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Some remarks on comparison functions

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Abstract. We answer some questions concerning Perron and Kamke comparison functions satisfying the Carathéodory condition. In particular, we show that a Perron function multiplied by a constant may not be a Perron function, and that not every comparison function is bounded by a comparison function with separated variables. Moreover, we investigate when a sum of Perron functions is a Perron function. We also present a criterion for comparison functions with separated variables.

The first and most applicable condition that guarantees uniqueness of solutions for a wide class of differential problems was introduced by Lipschitz. This result and the papers of Perron [5] and Kamke [2] initiated the development and applications of so-called comparison functions theory. In this paper we answer some questions concerning comparison functions satisfying the Carathéodory condition, in particular ones with separated variables.

This paper stems from conversations with other mathematicians who suggested that the class of Perron and Kamke functions should have some "nice" properties. They believed, for instance, that every comparison function was bounded by a comparison function with separated variables and that a Perron function multiplied by a constant was also a Perron function. We disprove these statements. We also study some other properties of the class of comparison functions, in particular we give a classification of comparison functions with separated variables and we consider the question whether the class of Perron functions is closed with respect to adding Lipschitz functions.

1. Preliminaries. We say that a function $\omega : (0,T] \times [0,a] \to \mathbb{R}_+ = [0,\infty)$ satisfies the *Carathéodory condition* if $\omega(t,\cdot) : [0,a] \to \mathbb{R}_+$ is continuous for almost every $t \in (0,T]$, $\omega(\cdot,x) : (0,T] \to \mathbb{R}_+$ is measurable for all $x \in [0,a]$, and $\omega(t,x) \le m(t)$ for a.e. $t \in (0,T]$, all $x \in [0,a]$ and some function $m : (0,T] \to \mathbb{R}_+$ integrable on $[\tau,T]$ for every $\tau > 0$.

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Suppose that ω satisfies the Carathéodory condition and $\omega(t,0)=0$. We say that ω is a *Perron function* if the zero function is the only absolutely continuous solution of the problem

$$x'(t)=\omega(t,x(t)) \quad \text{a.e. } (0,T), \qquad x(0)=0. \tag{1}$$

If the zero function is the only solution of the problem (1) under the additional condition $D^{+}x(0)=0$, where $D^{+}x$ is the right-hand upper Dini derivative, then ω is called a *Kamke function.*

We say that a function $\omega(t,x)=A(t)B(x)$ has *C-separated variables* if it satisfies the Carathéodory condition (that is, B is continuous and A is integrable on $[\tau,T]$ for each $\tau>0$) and for every $\epsilon\in(0,a]$ and $\gamma\in(0,\epsilon)$ we have

$$\int_{\gamma}^{\epsilon}\frac{dx}{B(x)}<\infty. \tag{2}$$

LEMMA 1. *Suppose that* $\omega:(0,T]\times[0,a]\to\mathbb{R}_{+}$ *satisfies the Carathéodory condition*, $\varphi:[0,T]\to\mathbb{R}_{+}$ *is continuous*, $\varphi(0)=0$ *and*

$$\varphi(t)\le\varphi(\tau)+\int_{\tau}^{t}\omega(s,\varphi(s))\,ds \qquad \text{for } 0<\tau<t<T. \tag{3}$$

Then for every $b\in(0,\varphi(T))$ *there exists an absolutely continuous function* $x:[0,T]\to\mathbb{R}_{+}$ *such that* $x(T)=b$, $x(t)\le\varphi(t)$ *for* $t\in[0,T]$ *and* $x'(t)=\omega(t,x(t))$ *a.e. in* $(0,T)$.

