Mixed 3-Sasakian structures and curvature

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Abstract. We deal with two classes of mixed metric 3-structures, namely the mixed 3-Sasakian structures and the mixed metric 3-contact structures. First, we study some properties of the curvature of mixed 3-Sasakian structures. Then we prove the identity between the class of mixed 3-Sasakian structures and the class of mixed metric 3-contact structures.

1. Introduction. The geometry of 3-Sasakian manifolds has been a well-known topic, since their introduction, independently, by Udrişte [22] and Kuo [19]. It was studied, in a first stage, by Ishihara, Kashiwada, Konishi, Kuo, Tachibana, Tanno, Yu and other geometers of the Japanese school, and then from different viewpoints by Boyer, Galicki and Mann; in particular, we mention the remarkable survey [4], to which we refer the reader for more details about such structures, as well as for historical remarks. On the other hand, studies of analogous odd-dimensional geometries related to the algebra of paraquaternionic numbers have begun very recently (see, for example, [1], [2], [8], [11] and [12]).

In analogy with an early result of Kashiwada [15] for Sasakian 3-structures, a first result we shall present in this paper is for manifolds endowed with mixed 3-Sasakian structures, which are also considered in [8], where they are called split three Sasakian structures. We give a direct proof that they are Einstein, which is analogous to the well-known fact that a paraquaternionic Kähler manifold is Einstein (cf. [10]). To this end, we shall need some formulas for the curvature tensor of a manifold with parasasakian structure and of a manifold with indefinite Sasakian structure. Some results recently proved in [23] will also be recovered.

The second result is concerned with the identity between the class of mixed metric 3-contact structures and the class of mixed 3-Sasakian struc-
tures (see Kashiwada [17] for the case of 3-contact metric manifolds). It is based on an extension of Kashiwada’s generalization of a lemma of Hitchin (cf. [16]) to the almost hyper parahermitian case.

The content of the paper is now briefly described.

In Section 2 we give some fundamental definitions and facts about paracomplex metric structures (cf. [9], [23]), which together with the notion of indefinite almost contact metric structure ([5]) are at the root of the notion of mixed metric 3-structure. We also recall a few definitions concerning almost hyper parahermitian structures. In Section 3, after introducing the notion of \([r]-Sasakian\) structure, \(r = \pm 1\), to mean an indefinite Sasakian structure for \(r = +1\), and a parasasakian structure for \(r = -1\), we consider some preliminary issues, needed to state, in Section 4, the result concerning the mixed 3-Sasakian manifolds. Finally, Section 5 is devoted to proving that a mixed metric 3-contact structure is in fact a mixed 3-Sasakian structure.

All manifolds and tensor fields are assumed to be smooth.

2. Preliminaries. We recall a few definitions about paracomplex and hyper paracomplex structures. For more details we refer the reader to [7] and [13].

**Definition 2.1.** An almost product structure on a manifold \(M\) is a \((1,1)\)-type tensor field \(F \neq \pm I\) satisfying \(F^2 = I\); the pair \((M,F)\) is then said to be an almost product manifold.

On an almost product manifold \((M,F)\) we have \(TM = T^+M \oplus T^-M\), where \(T^+M\) and \(T^-M\) are the eigensubbundles associated to the eigenvalues +1 and −1 of \(F\). \((M,F)\) is called an almost paracomplex manifold if \(\text{rank}(T^+M) = \text{rank}(T^-M)\). Finally, an almost product (resp. almost paracomplex) manifold \((M,F)\) is called a product (resp. paracomplex) manifold if \(N_F = 0\), \(N_F\) being the Nijenhuis tensor field of the structure \(F\). Any (almost) paracomplex manifold has even dimension.

An (almost) paracomplex manifold \((M,F)\) is called (almost) parahermitian if there exists a metric tensor \(g\) compatible with \(F\), i.e. such that \(g(FX,Y) + g(X,FY) = 0\) for any \(X,Y \in \Gamma(TM)\). Such a metric is necessarily semi-Riemannian, with neutral signature.

