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Periodic solutions for first order neutral functional differential equations with multiple deviating arguments

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Abstract. We consider first order neutral functional differential equations with multiple deviating arguments of the form

$$(x(t) + Bx(t-\delta))' = g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) + p(t).$$

By using coincidence degree theory, we establish some sufficient conditions on the existence and uniqueness of periodic solutions for the above equation. Moreover, two examples are given to illustrate the effectiveness of our results.

1. Introduction. Recently, Liu and Huang [LH1, LH2] discussed the problem of periodic solutions for first order neutral functional differential equations (NFDE) of the form

$$(x(t) + Bx(t-\delta))' = g_1(t, x(t)) + g_2(t, x(t-\tau(t))) + p(t), \tag{1.1}$$

where $\tau, p : \mathbb{R} \to \mathbb{R}$ and $g_1, g_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, B and δ are constants, p is T-periodic, g_1 and g_2 are T-periodic in the first argument, $|B| \neq 1$ and $T > 0$.

This kind of NFDE has been used in the study of distributed networks containing lossless transmission lines [HM, KN]. Hence, in recent years, the problem of the existence of periodic solutions for (1.1) has been extensively studied. We refer the reader to [GM, H, HM, KN, LH1, LH2, LG, Z] and the references cited therein for more details.

In particular, Liu and Huang [LH1, LH2] also provided a sufficient condition for the existence and uniqueness of T-periodic solutions of (1.1) under the following conditions:

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(A_0) one of the following conditions holds:

(1) $(g_i(t,u_1)-g_i(t,u_2))(u_1-u_2)>0$ for $i=1,2$, $u_i,t\in\mathbb{R}$ and $u_1\neq u_2$,

(2) $(g_i(t,u_1)-g_i(t,u_2))(u_1-u_2)<0$ for $i=1,2$, $u_i,t\in\mathbb{R}$ and $u_1\neq u_2$;

$(\overline{A}_0)$ there exist constants b_1 and b_2 such that

(1) $b_1T/\pi+\frac{1}{2}b_2T<\big|1-|B|\big|$,

(2) $|g_i(t,u_1)-g_i(t,u_2)|\le b_i|u_1-u_2|$ for $i=1,2$, $u_i,t\in\mathbb{R}$.

However, to the best of our knowledge, there exists not much work on the existence and uniqueness of periodic solutions of (1.1) without (A_0) and $(\overline{A}_0)$. Moreover, the existence and uniqueness of periodic solutions of NFDE with more than two delays have not been extensively studied. Motivated by this, we shall consider first order NFDE with multiple deviating arguments of the form

$$(x(t)+Bx(t-\delta))'=g_0(t,x(t))+\sum_{k=1}^{n}g_k(t,x(t-\tau_k))+p(t), \tag{1.2}$$

where (for all $k=1,\dots,n$) $\tau_k,p:\mathbb{R}\to\mathbb{R}$ and $g_0,g_k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, B and δ are constants, p and τ_k are T-periodic, g_0 and g_k are T-periodic in the first argument, $|B|\neq 1$ and $T>0$.

The purpose of this article is to investigate the existence and uniqueness of T-periodic solutions of (1.2). By using some differential inequality techniques and Mawhin's continuation theorem, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.2) without assuming the conditions (A_0) and $(\overline{A}_0)(1)$. Moreover, two illustrative examples are given in Section 4.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_2=\Big(\int_0^T|x(t)|^2\,dt\Big)^{1/2},\qquad |x|_\infty=\max_{t\in[0,T]}|x(t)|.$$

We also assume that $\tau_k\in C^1(\mathbb{R},\ \mathbb{R})$, $1-\tau_k'>0$ and $k=1,\dots,n$. Let

$$X=\{x\in C(\mathbb{R},\mathbb{R})\mid x(t+T)=x(t)\text{ for all }t\in\mathbb{R}\}$$

be a Banach space with the norm $\|x\|_X=|x|_\infty$. We will suppose that $\Lambda\subseteq I=\{1,\dots,n\}$ is a nonempty set. For $i\in\Lambda$, we assume that there exist a nonnegative constant b_i^- and an integer m_i such that

$$b_i^-|B|\,|x_1-x_2|^2\le B[g_i(t,x_1)-g_i(t,x_2)](x_1-x_2),\qquad \tau_i(t)\equiv\delta-m_iT, \tag{1.3}$$

for all $t,x_1,x_2\in\mathbb{R}$.

Define linear operators

$$A : X \to X, \qquad (Ax)(t) = x(t) + Bx(t-\delta)$$

and

$$L : D(L) \subset X \to X, \qquad Lx = (Ax)', \tag{1.4}$$

where $D(L) = \{x \in X \mid x' \in C(\mathbb{R},\mathbb{R})\}$.