Proof. Let $b\in(0,\varphi(T))$. For $n\in\mathbb{N}$ consider the initial-value problem

$$z'(t)=\omega_n(t,z(t)) \qquad \text{a.e. } (0,T), \qquad z(T)=b,$$

where

$$\omega_n(t,x)=\begin{cases}\omega(t,x), & x\le(1-1/n)\varphi(t),\\ n\Big(1-\dfrac{x}{\varphi(t)}\Big)\omega\Big(t,\Big(1-\dfrac{1}{n}\Big)\varphi(t)\Big) & \\ \quad+(n+1)\Big(\dfrac{x}{\varphi(t)}-1+\dfrac{1}{n}\Big)f_n(t), & 0<(1-1/n)\varphi(t)<x\le\varphi(t),\\ \dfrac{n+1}{n}f_n(t), & x\ge\varphi(t),\end{cases}$$

$$f_n(t)=\max\{\omega(t,y):(1-1/n)\varphi(t)\le y\le\varphi(t)\}.$$

It is clear that the above problem has a nondecreasing solution y_n defined on $[0,T]$, since $\omega_n(t,0)=0$. We show that $y_n\le\varphi$. Let $t_n=\max\{t: y_n(t)=\varphi(t)\}$. Suppose that $\varphi(t_n)>0$ and take $t>t_n$ such that $y_n(s)\ge\big(1-\frac{1}{n(n+1)}\big)\varphi(s)$ for $s\in[t_n,t]$. Then $\omega_n(s,y_n(s))\ge f_n(s)\ge\omega(s,\varphi(s))$ for

$s \in [t_n, t]$. We get

$$y_n(t) = \varphi(t_n) + \int_{t_n}^{t} \omega_n(s, y_n(s))\, ds \geq \varphi(t_n) + \int_{t_n}^{t} \omega(s, \varphi(s))\, ds \geq \varphi(t).$$

This contradicts the definition of t_n. Hence $\varphi(t_n) = 0,\ y_n(t) = 0 \leq \varphi(t)$ for $t < t_n$ and consequently $y_n \leq \varphi$.

The functions y_n are uniformly bounded and equicontinuous, hence there exists a subsequence $\{y_{n_k}\}$ uniformly convergent to a continuous function $x : [0, T] \to \mathbb{R}_+$. Since $\omega_{n_k}(s, y_{n_k}(s)) \to \omega(s, x(s))$ almost everywhere, standard arguments and the Lebesgue dominated convergence theorem imply the assertion. ∎

REMARK 1. The assumption $\varphi(0) = 0$ in the above lemma may be omitted. Moreover, the assertion is satisfied if $b = \varphi(T)$.

It is clear that condition (3) holds if $\varphi'(t) \leq \omega(t, \varphi(t))$ a.e. $(0, T)$. This implies that if ω is a Perron (or Kamke) function, then any function not greater than ω is also a Perron (resp. Kamke) function.

2. Results. We state a criterion for a function with separated variables to be a Kamke or Perron function.

THEOREM 1. *Suppose that $\omega(t, x) = A(t)B(x)$ satisfies the Carathéodory condition. If for every $\epsilon \in (0, T]$ there exists $\gamma \in (0, \epsilon)$ such that (2) is not valid, then ω is a Perron function.*

Assume that ω has C-separated variables. Then

(a) *ω is a Perron function iff $\int_0^T A(s)\, ds < \infty$ and $\int_0^a du/B(u) = \infty$.*

(b) *If $\int_0^T A(s)\, ds < \infty$, then ω is a Kamke function iff it is a Perron function.*

(c) *Suppose that $\int_0^T A(s)\, ds = \int_0^a du/B(u) = \infty$. Define*

$$I^m_{\delta,\epsilon}(t) = \frac{\int_t^\delta A(s)\, ds}{\int_{mt}^\epsilon du/B(u)}, \qquad I^m_{\delta,\epsilon} = \liminf_{t \to 0^+} I^m_{\delta,\epsilon}(t)$$

for $m > 0$, $t, \delta \in (0, T]$, $\epsilon \in (0, a]$. Then $I^m_{\delta,\epsilon} = I^m$ does not depend on δ, ϵ, and:

(i) *If ω is a Kamke function, then $I^m \leq 1$ for some $m_0 > 0$ and every $m \in (0, m_0]$.*

(ii) *If $I^m < 1$ for some $m > 0$, then ω is a Kamke function.*

Proof. Suppose that x is a solution of (1) and $\epsilon = x(t_1) > 0$. Take $t_2 \in (0, t_1)$ such that $x(t_2) > 0$ and (2) is not satisfied for $\gamma = x(t_2)$. We get $\infty = \int_\gamma^\epsilon dx/B(x) = \int_{t_2}^{t_1} A(t) dt < \infty$. The contradiction means that (2) has only the trivial solution.