**Definition 2.2.** An almost hyper parahermitian structure on a manifold \(M\) is a triple \((J_1, J_2, J_3)\) of \((1,1)\)-type tensor fields, together with a semi-Riemannian metric \(g\) satisfying:

(i) \((J_a)^2 = -\tau_a I\) for any \(a \in \{1,2,3\}\),
(ii) \(J_a J_b = \tau_c J_c = -J_b J_a\) for any cyclic permutation \((a,b,c)\) of \((1,2,3)\),
(iii) \(g(J_a X,Y) + g(X,J_a Y) = 0\) for any \(a \in \{1,2,3\}\) and \(X,Y \in \Gamma(TM)\).
where $\tau_1 = -1$, $\tau_2 = -1$ and $\tau_3 = +1$. Then $(M, J_1, J_2, J_3, g)$ will be said to be an almost hyper parahermitian manifold.

Such a manifold has dimension divisible by four and the metric has neutral signature. An almost hyper parahermitian structure on a manifold $M$ will be called hyper parahermitian if for any $a \in \{1, 2, 3\}$, the Nijenhuis tensor field $N_a$ vanishes, that is, each structure $J_a$ is integrable. Then $M$ will be called a hyper parahermitian manifold. An almost hyper parahermitian manifold is hyper parahermitian if and only if at least two of the Nijenhuis tensor fields vanish (cf. [13]).

**Definition 2.3.** Let $M$ be a manifold. An almost paracontact structure on $M$ is a triple $(\varphi, \xi, \eta)$, where $\varphi \in \mathfrak{T}^1(M)$, $\xi \in \Gamma(TM)$ and $\eta \in \wedge^1(M)$, satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$. Then $M$ is said to be an almost paracontact manifold, denoted by $(M, \varphi, \xi, \eta)$. An almost paracontact structure $(\varphi, \xi, \eta)$ will be called normal if $N_{\varphi} = 2d\eta \otimes \xi$, $N_{\varphi}$ being the Nijenhuis tensor field of $\varphi$.

Almost paracontact structures were originally introduced by I. Satō in [20] and [21], where he also studied the properties of manifolds endowed with such structures and with a Riemannian metric satisfying suitable compatibility conditions. Moreover, one may find similar definitions in [14] and [23], where the further condition that the restriction $\varphi|_{\text{Im}(\varphi)}$ is an almost paracomplex structure on the distribution $\text{Im}(\varphi)$ is required. The notion of normality for an almost paracontact structure is defined, as in the classical almost contact case (cf. [3]), through the integrability of the almost product structure $F$ canonically induced on the manifold $M \times \mathbb{R}$, defined by $F(X, f \frac{d}{dt}) := (\varphi X + f\xi, \eta(X)\frac{d}{dt})$ (cf. [14], [23]).

Other properties of almost paracontact manifolds $(M, \varphi, \xi, \eta)$, which are immediate consequences of the above definition, are $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\ker(\varphi) = \text{Span}(\xi)$, $\ker(\eta) = \text{Im}(\varphi)$ and $TM = \text{Im}(\varphi) \oplus \text{Span}(\xi)$.

Endowing an almost paracontact manifold with a metric tensor field and considering a suitable compatibility condition, we obtain the notion of almost paracontact metric manifold.

**Definition 2.4 ([23]).** Let $(M, \varphi, \xi, \eta)$ be an almost paracontact manifold and $g$ a metric tensor field on $M$, that is, a symmetric, nondegenerate $(0,2)$-type tensor field on $M$. Then $g$ is said to be compatible with the structure $(\varphi, \xi, \eta)$ if

$$g(\varphi X, \varphi Y) = -g(X, Y) + \varepsilon\eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$, with $\varepsilon = \pm 1$ according as $\xi$ is spacelike or timelike. Then $(\varphi, \xi, \eta, g)$ is said to be an almost paracontact metric structure. We shall call the structure positive or negative according as $\varepsilon = +1$ or $\varepsilon = -1$. 

Then \((M, \varphi, \xi, \eta, g)\) will be called an *almost paracontact metric manifold*. Such a structure \((\varphi, \xi, \eta, g)\) will be called *normal* if \(N_\varphi = 2d\eta \otimes \xi\).