We also define a nonlinear operator $N : X \to X$ by setting

$$Nx = g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k(t))) + p(t). \tag{1.5}$$

In Hale's terminology [H], a solution $u(t)$ of (1.2) is a function $u \in C(\mathbb{R},\mathbb{R})$ such that $Au \in C^1(\mathbb{R},\mathbb{R})$ and (1.2) is satisfied on $\mathbb{R}$. In general, $u \notin C^1(\mathbb{R},\mathbb{R})$. But from Lemma 1 of [LG], in view of $|B| \neq 1$, it is easy to see that $(Ax)' = Ax'$. So a T-periodic solution $u(t)$ of (1.2) must be in $C^1(\mathbb{R},\mathbb{R})$. Meanwhile, according again to Lemma 1 of [LG], we can easily see that $\operatorname{Ker} L = \mathbb{R}$ and $\operatorname{Im} L = \{x \in X \mid \int_0^T x(s)\, ds = 0\}$. Therefore, L is a Fredholm operator with index zero. Define continuous projectors $P : X \to \operatorname{Ker} L$ and $Q : X \to X/\operatorname{Im} L$ by setting

$$Px(t) = \frac{1}{T}\int_0^T x(s)\, ds \quad \text{and} \quad Qx(t) = \frac{1}{T}\int_0^T x(s)\, ds.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Set $L_P = L|_{D(L)\cap \operatorname{Ker} P}$. Then L_P has a continuous inverse L_P^{-1} defined by

$$L_P^{-1} y(t) = A^{-1}\biggl(\frac{1}{T}\int_0^T s y(s)\, ds + \int_0^t y(s)\, ds\biggr). \tag{1.6}$$

Therefore, it is easy to see from (1.5) and (1.6) that N is L-compact on $\overline{\Omega}$, where Ω is an open bounded set in X.

2. Preliminary results. In view of (1.4) and (1.5), the operator equation

$$Lx = \lambda N x$$

where $\lambda \in (0,1)$ is equivalent to

$$x'(t) + Bx'(t-\delta) = \lambda\Bigl[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) + p(t)\Bigr]. \tag{2.1}$$

For later use, we introduce the continuation theorem from [GM]:

LEMMA 2.1. *Let X be a Banach space. Suppose that $L : D(L) \subset X \to X$ is a Fredholm operator with index zero and $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$,* *where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$ *for all* $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$*;*
(2) $Nx \notin \operatorname{Im} L$ *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$*;*
(3) *the Brouwer degree satisfies*

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ *has a solution on* $\overline{\Omega} \cap D(L)$*.*

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2. *Suppose that*

(A_1) *there exist nonnegative constants* $b_0^+, b_1^+, \ldots, b_n^+$ *such that*

$$|g_k(t,x_1) - g_k(t,x_2)| \le b_k^+|x_1 - x_2| \quad \text{for all } t, x_1, x_2 \in \mathbb{R},\ k = 0,1,\ldots,n.$$

Moreover, assume that one of the following conditions is satisfied:

(1) *there exists a positive constant* b_0^- *such that*

$$b_0^-|x_1 - x_2|^2 \le -(g_0(t,x_1) - g_0(t,x_2))(x_1 - x_2) \quad \text{for all } t, x_1, x_2 \in \mathbb{R}$$

and

$$b_0^- > |B|b_0^+ + \sum_{k=1}^{n} b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2} + |B|\sum_{k=1}^{n} b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2};$$

(2) *we have*

$$\sum_{k\in\Lambda}|B|b_k^- > b_0^+ + |B|b_0^+ + \sum_{k=1}^{n} b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2} + |B|\sum_{k\in I-\Lambda} b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2}.$$

Then (1.2) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.2). Set $Z(t) = x_1(t) - x_2(t)$. Then

$$(Z(t) + BZ(t-\delta))' - [g_0(t,x_1(t)) - g_0(t,x_2(t))] - \sum_{k=1}^{n}[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))] = 0. \tag{2.2}$$

Multiplying both sides of (2.2) by $Z(t) + BZ(t-\delta)$ and then integrating from 0 to T, we have

$$(2.3)\quad -\int_0^T Z(t)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$= B\int_0^T Z(t-\delta)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$+ \sum_{k=1}^n \int_0^T Z(t)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$
$$+ B\sum_{k=1}^n \int_0^T Z(t-\delta)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$

and

$$(2.4)\quad \sum_{k\in\Lambda} \int_0^T BZ(t-\delta)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$
$$= -\int_0^T Z(t)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$- B\int_0^T Z(t-\delta)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$- \sum_{k=1}^n \int_0^T Z(t)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$
$$- \sum_{k\in I-\Lambda} B\int_0^T Z(t-\delta)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt.$$

In view of $(A_1)(1)$ and $(A_1)(2)$, we shall consider two cases.