Suppose that (2) holds for every $0<\gamma<\epsilon\le a$. Every non-zero solution of the equation $x'(t)=A(t)B(x(t))$ is determined by the relation

$$\int\limits_{x(t)}^{\epsilon}\frac{du}{B(u)}=\int\limits_{t}^{\delta}A(s)\,ds \tag{4}$$

for $\epsilon=x(\delta)\ne 0$ and $t\in(0,T]$ such that $x(t)\ne 0$. Define $t_x=\max\{t : x(t)=0\}$. Suppose that $t_x<T$. Then

$$\int\limits_{0}^{x(t)}\frac{du}{B(u)}=\int\limits_{t_x}^{t}A(s)\,ds \quad \text{for } t>t_x. \tag{5}$$

If $\int_0^T A(s)\,ds<\infty$ and $\int_0^a du/B(u)=\infty$, then (5) cannot be satisfied, hence the zero function is the only solution of (1). In other cases, (4) defines a non-zero solution x for $\delta=T$ and $\epsilon>0$ such that $\int_0^\epsilon du/B(u)\le\int_0^T A(s)\,ds$. If $\int_0^a du/B(u)<\infty$ and $\int_0^\epsilon du/B(u)<\int_0^T A(s)\,ds$, then $t_x>0$ for x defined by (4). The above remarks prove assertions (a) and (b).

To prove (c), suppose $\int_0^a du/B(u)=\int_0^T A(s)\,ds=\infty$. If $\delta,\delta'\in(0,T]$ and $\epsilon,\epsilon'\in(0,a]$, then

$$I^m_{\delta',\epsilon'}(t)=\left(1+\frac{\int_\epsilon^{\epsilon'}du/B(u)}{\int_{mt}^{\epsilon}du/B(u)}\right)^{-1}\left(I^m_{\delta,\epsilon}(t)+\frac{\int_\delta^{\delta'}A(s)\,ds}{\int_{mt}^{\epsilon}du/B(u)}\right),$$

hence $I^m_{\delta',\epsilon'}=I^m_{\delta,\epsilon}=I^m$.

It follows from (a) that there exists a non-zero solution x of (1). We have

$$\int\limits_{x(t)}^{\epsilon}\frac{du}{B(u)}=I^m_{\delta,\epsilon}(t)\int\limits_{mt}^{\epsilon}\frac{du}{B(u)}. \tag{6}$$

If ω is a Kamke function, then there exists $m_0>0$ such that for every $t_0>0$ one can find $t\in(0,t_0)$ such that $x(t)/t\ge m_0$. Hence

$$\int\limits_{x(t)}^{\epsilon}\frac{du}{B(u)}\le\int\limits_{m_0t}^{\epsilon}\frac{du}{B(u)}.$$

The above and (6) imply $I^{m_0}_{\delta,\epsilon}(t)\le 1$. Consequently, $I^{m_0}\le 1$. It is clear that I^m is nondecreasing with respect to m, hence assertion (i) follows.