In [9], the author refers to the same kind of structure called almost paracontact hyperbolic metric structure.

As a consequence of the above definition, for an almost paracontact metric manifold \((M, \varphi, \xi, \eta, g)\), the pair \((F, g)\), where \(F := \varphi|_{\text{Im(}\varphi)}\), is an almost parahermitian structure on the distribution \(\text{Im}(\varphi)\). Hence \(\text{rank}(\text{Im}(\varphi)) = 2m\) and \(\dim(M) = 2m + 1\). Furthermore, the signature of \(g\) on \(\text{Im}(\varphi)\) is \((m, m)\), where we put first the minus signs, and the signature of \(g\) on \(TM\) is \((m, m + 1)\) or \((m + 1, m)\) according as \(\xi\) is spacelike (the structure is positive) or timelike (the structure is negative). It follows that \(g\) is a Lorentzian metric only if \(m = 1\) and \(\dim(M) = 3\).

We know that \(TM\) is the orthogonal direct sum of \(\text{Im}(\varphi)\) and \(\text{Span}(\xi)\), and finally that \(\eta(X) = \varepsilon g(X, \xi)\) and \(g(\varphi X, Y) + g(X, \varphi Y) = 0\) for any \(X, Y \in \Gamma(TM)\).

Particular classes of almost paracontact metric structures are defined as follows.

**Definition 2.5 ([9], [23])**. Let \((M, \varphi, \xi, \eta, g)\) be an almost paracontact metric manifold. Then it is said to be a

(i) *paracontact metric manifold* if \(d\eta = \Phi\);

(ii) *parasasakian manifold* if \(d\eta = \Phi\) and the structure is normal;

(iii) *para-K-contact manifold* if \(d\eta = \Phi\) and \(\xi\) is a Killing vector field,

where \(\Phi(X, Y) := g(X, \varphi Y)\) is the fundamental 2-form associated with the almost paracontact metric structure.

Furthermore, we recall the following result.

**Proposition 2.6**. Let \((M, \varphi, \xi, \eta, g)\) be an almost paracontact metric manifold. Then it is a parasasakian manifold if and only if

\[
(\nabla_X \varphi)(Y) = -g(X, Y)\xi + \varepsilon\eta(Y)X
\]

for any \(X, Y \in \Gamma(TM)\), where \(\varepsilon = g(\xi, \xi) = \pm 1\).

We assume the following definition of mixed (metric) 3-structure, which is introduced in [11] and [12], although in a different form.

**Definition 2.7**. Let \(M\) be a manifold. A *mixed 3-structure* on \(M\) is a triple of structures \((\varphi_a, \xi_a, \eta^a)\), \(a \in \{1, 2, 3\}\), which are almost paracontact structures for \(a = 1, 2\) and an almost contact structure for \(a = 3\), satisfying

\[
\begin{align*}
(1) & \quad \varphi_a \varphi_b - \tau_a \eta^b \otimes \xi_a = \tau_c \varphi_c = -\varphi_b \varphi_a + \tau_b \eta^a \otimes \xi_b, \\
(2) & \quad \eta^a \circ \varphi_b = \tau_c \eta^c = -\eta^b \circ \varphi_a, \\
(3) & \quad \varphi_a(\xi_b) = \tau_b \xi_c, \quad \varphi_b(\xi_a) = -\tau_a \xi_c,
\end{align*}
\]
for any cyclic permutation \((a, b, c)\) of \((1, 2, 3)\), with \(\tau_1 = \tau_2 = -1 = -\tau_3\). A mixed metric 3-structure on \(M\) is a triple of structures \((\varphi_a, \xi_a, \eta^a, g)\), \(a \in \{1, 2, 3\}\), which are almost paracontact metric structures for \(a = 1, 2\), and an almost contact metric structure for \(a = 3\), satisfying (1)–(3).

From now on, a mixed 3-structure and a mixed metric 3-structure on a manifold \(M\) will be denoted simply by \((\varphi_a, \xi_a, \eta^a)\) and \((\varphi_a, \xi_a, \eta^a, g)\), with the condition \(a \in \{1, 2, 3\}\) understood.