CASE (i). If $(A_1)(1)$ holds, using (2.3) and the Schwarz inequality, we get

$$(2.5)\quad b_0^-|Z|_2^2 \le -\int_0^T Z(t)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$= B\int_0^T Z(t-\delta)[g_0(t,x_1(t)) - g_0(t,x_2(t))]\,dt$$
$$+ \sum_{k=1}^n \int_0^T Z(t)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$
$$+ B\sum_{k=1}^n \int_0^T Z(t-\delta)[g_k(t,x_1(t-\tau_k(t))) - g_k(t,x_2(t-\tau_k(t)))]\,dt$$

$$
\begin{aligned}
&\le |B| b_0^+ \int_0^T |Z(t-\delta)|\,|Z(t)|\,dt \\
&\quad + \sum_{k=1}^n b_k^+ \int_0^T |Z(t)|\,|Z(t-\tau_k(t))|\,dt \\
&\quad + |B| \sum_{k=1}^n b_k^+ \int_0^T |Z(t-\delta)|\,|Z(t-\tau_k(t))|\,dt \\
&\le |B| b_0^+ \Big(\int_0^T |Z(t-\delta)|^2\,dt \Big)^{1/2} \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \\
&\quad + \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \Big(\int_0^T |Z(t-\tau_k(t))|^2\,dt \Big)^{1/2} \\
&\quad + |B| \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t-\delta)|^2\,dt \Big)^{1/2} \Big(\int_0^T |Z(t-\tau_k(t))|^2\,dt \Big)^{1/2} \\
&= |B| b_0^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \\
&\quad + \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \bigg(\int_{-\tau_k(0)}^{T-\tau_k(0)} |Z(s)|^2 \frac{1}{1-\tau_k'(t)}\,ds \bigg)^{1/2} \\
&\quad + |B| \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \bigg(\int_{-\tau_k(0)}^{T-\tau_k(0)} |Z(s)|^2 \frac{1}{1-\tau_k'(t)}\,ds \bigg)^{1/2} \\
&= |B| b_0^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \\
&\quad + \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \bigg(\int_0^T |Z(s)|^2 \frac{1}{1-\tau_k'(t)}\,ds \bigg)^{1/2} \\
&\quad + |B| \sum_{k=1}^n b_k^+ \Big(\int_0^T |Z(t)|^2\,dt \Big)^{1/2} \bigg(\int_0^T |Z(s)|^2 \frac{1}{1-\tau_k'(t)}\,ds \bigg)^{1/2} \\
&\le \bigg(|B| b_0^+ + \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}} \Big(\frac{1}{1-\tau_k'(t)} \Big)^{1/2} \\
&\qquad\qquad + |B| \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}} \Big(\frac{1}{1-\tau_k'(t)} \Big)^{1/2} \bigg) |Z|_2^2,
\end{aligned}
$$

which implies that

$$Z(t) \equiv 0 \quad \text{for all } t \in \mathbb{R}.$$

Hence, $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$. Therefore, (1.2) has at most one T-periodic solution.

Case (ii). If $(A_1)(2)$ holds, using (2.4), (1.3) and the Schwarz inequality, we obtain

$$
\begin{aligned}
(2.6)\quad & \sum_{k\in\Lambda} |B| b_k^- |Z|_2^2 \\
&= \sum_{k\in\Lambda} |B| b_k^- \int_0^T |Z(t)|^2\,dt \\
&= \sum_{k\in\Lambda} |B| b_k^- \int_0^T |Z(t-\delta)|\,|Z(t-\tau_k(t))|\,dt \\
&\le \sum_{k\in\Lambda} \int_0^T BZ(t-\delta)[g_k(t, x_1(t-\tau_k(t))) - g_k(t, x_2(t-\tau_k(t)))]\,dt \\
&= -\int_0^T Z(t)[g_0(t, x_1(t)) - g_0(t, x_2(t))]\,dt \\
&\quad - B\int_0^T Z(t-\delta)[g_0(t, x_1(t)) - g_0(t, x_2(t))]\,dt \\
&\quad - \sum_{k=1}^n \int_0^T Z(t)[g_k(t, x_1(t-\tau_k(t))) - g_k(t, x_2(t-\tau_k(t)))]\,dt \\
&\quad - \sum_{k\in I-\Lambda} B\int_0^T Z(t-\delta)\big[g_k(t, x_1(t-\tau_k(t))) \\
&\qquad\qquad - g_k(t, x_2(t-\tau_k(t)))\big]\,dt \\
&\le \Bigg(b_0^+ + |B| b_0^+ + \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}} \bigg(\frac{1}{1-\tau_k'(t)}\bigg)^{1/2} \\
&\qquad + |B| \sum_{k\in I-\Lambda} b_k^+ \max_{t\in\mathbb{R}} \bigg(\frac{1}{1-\tau_k'(t)}\bigg)^{1/2}\Bigg) |Z|_2^2.
\end{aligned}
$$