Assume that $I^m<1$ for some $m>0$. Then for any $\delta,\epsilon,t_0>0$ there exists $t\in(0,t_0)$ such that $I^m_{\delta,\epsilon}(t)\le 1$. If x is defined by (4), then (6) implies

$$\int\limits_{x(t)}^{\epsilon}\frac{du}{B(u)}\le\int\limits_{mt}^{\epsilon}\frac{du}{B(u)}.$$

This means that $x(t)\ge mt$ and $D^+x(0)\ge m>0$. Hence, any non-zero solution of (1) satisfies $D^+x(0)\ne 0$. Consequently, ω is a Kamke function. ∎

REMARK 2. If $I^m = 1$ for every $m > 0$ small enough, then ω may be a Kamke function or not. Indeed, it is known that $\omega(t,x) = x/t$ is a Kamke function and it is easy to verify that $I^m = 1$ for every $m > 0$. On the other hand, $\omega(t,x) = (1 - 1/\ln t)x/t$ is not a Kamke function (problem (1) has the solution $-t/\ln t$), but for this function we also have $I^m = 1$ for all $m > 0$.

It is well known that $\omega(t,x) = \lambda x/t$ is a Kamke function iff $\lambda \le 1$. This example shows that a Kamke function multiplied by a constant may not be a Kamke function. We prove an analogous result for Perron functions.

PROPOSITION 1. *Let $\rho : (0,1] \times [0,1] \to \mathbb{R}$ be defined by the formula*

$$\rho(t,x) = \min\{\sqrt{x}, x/t\}. \tag{7}$$

Then $\lambda\rho$ is a Perron function iff $\lambda < 2$. If $\lambda \ge 2$ then $\lambda\rho$ is not even a Kamke function.

Proof. Suppose that $x'(t) = \lambda\rho(t,x(t))$ for $t \in (0,T]$ and $x(0) = 0$. Then $x'(t) \le \lambda\sqrt{x(t)}$, hence $x(t) \le (\lambda t/2)^2$. If $\lambda < 2$, then $x(t) \le t^2$ for $t \in [0,T]$, so $\rho(t,x(t)) = x(t)/t$ and $x'(t) = (\lambda/t)x(t)$. Consequently, $x(t) = ct^\lambda$ for some $c \ge 0$. We get $c = 0$, since $x(t) \le t^2$. We conclude that the zero function is the only solution of (1) for $\lambda < 2$ and $\omega = \lambda\rho$.

If $\lambda \ge 2$ then $x(t) = t^\lambda$ is a solution of (1) with $D^+x(0) = 0$. ∎

All well known Perron and Kamke functions have separated variables. We present elementary comparison functions that are not even bounded by comparison functions with separated variables.

PROPOSITION 2. *Suppose that $\rho(t,x) \le \omega(t,x)$, where ρ is defined by (7), and the function ω has C-separated variables. Then ω is not a Perron function.*

Proof. Suppose that ω is a Perron function. Then it follows from Theorem 1 that 2ω is a Perron function. Lemma 1 implies that 2ρ is also a Perron function. This contradicts Proposition 1. ∎

PROPOSITION 3. *Define*

$$\rho(t,x) = \max\left\{\frac{x}{t}, 1 + \frac{x-t}{t^2}\right\} \quad \textit{for } t \in (0,1],\ x \in [0,1].$$

Then ρ is a Kamke function. Suppose that $\rho(t,x) \le \omega(t,x) = A(t)B(x)$, where ω has C-separated variables. Then ω is not a Kamke function.

Proof. It is clear that $\rho(t,x) = x/t$ for $x \le t$ and $\rho(t,x) = 1 + (x-t)/t^2$ for $x \ge t$. Now it is easily seen that if $x'(t) = \rho(t,x(t))$ for $t > 0$, then either $x(t) \ge t$ on $[0,1]$ or $x(t) = kt$ for some $k \ge 0$. Hence ρ is a Kamke function.