**Remark 2.8.** Equivalently, a mixed metric 3-structure on a manifold \(M\) is given by a mixed 3-structure \((\varphi_a, \xi_a, \eta^a)\), together with a metric tensor \(g\) satisfying the following compatibility condition:

\[
g(\varphi_a X, \varphi_a Y) = \tau_a g(X, Y) - \varepsilon_a \eta^a(X) \eta^a(Y)
\]

for any \(a \in \{1, 2, 3\}\), and any \(X, Y \in \Gamma(TM)\), where \(\varepsilon_a = g(\xi_a, \xi_a) = \pm 1\).

**Remark 2.9.** We point out that the above definition of mixed 3-structure, without the metric compatibility, is equivalent to the definition given in [11], and very recently in [12], providing that one substitutes the structures \((\varphi_1, \xi_1, \eta^1)\), \((\varphi_2, \xi_2, \eta^2)\) and \((\varphi_3, \xi_3, \eta^3)\) of [11] and [12] with \((\varphi_3, \xi_3, \eta^3)\), \((\varphi_1, \xi_1, \eta^1)\) and \((\varphi_2, \xi_2, \eta^2)\), respectively, and then the vector fields \(\xi_1\) and \(\xi_2\) with their negatives.

We remark that the conditions (1)–(4) are compatible. We first observe that from (3), one has \(\eta^a(\xi_c) = 0\) whenever \(a \neq c\), and by the definition of almost (para)contact metric structures, one gets

\[
\eta^a(\xi_c) = \delta^a_c
\]

for any \(a, c \in \{1, 2, 3\}\). Moreover, since each structure \((\varphi_a, \xi_a, \eta^a, g)\) is almost (para)contact metric, one has

\[
\eta^a(X) = \varepsilon_a g(X, \xi_a)
\]

for any \(X \in \Gamma(TM)\), and any \(a \in \{1, 2, 3\}\). From (3) one has \(\varphi_2(\xi_3) = \xi_1 = \varphi_3(\xi_2)\), and using (4) and (5) we find, on one hand,

\[
g(\xi_1, \xi_1) = g(\varphi_2(\xi_3), \varphi_2(\xi_3)) = -g(\xi_3, \xi_3),
\]

and on the other hand,

\[
g(\xi_1, \xi_1) = g(\varphi_3(\xi_2), \varphi_3(\xi_2)) = g(\xi_2, \xi_2).
\]

Thus, \(\varepsilon_1 = \varepsilon_2 = -\varepsilon_3\). Analogously, starting from \(\varphi_1(\xi_2) = -\xi_3 = -\varphi_2(\xi_1)\) and from \(\varphi_3(\xi_1) = -\xi_2 = \varphi_1(\xi_3)\), we obtain the same restrictions on the values of \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\). Let us now verify that (4) makes sense for arbitrary choices of \(\xi_1, \xi_2\) and \(\xi_3\). Fixing a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\) with \((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+1, +1, -1)\), for \(a = 1\) the condition (4) becomes

\[
g(\varphi_1 X, \varphi_1 Y) = -g(X, Y) + \eta^1(X) \eta^1(Y),
\]
and using (3), (5) and (6), putting \((X, Y) = (\xi_1, \xi_1)\), we have \(0 = -g(\xi_1, \xi_1) + \eta^1(\xi_1)\eta^1(\xi_1) = -\varepsilon_1 + 1 = 0\). If the mixed metric 3-structure is such that \((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, +1)\), it is easy to check that we get the same identity. Analogously, one verifies the consistency for the other choices of \((\xi_b, \xi_c)\), choosing \(a = 2\) and \(a = 3\) in (4).