Then using a similar argument to that in Case (i), from (2.6) and $(A_2)(2)$, we conclude that (1.2) has at most one T-periodic solution. The proof of Lemma 2.2 is now complete.

Lemma 2.3. *Assume that*

(A_2) *there exists a constant* $\overline{d} > 0$ *such that*

(1) $\sum_{k=0}^{n} g_k(t, x_k) + p(t) < 0$ *for* $x_k > \overline{d}$, $t \in \mathbb{R}$, $k = 0, 1, \ldots, n$;
(2) $\sum_{k=0}^{n} g_k(t, x_k) + p(t) > 0$ *for* $x_k < -\overline{d}$, $t \in \mathbb{R}$, $k = 0, 1, \ldots, n$.

Let $x(t)$ *be a* T*-periodic solution of* (2.1). *Then*

$$|x|_\infty \le \overline{d} + \frac{1}{2}\int_0^T |x'(s)|\,ds. \tag{2.7}$$

Proof. Integrating (2.1) over $[0, T]$, we have

$$\int_0^T \Big[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) + p(t)\Big]\,dt = 0.$$

Using the integral mean-value theorem, it follows that there exists $t_1 \in [0, T]$ such that

$$g_0(t_1, x(t_1)) + \sum_{k=1}^{n} g_k(t_1, x(t_1 - \tau_k(t_1))) + p(t_1) = 0. \tag{2.8}$$

We first claim that there exists a constant $t_2 \in \mathbb{R}$ such that

$$|x(t_2)| \le \overline{d}. \tag{2.9}$$

Assume, on the contrary, that (2.9) does not hold. Then

$$|x(t)| > \overline{d} \quad \text{for all } t \in \mathbb{R}. \tag{2.10}$$

Let $\tau_0 \equiv 0$ and $t_1 \in [0, T]$ be the constant prescribed in (2.8). Using (A_2), (2.8) and (2.10), we see that there exist $0 \le i, j \le n$ such that

$$x(t_1 - \tau_i(t_1)) = \max_{0\le k\le n} x(t_1 - \tau_k(t_1)) \ge \min_{0\le k\le n} x(t_1 - \tau_k(t_1)) = x(t_1 - \tau_j(t_1)),$$

which, together with (2.10), implies that

$$-\overline{d} > x(t_1 - \tau_i(t_1)) = \max_{0\le k\le n} x(t_1 - \tau_k(t_1))$$

or

$$x(t_1 - \tau_j(t_1)) = \min_{0\le k\le n} x(t_1 - \tau_k(t_1)) > \overline{d}.$$

Without loss of generality, we may assume that the latter condition holds (for the former, the situation is analogous). Then

$$x(t_1-\tau_i(t_1)) \ge x(t_1-\tau_k(t_1)) \ge x(t_1-\tau_j(t_1)) > \overline{d}, \qquad k = 0, 1, \ldots, n. \tag{2.11}$$

According to (2.11) and (A_2), we obtain

$$0 > \sum_{k=0}^{n} g_k(t_1, x(t_1 - \tau_k(t_1))) + p(t_1),$$

which contradicts (2.8); thus (2.9) is true.

Let $t_2 = mT + t_0$ where $t_0 \in [0, T]$ and m is an integer. Then

$$|x(t_2)| = |x(t_0)| \le \overline{d},$$

$$|x(t)| = \Big|x(t_0) + \int_{t_0}^{t} x'(s)\,ds\Big| \le \overline{d} + \int_{t_0}^{t} |x'(s)|\,ds, \qquad t \in [t_0, t_0 + T],$$

and

$$|x(t)| = |x(t-T)| = \Big|x(t_0) - \int_{t-T}^{t_0} x'(s)\,ds\Big|$$

$$\le \overline{d} + \int_{t-T}^{t_0} |x'(s)|\,ds, \qquad t \in [t_0, t_0 + T].$$

Combining the above two inequalities and using the Schwarz inequality, for any T-periodic solution $x(t)$ of (2.2), we have

$$|x|_\infty \le \max_{t\in[t_0,t_0+T]} \Big\{\overline{d} + \frac{1}{2}\Big(\int_{t_0}^{t} |x'(s)|\,ds + \int_{t-T}^{t_0} |x'(s)|\,ds\Big)\Big\} = \overline{d} + \frac{1}{2}\int_0^T |x'(s)|\,ds.$$

This completes the proof of Lemma 2.3.