We now prove that ω is not a Kamke function. Since $A(t) \ge (B(1)t)^{-1}$ a.e. $(0,1)$, we get $\int_0^1 A(s)\,ds = \infty$. If $\int_0^1 du/B(u) < \infty$, then by Theorem 1, ω is not a Kamke function. Suppose that $\int_0^1 du/B(u) = \infty$. There exists a constant $b > 0$ such that $B(x) \ge bx$ for all $x \in [0,1]$, since $A(t)B(x) \ge x/t$. For all $\epsilon \in (0,1)$ and $m > 0$ we have

$$I^m_{\epsilon,\epsilon}(t) \ge \frac{\int_t^\epsilon \frac{\omega(s,\epsilon)}{B(\epsilon)}\,ds}{\int_{mt}^\epsilon \frac{du}{bu}} \ge \frac{b\int_t^\epsilon \frac{\epsilon - s}{s^2}\,ds}{B(\epsilon)\int_{mt}^\epsilon \frac{du}{u}} = \frac{b}{B(\epsilon)}\,\frac{\epsilon - t + t\ln\frac{t}{\epsilon}}{t\ln\frac{\epsilon}{mt}} \to \infty \quad \text{as } t \to 0^+,$$

hence $I^m = \infty$. The assertion follows from Theorem 1. ∎

The uniqueness of the zero solution of problem (1) is considered either on the interval $[0,T]$ or on every interval $[0,T']$ for $T' < T$. We show that both notions mean the same.

Proposition 4. *Suppose that $\omega : (0,T] \times [0,a] \to \mathbb{R}$ is a Perron (resp. Kamke) function. Then ω is a Perron (resp. Kamke) function on $[0,T']$ for every $T' \le T$.*

Proof. Let $y'(t) = \omega(t,y(t))$ for $t \in (0,T')$ with $T' < T$ and $y(0) = 0$. Suppose that $y \not\equiv 0$. Define $\varphi(t) = y(t)$ for $t \in [0,T']$ and $\varphi(t) = y(T')$ for $t > T'$. It is clear that the assumptions of Lemma 1 are satisfied. Since $\varphi(T) \ne 0$, there exists a non-trivial solution x of (1) defined on $[0,T]$ such that $x \le y$ on $[0,T']$. This implies that $D^+x(0) = 0$ if $D^+y(0) = 0$. The contradiction gives $y \equiv 0$. ∎

Remark 3. One can generalize Proposition 4 to functional comparison functions $\sigma : (0,T] \times C([0,T];\mathbb{R}) \to \mathbb{R}$ which are continuous and nondecreasing with respect to the second variable. It is enough to apply Theorems 1 and 2 from [1] instead of Lemma 1.

The simplest Perron function is the Lipschitz function $L(t)x$, where L is integrable on $[0,T]$. We know that a sum of Perron functions may not be a Perron function (see Proposition 1). Let us check what happens if one of the summands is a Lipschitz function. Define

$$B(x) = \begin{cases} n^{-1/2}, & x \in \left[\frac{1}{n+1}, \frac{1}{n} - \frac{3}{n^3}\right], \\ n^{-2}, & x \in \left[\frac{1}{n} - \frac{2}{n^3}, \frac{1}{n} - \frac{1}{n^3}\right], \\ \text{linear otherwise.} \end{cases} \tag{8}$$

Proposition 5. *The function B defined by (8) is a Perron function. The functions $B(x) + x$ and $\sup\{B(y) : y \le x\}$ are not even Kamke functions.*

Proof. Since $B(x)+x \ge n^{-1/2}$ for $x \in \left[\frac{1}{n+1}, \frac{1}{n} - \frac{3}{n^3}\right]$, $B(x)+x \ge (n+1)^{-1}$ for $x \in \left[\frac{1}{n} - \frac{3}{n^3}, \frac{1}{n}\right]$ and $\sup_{y\le x} B(y) \ge n^{-1/2}$ for $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, we have

$$\int_0^{1/N} \frac{dx}{B(x)} \ge \sum_{n=N}^{\infty} \int_{1/n-2/n^3}^{1/n-1/n^3} n^2\,dx = \sum_{n=N}^{\infty} \frac{1}{n} = \infty,$$

$$\int_0^{1/N} \frac{dx}{B(x)+x} \le \sum_{n=N}^{\infty}\Big(\int_{1/n+1}^{1/n} \sqrt{n}\,dx + \int_{1/n-3/n^3}^{1/n} (n+1)\,dx\Big) < \infty,$$

$$\int_0^{1/N} \frac{dx}{\sup_{y\le x} B(y)} \le \sum_{n=N}^{\infty} \int_{1/(n+1)}^{1/n} \sqrt{n}\,dx < \infty.$$

The assertion follows from Theorem 1. ∎

If we assume that a non-Lipschitzian summand is a Perron function with separated variables and it satisfies some monotonicity condition, then the sum is still a Perron function.