Let us check that (1) and (4) are compatible for any \(X, Y \in \Gamma(TM)\). If we fix \((a, b, c) = (1, 2, 3)\), then (1) becomes

\[
\varphi_1 \varphi_2 + \eta^2 \otimes \xi_1 = \varphi_3 = -\varphi_2 \varphi_1 - \eta^1 \otimes \xi_2. 
\]

From (4), on one hand we have

\[
g(\varphi_3 X, \varphi_3 Y) = g(X, Y) - \varepsilon_3 \eta^3(X)\eta^3(Y),
\]

and on the other hand, by (2), (4) and (7),

\[
g(\varphi_3 X, \varphi_3 Y) = g(\varphi_1 \varphi_2 X + \eta^2(X)\xi_1, \varphi_1 \varphi_2 Y + \eta^2(Y)\xi_1)
\]

\[
= g(\varphi_1 \varphi_2 X, \varphi_1 \varphi_2 Y) + \varepsilon_1 \eta^2(X)\eta^2(Y)
\]

\[
= -g(\varphi_2 X, \varphi_2 Y) + \varepsilon_1 \eta^1(\varphi_2 X)\eta^1(\varphi_2 Y) + \varepsilon_1 \eta^2(X)\eta^2(Y)
\]

\[
= g(X, Y) - \varepsilon_2 \eta^2(X)\eta^2(Y)
\]

\[
+ \varepsilon_1 \eta^1(\varphi_2 X)\eta^1(\varphi_2 Y) + \varepsilon_1 \eta^2(X)\eta^2(Y),
\]

from which, using \(\varepsilon_1 = \varepsilon_2 = -\varepsilon_3\), (8) follows. Again, by (2), (4) and (7), we have

\[
g(\varphi_3 X, \varphi_3 Y) = g(\varphi_2 \varphi_1 X + \eta^1(X)\xi_2, \varphi_2 \varphi_1 Y + \eta^1(Y)\xi_2)
\]

\[
= g(\varphi_2 \varphi_1 X, \varphi_2 \varphi_1 Y) + \varepsilon_2 \eta^1(X)\eta^1(Y)
\]

\[
= -g(\varphi_1 X, \varphi_1 Y) + \varepsilon_2 \eta^2(\varphi_1 X)\eta^2(\varphi_1 Y) + \varepsilon_2 \eta^1(X)\eta^1(Y)
\]

\[
= g(X, Y) - \varepsilon_1 \eta^1(X)\eta^1(Y)
\]

\[
+ \varepsilon_2 \eta^2(\varphi_1 X)\eta^2(\varphi_1 Y) + \varepsilon_2 \eta^1(X)\eta^1(Y),
\]

from which, using \(\varepsilon_1 = \varepsilon_2 = -\varepsilon_3\), (8) follows again. Analogously, one verifies that the consistency also holds starting from the other two cyclic permutations of \((a, b, c)\).

Let \(M\) be a manifold endowed with a mixed 3-structure \((\varphi_a, \xi_a, \eta^a)\). Considering the two distributions \(\mathcal{H} := \bigcap_{a=1}^3 \ker(\eta^a)\) and \(\mathcal{V} := \text{Span}(\xi_1, \xi_2, \xi_3)\), one has the decomposition \(TM = \mathcal{H} \oplus \mathcal{V}\). It follows that \((\varphi_1|_{\mathcal{H}}, \varphi_2|_{\mathcal{H}}, \varphi_3|_{\mathcal{H}})\) is an almost hyper para complex structure on the distribution \(\mathcal{H}\). Hence \(\text{rank}(\mathcal{H}) = 2n\) and \(\text{dim}(M) = 2n + 3\). Furthermore, if we have a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\) on \(M\), then \((\varphi_a|_{\mathcal{H}}, g), a \in \{1, 2, 3\}\), becomes an almost hyper para hermitian structure on the distribution \(\mathcal{H}\). Hence \(\text{rank}(\mathcal{H}) = 4m\) and \(\text{dim}(M) = 4m + 3\). As an obvious consequence we have the following result.

**Proposition 2.10.** Let \(M\) be a manifold with \(\text{dim}(M) = 2n + 3\), endowed with a mixed 3-structure \((\varphi_a, \xi_a, \eta^a)\). If \(n \neq 2m\), then there is no metric
tensor field \( g \) on \( M \) compatible with the mixed 3-structure, and \( M \) cannot have any mixed metric 3-structure.