3. Main results

THEOREM 3.1. *Let* (A_2) *hold. Assume that either the condition* $(A_1)(1)$ *or* $(A_1)(2)$ *is satisfied. Then* (1.2) *has a unique* T*-periodic solution.*

Proof. From Lemma 2.3, together with (A_1) and (A_2), it follows easily that (1.2) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that (1.2) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1.

First, we claim that the set of all possible T-periodic solutions of (2.1) is bounded.

Let $x(t)$ be a T-periodic solution of (2.1). Multiplying $x(t) + Bx(t-\delta)$ and (2.1), and then integrating from 0 to T, we have

$$\begin{aligned}(3.1)\qquad &\int_0^T x(t)\Big[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) + p(t)\Big]\,dt\\ &+ \int_0^T Bx(t-\delta)\Big[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k(t))) + p(t)\Big]\,dt = 0.\end{aligned}$$

Now, we shall consider two cases.

CASE (i). If $(A_1)(1)$ holds, then from (3.1) and the Schwarz inequality, we have

$$
\begin{aligned}
b_0^-|x|_2^2 &\le -\int_0^T (x(t)-0)[g_0(t,x(t))-g_0(t,0)]\,dt \\
&= B\int_0^T x(t-\delta)[g_0(t,x(t))-g_0(t,0)]\,dt \\
&\quad + \sum_{k=1}^n \int_0^T x(t)[g_k(t,x(t-\tau_k(t)))-g_k(t,0)]\,dt \\
&\quad + B\sum_{k=1}^n \int_0^T x(t-\delta)[g_k(t,x(t-\tau_k(t)))-g_k(t,0)]\,dt \\
&\quad + \sum_{k=0}^n \int_0^T x(t)g_k(t,0)\,dt + B\sum_{k=0}^n \int_0^T x(t-\delta)g_k(t,0)\,dt \\
&\quad + \int_0^T x(t)p(t)\,dt + B\int_0^T x(t-\delta)p(t)\,dt \\
&\le |B|b_0^+\int_0^T |x(t-\delta)|\,|x(t)|\,dt + \sum_{k=1}^n b_k^+\int_0^T |x(t)|\,|x(t-\tau_k(t))|\,dt \\
&\quad + |B|\sum_{k=1}^n b_k^+\int_0^T |x(t-\delta)|\,|Z(t-\tau_k(t))|\,dt \\
&\quad + (1+|B|)\sqrt{T}\sum_{k=0}^n |g_k(t,0)|_\infty|x|_2 + (1+|B|)\sqrt{T}\,|p|_\infty|x|_2 \\
&\le \Bigg(|B|b_0^+ + \sum_{k=1}^n b_k^+\max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2} \\
&\qquad\qquad + |B|\sum_{k=1}^n b_k^+\max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}\Bigg)|x|_2^2 \\
&\quad + (1+|B|)\sqrt{T}\sum_{k=0}^n |g_k(t,0)|_\infty|x|_2 + (1+|B|)\sqrt{T}\,|p|_\infty|x|_2.
\end{aligned}
\tag{3.2}
$$

Now, let

$$
D_1 = \frac{(1+|B|)\sqrt{T}\sum_{k=0}^n |g_k(t,0)|_\infty + (1+|B|)\sqrt{T}\,|p|_\infty}{b_0^- - \left(|B|b_0^+ + (1+|B|)\sum_{k=1}^n b_k^+\max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}\right)}.
\tag{3.3}
$$

In view of (3.2) and (3.3), we obtain

$$|x|_2 \le D_1. \tag{3.4}$$

CASE (ii). If $(A_1)(2)$ holds, from (1.3), (3.1) and the Schwarz inequality, we get