PROPOSITION 6. *Suppose that a Perron function $A(t)B(x)$ has C-separated variables, $L(t)x$ is a Lipschitz function and the function B is nondecreasing on the set $\{x : B(x) \ge Kx\}$ for some $K > 1$. Then $A(t)B(x)+L(t)x$ and $A(t)\sup_{y\le x} B(y)$ are Perron functions.*

Proof. Set $\omega(t,x) = (A(t) + L(t))(B(x) + Kx)$. Both functions in the conclusion are not greater than ω. In view of Remark 1 and Theorem 1 it is enough to prove that $\int_0^a (B(x)+Kx)^{-1}\,dx = \infty$ assuming $\int_0^a B(x)^{-1}\,dx = \infty$. Suppose that $\int_0^a (B(x) + Kx)^{-1}\,dx < \infty$. If $B(x) \ge Kx$ for all $x \in (0,a')$ and some $a' \in (0,a)$, then $\int_0^a (B(x) + Kx)^{-1}\,dx \ge \frac{1}{2}\int_0^{a'} B(x)^{-1}\,dx = \infty$. Analogously we prove that the condition $B(x) \le Kx$ for all $x \in (0,a')$ is also impossible. This means that there exists a decreasing sequence $\{\alpha_j\} \subset (0,a)$ such that $B(\alpha_j) = K\alpha_j$ and $\lim_{j\to\infty}\alpha_j = 0$. We have

$$(\alpha_{j+1},\alpha_j) = Q_j \cup \bigcup_{p=1}^{N_j} (x_{jp}, y_{jp}),$$

where $B(x_{jp}) = Kx_{jp}$, $B(y_{jp}) = Ky_{jp}$, the function $B(x) - Kx$ does not change sign on (x_{jp}, y_{jp}) and $\int_{Q_j} dx/x \le K\,2^{-j}$. Let us arrange the set $\{x_{jp}, y_{jp}\}_{j,p}$ in a decreasing sequence $\{a_k\}_{k=1}^{\infty}$ and define

$$I_1 = \{k : B(x) \le Kx \text{ for } x \in (a_{k+1}, a_k)\},$$
$$I_2 = \{k \notin I_1 : B(x) \ge Kx \text{ for } x \in (a_{k+1}, a_k)\},$$
$$I_3 = \mathbb{N}\setminus(I_1 \cup I_2) = \{k : (\exists_j)\ (a_{k+1}, a_k) \subset Q_j\}\setminus(I_1\cup I_2).$$

We obtain

$$\int_0^a \frac{dx}{B(x)+Kx} \ge \sum_{k\in I_1}\int_{a_{k+1}}^{a_k}\frac{dx}{2Kx} + \sum_{k\in I_2}\int_{a_{k+1}}^{a_k}\frac{dx}{2B(x)} + \sum_{k\in I_3}\int_{a_{k+1}}^{a_k}\frac{dx}{2Kx} - 1$$
$$\ge \sum_{k\in I_1\cup I_3}\frac{a_k-a_{k+1}}{2Ka_k} + \sum_{k\in I_2}\frac{a_k-a_{k+1}}{2B(a_k)} - 1 = \frac{1}{2K}\sum_{k=1}^{\infty}\Big(1-\frac{a_{k+1}}{a_k}\Big) - 1,$$

hence $\sum_{k=1}^{\infty}(1-a_{k+1}/a_k)<\infty$. Consequently, $\sum_{k=1}^{\infty}\ln(a_k/a_{k+1})<\infty$, but

$$\sum_{k=1}^{N}\ln\frac{a_k}{a_{k+1}}=\ln\frac{a_1}{a_{N+1}}\to\infty\qquad\text{as } N\to\infty.$$