The compatibility condition (4) between a metric tensor \( g \) and a mixed 3-structure \((\varphi_a, \xi_a, \eta^a)\) on a \((4m + 3)\)-dimensional manifold \( M \), together with (3), has some consequences on the signature of the metric \( g \) too. Since \( g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3) \), the vector fields \( \xi_1 \) and \( \xi_2 \) related to the almost paracontact metric structures are either both spacelike or both timelike. We may therefore distinguish between positive and negative mixed metric 3-structures according as \( \xi_1 \) and \( \xi_2 \) are both spacelike \((\varepsilon_1 = \varepsilon_2 = +1)\) or both timelike \((\varepsilon_1 = \varepsilon_2 = -1)\). This forces the causal character of the third vector field \( \xi_3 \). Since the signature of \( g \) on \( \mathcal{H} \) is necessarily neutral \((2m, 2m)\), we have only the following two possibilities:

(i) the signature of \( g \) on \( TM \) is \((2m + 1, 2m + 2)\) if the mixed metric 3-structure is positive \((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+1, +1, -1)\);

(ii) the signature of \( g \) on \( TM \) is \((2m + 2, 2m + 1)\) if the mixed metric 3-structure is negative \((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, +1)\).

We point out that any metric \( g \) which is compatible with a mixed 3-structure, in the sense of (4), can never be Lorentzian and that the definition of mixed metric 3-structure given in [12] is equivalent to that of a negative mixed metric 3-structure.

**Example 2.11 ([6]).** Let \( M^{4m+3} \) be any orientable nondegenerate hypersurface of an almost hyper parahermitian manifold \((M^{4m+4}, J_a, G)_{a=1,2,3}\). If \( N \in \Gamma(TM^\perp) \) is a unit normal vector field such that \( G(N, N) = s = \pm 1 \), put \( \xi_a := -\tau_a J_a N \) for any \( a \in \{1, 2, 3\} \), and define three \((1, 1)\)-type tensor fields \( \varphi_a \) and three 1-forms \( \eta^a \) on \( M \) such that \( J_a X = \varphi_a X + \eta^a(X) N \), for any \( X \in \Gamma(TM) \) and any \( a \in \{1, 2, 3\} \). Then, denoting by \( g \) the metric induced on \( M \) from \( G \), it is easy to check that \((\varphi_a, \xi_a, \eta^a, g)\) is a mixed metric 3-structure on \( M \) with sign \( \sigma = -s \).

Finally, we adopt the following definition of mixed 3-Sasakian structure on a manifold, which is already given in [8], although in a different form and called split three Sasakian structure.

**Definition 2.12.** Let \( M \) be a manifold with a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\). This structure will be said to be a **mixed 3-Sasakian structure** if \((\varphi_1, \xi_1, \eta^1, g)\) and \((\varphi_2, \xi_2, \eta^2, g)\) are both parasasakian structures, and \((\varphi_3, \xi_3, \eta^3, g)\) is an indefinite Sasakian structure. Then \((M, \varphi_a, \xi_a, \eta^a, g)\) will be called **mixed 3-Sasakian manifold**.

**Remark 2.13.** The previous definition is equivalent to the notion of split three Sasakian structure given in [8], providing that one replaces the structures \((\Phi_1, \xi^1), (\Phi_2, \xi^2)\) and \((\Phi_3, \xi^3)\) of [8] with \((\varphi_3, \xi_3, \eta^3), (\varphi_2, \xi_2, \eta^2)\).
and \((\varphi_1, \xi_1, \eta^1)\), and the vector field \(\xi_3\) with its negative, taking the vector fields \(\xi_1, \xi_2\) and \(\xi_3\) with \(g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -1\) and \(g(\xi_3, \xi_3) = 1\), that is, \((\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, +1)\).

**Remark 2.14.** By Proposition 2.6, a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\) on a manifold \(M\) is mixed 3-Sasakian if and only if
\[
(\nabla_X \varphi_a)(Y) = \tau_a(g(X, Y)\xi_a - \varepsilon_a \eta^a(Y)X)
\]
for any \(X, Y \in \Gamma(TM)\) and any \(a \in \{1, 2, 3\}\), with \(\tau_1 = \tau_2 = -1 = -\tau_3\).