$$\begin{aligned}
\sum_{k\in\Lambda} |B|b_k^-|x|_2^2 &= \sum_{k\in\Lambda} |B|b_k^- \int_0^T |x(t)|^2\,dt \\
&= \sum_{k\in\Lambda} |B|b_k^- \int_0^T |x(t-\delta)|\,|x(t-\tau_k(t))|\,dt \\
&\le \sum_{k\in\Lambda} \int_0^T Bx(t-\delta)[g_k(t,x(t-\tau_k(t))) - g_k(t,0)]\,dt \\
&= -\int_0^T x(t)[g_0(t,x(t)) - g_0(t,0)]\,dt \\
&\quad - B\int_0^T x(t-\delta)[g_0(t,x(t)) - g_0(t,0)]\,dt \\
&\quad - \sum_{k=1}^n \int_0^T x(t)[g_k(t,x(t-\tau_k(t))) - g_k(t,0)]\,dt \\
&\quad - B\sum_{k\in I-\Lambda} \int_0^T x(t-\delta)[g_k(t,x(t-\tau_k(t))) - g_k(t,0)]\,dt \\
&\quad - \sum_{k=0}^n \int_0^T x(t)g_k(t,0)\,dt - B\sum_{k=0}^n \int_0^T x(t-\delta)g_k(t,0)\,dt \\
&\quad - \int_0^T x(t)p(t)\,dt - B\int_0^T x(t-\delta)p(t)\,dt \\
&\le \biggl(b_0^+ + |B|b_0^+ + \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2} \\
&\qquad + |B|\sum_{k\in I-\Lambda} b_k^+ \max_{t\in\mathbb{R}}\biggl(\frac{1}{1-\tau_k'(t)}\biggr)^{1/2}\biggr)|x|_2^2 \\
&\quad + (1+|B|)\sqrt{T}\sum_{k=0}^n |g_k(t,0)|_\infty |x|_2 + (1+|B|)\sqrt{T}\,|p|_\infty |x|_2.
\end{aligned} \tag{3.5}$$

Now, let

$$\varPhi = \sum_{k=1}^{n} b_k^{+} \max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2} + |B| \sum_{k\in I-\varLambda} b_k^{+} \max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}$$

and

$$\overline{D}_1 = \frac{(1+|B|)\sqrt{T}\sum_{k=0}^{n}|g_k(t,0)|_\infty + (1+|B|)\sqrt{T}\,|p|_\infty}{\sum_{k\in\varLambda}|B|b_k^{-} - b_0^{+} - |B|b_0^{+} - \varPhi}. \tag{3.6}$$

According to (3.5) and (3.6), we obtain

$$|x|_2 \le \overline{D}_1. \tag{3.7}$$

If $1 > |B|$, in view of (2.1), we get

$$\begin{aligned}
\int_0^T |x'(t)|\,dt &= \int_0^T \Big|-Bx'(t-\delta) + \lambda\Big[g_0(t,x(t)) \\
&\qquad + \sum_{k=1}^{n} g_k(t,x(t-\tau_k(t))) + p(t)\Big]\Big|\,dt \\
&\le |B|\int_0^T |x'(t-\delta)|\,dt + \int_0^T |g_0(t,x(t)) - g_0(t,0)|\,dt \\
&\quad + \sum_{k=1}^{n}\int_0^T |g_k(t,x(t-\tau_k(t))) - g_k(t,0)|\,dt \\
&\quad + \sum_{k=0}^{n}\int_0^T |g_k(t,0)|\,dt + \int_0^T |p(t)|\,dt \\
&\le |B|\int_0^T |x'(t)|\,dt + \sqrt{T}\left(b_0^{+} + \sum_{k=1}^{n} b_k^{+}\max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}\right)|x|_2 \\
&\quad + \sum_{k=0}^{n} T|g_k(t,0)|_\infty + T|p|_\infty,
\end{aligned}$$

which together with (3.4) and (3.7) implies that

$$\begin{aligned}
\int_0^T |x'(t)|\,dt &\le \frac{1}{1-|B|}\sqrt{T}\left(b_0^{+} + \sum_{k=1}^{n} b_k^{+}\max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}\right)(D_1+\overline{D}_1) \\
&\quad + \frac{1}{1-|B|}\left(\sum_{k=0}^{n} T|g_k(t,0)|_\infty + T|p|_\infty\right) \\
&:= M_1.
\end{aligned} \tag{3.8}$$

If $1 < |B|$, from (2.1), it follows that

$$\begin{aligned}
|B|\int_0^T |x'(t)|\,dt &= \int_0^T |Bx'(t-\delta)|\,dt\\
&= \int_0^T \Big|-x'(t) + \lambda\Big[g_0(t,x(t)) + \sum_{k=1}^n g_k(t,x(t-\tau_k(t))) + p(t)\Big]\Big|\,dt\\
&\le \int_0^T |x'(t)|\,dt + \int_0^T |g_0(t,x(t)) - g_0(t,0)|\,dt\\
&\quad + \sum_{k=1}^n \int_0^T |g_k(t,x(t-\tau_k(t))) - g_k(t,0)|\,dt\\
&\quad + \sum_{k=0}^n \int_0^T |g_k(t,0)|\,dt + \int_0^T |p(t)|\,dt\\
&\le \int_0^T |x'(t)|\,dt + \sqrt{T}\bigg(b_0^+ + \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}}\Big(\frac{1}{1-\tau_k'(t)}\Big)^{1/2}\bigg)|x|_2\\
&\quad + \sum_{k=0}^n T|g_k(t,0)|_\infty + T|p|_\infty,
\end{aligned}$$