The contradiction proves the assertion. ■

If both summands are non-Lipschitzian, then the monotonicity does not guarantee that the sum is a Perron function, even for summands independent of the first variable. Let $a_n=1/n!$ and $b_n=na_{n-2}$ for $n=3,4,\ldots$ and define

$$B_1(x)=\begin{cases} b_{2n+1} & \text{for } x\in[a_{2n+1},a_{2n-1}(1-1/n)],\\ \text{linear otherwise,} & \end{cases}$$
$$B_2(x)=\begin{cases} b_{2n} & \text{for } x\in[a_{2n},a_{2n-2}(1-1/n)],\\ \text{linear otherwise.} & \end{cases}$$

PROPOSITION 7. *The functions B_1, B_2 are Perron functions, but $B(x)=\max\{B_1(x),B_2(x)\}$ and $B_1(x)+B_2(x)$ are not even Kamke functions.*

Proof. We have

$$\int_0^{a_3}\frac{dx}{B_1(x)}\ge\sum_{n=2}^{\infty}\frac{1}{b_{2n+1}}\Big(a_{2n-1}-a_{2n+1}-\frac{1}{n}a_{2n-1}\Big)$$
$$=\sum_{n=2}^{\infty}\frac{1}{2n+1}\Big(1-\frac{1}{2n(2n+1)}-\frac{1}{n}\Big)=\infty,$$
$$\int_0^{a_4}\frac{dx}{B_2(x)}\ge\sum_{n=3}^{\infty}\frac{1}{b_{2n}}\Big(a_{2n-2}-a_{2n}-\frac{1}{n}a_{2n-2}\Big)$$
$$=\sum_{n=3}^{\infty}\frac{1}{2n}\Big(1-\frac{1}{2n(2n-1)}-\frac{1}{n}\Big)=\infty.$$

By Theorem 1, B_1 and B_2 are Perron functions. Since $B(x)\ge b_n$ for $x\in[a_n,a_{n-1}]$, we get

$$\int_0^{a_3}\frac{dx}{B(x)}\le\sum_{n=4}^{\infty}\frac{1}{b_n}(a_{n-1}-a_n)=\sum_{n=4}^{\infty}\frac{1}{n^2}<\infty,$$

hence B is not a Perron function. The inequality $B_1(x)+B_2(x)\ge B(x)$ completes the proof. ■

To prove convergence of numerical methods for the equation

$$\partial_t z(t,x)=\sum_{j=1}^{n}f_j(t,x,z(\cdot))\partial_{x_j}z(t,x)+g(t,x,z(\cdot))$$

one can consider the conditions

$$\|f(t,x,y)-f(t,x,z)\| \le \sigma_1(t,\|y-z\|_t),$$
$$|g(t,x,y)-g(t,x,z)| \le \sigma_2(t,\|y-z\|_t),$$

where $\|u\|_t = \sup_{s\le t}|u(s,x)|$. It is sometimes assumed that $\sigma_1=\sigma_2=\sigma$ and $c\sigma$ is a Perron function for every $c>0$ (see, for instance, [3, Theorem 4.1]). The convergence can be proved under the assumption that $k\sigma_1+\sigma_2$ is a Perron function for every $k>0$. We show that the last assumption is weaker.

PROPOSITION 8. *For every $k>0$ the function $\rho(t,x)+kx$ is a Perron function, where ρ is defined by* (7). *The function $\omega(t,x)=2\max\{\rho(t,x),x\}$ is not a Perron function.*

Proof. Since $\omega(t,x)\ge 2\rho(t,x)$ and 2ρ is not a Perron function (see Proposition 1) the second assertion is proved. Suppose that $x'(t)=\rho(t,x(t))$ $+\,kx(t)$, $x(0)=0$. If $x(t)=t^2$ for some $t\in(0,1/k)$, then $x'(t)=t+kt^2<2t=(t^2)'$. Hence only two cases are possible:

(a) $x(t)>t^2$ for every $t\in(0,T)$ and some $T\in(0,1/k)$.
(b) $x(t)<t^2$ for every $t\in(0,1/k)$.

