}$$

And if $\varphi_i(x) \in [\varphi_0(\lambda), \alpha_0]$, then (A.10) is equivalent to

$$\varphi_i(x) = \psi_-^{-1}(f(\varphi(f^{-1}(x)))), \qquad x \in \Delta_{m+1}. \tag{A.12}$$

We claim that $\varphi_1(x)$ and $\varphi_2(x)$ either both satisfy (A.11) or both satisfy (A.12). Suppose that there exists $x_0 \in \Delta_{m+1}$ such that

$$\varphi_1(x_0) = \psi_+^{-1}(f(\varphi(f^{-1}(x_0)))), \qquad \varphi_2(x_0) = \psi_-^{-1}(f(\varphi(f^{-1}(x_0)))).$$

Then there exist $x_1 > x_0$ and $x_1 \in \Delta_{m+1}$ such that

$$\varphi_1(x_0) > \varphi_1(x_1) > \alpha_0 > \varphi_2(x_0) > \varphi_2(x_1).$$

Since φ_0^{p-1} is strictly monotone on $[\varphi_0(\lambda), \alpha_0]$ and on $[\alpha_0, 1]$, by (A.10) we have

$$\varphi_0^{p-1}(\varphi_1(x_1)) < \varphi_0^{p-1}(\varphi_1(x_0)) = \varphi_0^{p-1}(\varphi_2(x_0)) < \varphi_0^{p-1}(\varphi_2(x_1)).$$

This contradicts (A.10) ($x = x_1$). Thus $\varphi_1(x) = \varphi_2(x)$ on Δ_{m+1}. By induction, $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \ge 0$.

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Min Zhang (corresponding author)
School of Science
China University of Petroleum
266555 Qingdao, Shandong
People's Republic of China
E-mail: zhangminmath@163.com

Jianguo Si
School of Mathematics
Shandong University
250100 Jinan, Shandong
People's Republic of China

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