which together with (3.4) and (3.7) implies that

$$\begin{aligned}
\int_0^T |x'(t)|\,dt \le{}& \frac{1}{|B|-1}\sqrt{T}\bigg(b_0^+ + \sum_{k=1}^n b_k^+ \max_{t\in\mathbb{R}}\Big(\frac{1}{1-\tau_k'(t)}\Big)^{1/2}\bigg)(D_1+\overline{D}_1)\\
&+ \frac{1}{|B|-1}\Big(\sum_{k=0}^n T|g_k(t,0)|_\infty + T|p|_\infty\Big)\\
:={}& M_2.
\end{aligned} \tag{3.9}$$

Thus, from (2.7), (3.8) and (3.9) there exists a positive constant M such that for all $t \in \mathbb{R}$,

$$|x|_\infty \le \overline{d} + \tfrac{1}{2}(M_1 + M_2) < M. \tag{3.10}$$

Set

$$\Omega = \{x \in X : |x|_\infty < M\}.$$

Then we know that equation (2.1) has no T-periodic solution for $\partial\Omega$ for $\lambda \in (0,1)$. When $x(t) \in \partial\Omega \cap R$, $x(t) = M$ or $x(t) = -M$, from (A_2) we can see that

$$-\frac{1}{T}\int_0^T \Big\{g_0(t,M)+\sum_{k=1}^n g_k(t,M)+p(t)\Big\}\,dt>0,$$

$$-\frac{1}{T}\int_0^T \Big\{g_0(t,-M)+\sum_{k=1}^n g_k(t,-M)+p(t)\Big\}\,dt<0,$$

so condition (ii) of Lemma 2.1 is also satisfied. Set

$$H(x,\mu)=-(1-\mu)x+\mu\frac{1}{T}\int_0^T \Big[g_0(t,x)+\sum_{k=1}^n g_k(t,x)+p(t)\Big]\,dt.$$

When $x\in\partial\Omega\cap R,\ \mu\in[0,1]$, we have

$$xH(x,\mu)=-(1-\mu)x^2+\mu x\frac{1}{T}\int_0^T \Big[g_0(t,x)+\sum_{k=1}^n g_k(t,x)+p(t)\Big]\,dt<0.$$

Thus $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned}\deg\{-x,\Omega\cap\mathbb{R},0\}&=\deg\Big\{\frac{1}{T}\int_0^T \Big[g_0(t,x)+\sum_{k=1}^n g_k(t,x)+p(t)\Big]\,dt,\Omega\cap\mathbb{R},0\Big\}\\&=\deg\{x,\Omega\cap\mathbb{R},0\}\neq 0,\end{aligned}$$

so condition (iii) of Lemma 2.1 is satisfied. In view of that lemma, equation (1.1) has a solution with period T. This completes the proof.

Similar to the proof of Theorem 3.1, one can prove the following result:

THEOREM 3.2. *Suppose that*

(A_1^*) *there exist nonnegative constants* $b_0^+,b_1^+,\dots,b_n^+$ *such that*

$$|g_k(t,x_1)-g_k(t,x_2)|\le b_k^+|x_1-x_2| \quad \text{for all } t,x_1,x_2\in\mathbb{R},\ k=0,1,\dots,n.$$

Moreover, assume that one of the following conditions is satisfied:

(1) *there exists a positive constant* b_0^- *such that*

$$b_0^-|x_1-x_2|^2\le[g_0(t,x_1)-g_0(t,x_2)](x_1-x_2) \quad \text{for all } \ t,x_1,x_2\in\mathbb{R},$$

and

$$b_0^->|B|b_0^++\sum_{k=1}^n b_k^+\max_{t\in\mathbb{R}}\Big(\frac{1}{1-\tau_k'(t)}\Big)^{1/2}+|B|\sum_{k=1}^n b_k^+\max_{t\in\mathbb{R}}\Big(\frac{1}{1-\tau_k'(t)}\Big)^{1/2};$$