In case (a), we have $x'(t)=\sqrt{x(t)}+kx(t)$, so $x(t)=k^{-2}(\exp(kt/2)-1)^2=t^2/4+kt^3/16+\cdots<t^2$ for sufficiently small t. The contradiction shows that $x(t)<t^2$ for every $t\in(0,1/k)$. This implies that $x'(t)=x(t)/t+kx(t)$ and $x(t)=Ct\exp(kt)<t^2$ for $t\in(0,1/k)$. It is obvious that $C=0$ and $x(t)\equiv 0$ on $[0,1]$. ∎

In [4] it is shown that if

$$|f(t,x)-f(t,y)|\le\omega(t,|x-y|) \tag{9}$$

for any Kamke function ω and the function f is continuous, then the function

$$\omega_f(t,u)=\sup_{|x-y|=u}|f(t,x)-f(t,y)| \tag{10}$$

is a Perron function.

On the other hand, for f discontinuous the condition (9) with some Kamke function ω does not give uniqueness of solutions of the Cauchy problem

$$x'(t)=f(t,x(t)),\qquad x(t_0)=x_0; \tag{11}$$

take for instance $f(t,x)=x/t$, $t_0=x_0=0$. We obtain uniqueness if we additionally assume the following condition (the weakest known to the author):

$$D^+|x-y|(t_0)=\limsup_{t\to t_0+}\frac{1}{t-t_0}\left|\int_{t_0}^{t}[f(s,x(s))-f(s,y(s))]\,ds\right|=0 \tag{12}$$

for any solutions x,y of (11). In general, if we have no information about the properties of f except condition (9), then the only possibility to verify condition (12) is the assumption that $\lim_{t\to 0+} \frac{1}{t}\int_0^t \omega(s,u(s))\,ds = 0$ for every continuous function u such that $u(0)=0$. But this implies that ω is a Perron function.

The above suggests that Kamke functions are not important from the practical point of view. However, let us consider the following example.

Example 1. Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be any function satisfying the Carathéodory condition such that

$$f(t,x) = \frac{|x|-\sqrt{t}}{t} + f\left(t, \frac{x}{|x|}\sqrt{t}\right) \quad \text{for } |x| > \sqrt{t},$$
$$|f(t,x)-f(t,y)| \le |x-y|/t \quad \text{for } |x|,|y| \le \sqrt{t},$$
$$xf(t,x) \le 0 \quad \text{for } t^2 \le |x| \le \sqrt{t}.$$

For instance, we can take $f(t,x) = -x$ for $|x| \le \sqrt{t}$. One can verify that condition (12) is satisfied for $t_0 = x_0 = 0$ since there is no solution of (11) passing through the point (t,η) with $|\eta| \ge t^2$. Moreover, $\omega_f(t,u) = u/t$ for every $a\in(0,1]$ and $t<a^2$, $u \le a-\sqrt{t}$, hence ω_f is not a Perron function.

The above example suggests that Kamke functions satisfying the Carathéodory condition are still valuable.

References

[1] A. Augustynowicz and Z. Kamont, *On Kamke's functions in uniqueness theorems for first order partial differential equations*, Nonlinear Anal. 14 (1990), 837–850.

[2] E. Kamke, *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen, II*, Acta Math. 58 (1932), 57–85.

[3] A. Kępczyńska, *Implicit difference methods for quasilinear differential functional equations on the Haar pyramid*, Z. Anal. Anwend., to appear.

[4] C. Olech, *Remarks concerning criteria for uniqueness of solutions of ordinary differential equations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 661–666.

[5] O. Perron, *Ein neuer Existenzbeweis für die Integrale der Differentialgleichung* $y' = f(x,y)$, Math. Ann. 76 (1915), 471–484.

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