(2) $\varLambda \subseteq \{1, \dots, n\}$ *is a nonempty set such that*

$$\sum_{k\in\varLambda} |B| b_k^- > b_0^+ + |B| b_0^+ + \sum_{k=1}^{n} b_k^+ \max_{t\in\mathbb{R}} \left(\frac{1}{1-\tau_k'(t)}\right)^{1/2} + |B| \sum_{k\in I-\varLambda} b_k^+ \max_{t\in\mathbb{R}} \left(\frac{1}{1-\tau_k'(t)}\right)^{1/2}.$$

(A_2^*) *there exists a constant* $\overline{d} > 0$ *such that*

(1) $\sum_{k=0}^{n} g_k(t, x_k) + p(t) > 0$ *for* $x_k > \overline{d}$, $t \in \mathbb{R}$, $k = 0, 1, \dots, n$;
(2) $\sum_{k=0}^{n} g_k(t, x_k) + p(t) < 0$ *for* $x_k < -\overline{d}$, $t \in \mathbb{R}$, $k = 0, 1, \dots, n$.

Then (1.2) *has a unique* T*-periodic solution.*

4. Examples and remarks

Example 4.1. The first order NFDE

$$\big(x(t) + \tfrac{1}{30}x(t-2)\big)' = -100(2+\sin t)x + \big|\cos x\big(t - \tfrac{1}{2}\sin t\big)\big| + e^{\cos t} \tag{4.1}$$

has a unique 2π-periodic solution.

Proof. From (4.1), we have $B = 1/30$, $g_0(t,x) = -100(2+\sin t)x$, $g_1(t,x) = |\cos x|$ and $p(t) = e^{\cos t}$. Then, $b_0^- = 100$, $b_0^+ = 300$, $b_1^+ = 1$. It follows that (4.1) satisfies the conditions $(A_1)(1)$ and (A_2). Therefore, from Theorem 3.1, (4.1) has a unique 2π-periodic solution.

Remark 4.1. Since $B = 1/30$, $p(t) = e^{\cos t}$, $g_1(t,x) = |\cos x|$ and $g_0(t,x) = -100(2+\sin t)x$ with $b_0^- = 100 > 2(1-B) = 29/15$, one can observe that the conditions (A_0) and $(\overline{A}_0)(1)$ in [1-2] are not satisfied. Hence, the results obtained in [LH1, LH2] and the references cited therein are not applicable to Example 4.1.

Example 4.2. The first order NFDE

$$\begin{aligned}(x(t) - 100x(t-\pi/2))' = -\tfrac{1}{300}e^{\sin t}\cos x(t) - x(t-3\pi/2)\\ + \tfrac{1}{100}\big|\sin x\big(t-\tfrac{1}{2}\sin t\big)\big| + \cos t\end{aligned} \tag{4.2}$$

has a unique 2π-periodic solution.

Proof. From (4.2), we obtain

$$B = -100, \quad g_0(t,x) = -\tfrac{1}{300}e^{\sin t}\cos x,$$
$$g_1(t,x) = -x, \quad g_2(t,x) = \tfrac{1}{100}|\sin x|$$

and

$$p(t) = \cos t, \quad \delta = \pi/2, \quad \tau_1(t) \equiv= 3\pi/2,$$
$$\tau_2(t) = \tfrac{1}{2}\sin t, \quad 1 - \tau_2'(t) \geq 1/2.$$

Then

$$\Lambda = \{1\} \subset I = \{1,2\}, \quad b_1^- |B| = 100, \quad b_0^+ \le 1/100, \quad b_1^+ = 1, \quad b_2^+ \le 1/100$$

and

$$\begin{aligned}|B|b_1^- &= 100 \\ &> \frac{1}{100} + 100 \times \frac{1}{100} + 1 \times 1 + \frac{1}{100}\left(\frac{1}{1-1/2}\right)^{1/2} + 100 \times \frac{1}{100}\left(\frac{1}{1-1/2}\right)^{1/2} \\ &\ge b_0^+ + |B|b_0^+ + b_1^+ \max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_1'(t)}\right)^{1/2} + b_2^+ \max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_2'(t)}\right)^{1/2} \\ &\quad + |B|b_2^+ \max_{t\in\mathbb{R}}\left(\frac{1}{1-\tau_2'(t)}\right)^{1/2}.\end{aligned}$$

This implies that (4.2) satisfies the conditions $(A_1)(2)$ and (A_2). Therefore, by Theorem 3.1, (4.2) has a unique 2π-periodic solution.

Remark 4.2. Obviously, $g_0(t,x) = -\frac{1}{300}e^{\sin t}\cos x$ and $g_2(t,x) = \frac{1}{100}|\sin x|$ do not satisfy the condition (A_0). Since the results in [LH1, LH2] were established for first order NFDE with only one deviating argument, they also cannot be applied to (4.2) to obtain the existence and uniqueness of 2π-periodic solutions.

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