We remark that Definition 2.12 is not equivalent to that given in [12]. More precisely, referring to [12], the condition \((\nabla_X \varphi_2)(Y) = g(\varphi_2 X, \varphi_2 Y)\xi_2 + \eta^2(Y)(\varphi_2)^2(X)\) in Definition 4.3, using the compatibility condition (29) there, may be rewritten in the form \((\nabla_X \varphi_1)(Y) = -g(X, Y)\xi_1 + \eta^1(Y)X\), which corresponds to
\[
(\nabla_X \varphi_1)(Y) = g(X, Y)\xi_1 + \eta^1(Y)X
\]
in our notation. Since the definition of mixed metric 3-structure given in [12] is equivalent to that of negative mixed metric 3-structure, writing the condition (9) for \(\tau_a = \tau_1 = -1\) and \(\varepsilon_a = \varepsilon_1 = -1\), we get \((\nabla_X \varphi_1)(Y) = -g(X, Y)\xi_1 - \eta^1(Y)\), which is clearly the negative of (10). One obtains an analogous result considering the condition on \((\nabla_X \varphi_3)(Y)\) of [12].

**3. On the curvature of \([r]\)-Sasakian structures.** In this section, we prove some useful formulas concerning the curvature of both parasa-

sakian structures and indefinite Sasakian structures. To treat both cases simultaneously, we introduce the synthetic notation of \([r]\)-Sasakian struc-
ture on a manifold \(M\), considering a system \((\varphi, \xi, \eta, g)\) where \(\varphi \in \mathfrak{T}_1^1(M)\), \(\xi \in \Gamma(TM)\), \(\eta \in \bigwedge^1(M)\) and \(g \in \mathfrak{T}_0^0(M)\) is a metric tensor field, such that \(g(\xi, \xi) = \varepsilon = \pm 1\), \(\varphi^2 = r(-I + \eta \otimes \xi)\), \(\eta(\xi) = 1\) and
\[
g(\varphi X, \varphi Y) = r(g(X, Y) - \varepsilon \eta(X)\eta(Y))
\]
\[
(\nabla_X \varphi)(Y) = r(g(X, Y)\xi - \varepsilon \eta(Y)X).
\]
Thus, we obtain an indefinite Sasakian structure for \(r = +1\) and a parasa-
sakian structure for \(r = -1\). From (12) it follows that \(\nabla_X \xi = -\varepsilon \varphi(X)\) for any \(X \in \Gamma(TM)\).

Following [18], the curvature tensor field \(R \in \mathfrak{T}_3^1(M)\) of the Levi-Civita connection \(\nabla\), the Riemannian curvature tensor field \(R \in \mathfrak{T}_4^1(M)\), and the Ricci curvature tensor field \(\rho \in \mathfrak{T}_2^0(M)\) are defined by
\[
R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,
\]
\[
R(X, Y, Z, W) := g(R(Z, W)Y, X) = -g(R(X, Y)W, Z),
\]
\[
\rho(X, Y) := \text{tr}_g\{Z \mapsto R(Z, X)Y\} = \sum_{i=1}^m \varepsilon_i g(R(E_i, X)Y, E_i),
\]
where \((E_i)_{1 \leq i \leq m}\) is a local orthonormal frame, \(\varepsilon_i = g(E_i, E_i)\) and \(m = \text{dim}(M)\).

Lemma 3.1. Let \(M\) be a manifold endowed with an \([r]\)-Sasakian structure \((\varphi, \xi, \eta, g)\). Then, for any \(X, Y, Z, W \in \Gamma(TM)\),
\[
g(R(X, Y)Z, \varphi W) + g(R(X, Y)\varphi Z, W) = -r\varepsilon P(X, Y, Z, W),
\]
where \(P \in \mathcal{T}^1_1(M)\) is the tensor field defined by
\[
P(X, Y, Z, W) := d\eta(X, Z)g(Y, W) - d\eta(X, W)g(Y, Z) - d\eta(Y, Z)g(X, W) + d\eta(Y, W)g(X, Z).
\]

Proof. Denoting by \(\Phi\) the fundamental 2-form defined by \(\Phi(X, Y) := g(X, \varphi Y)\), let us consider the derivation \(R_{XY}\) of the tensor algebra \(\mathfrak{T}(M)\), canonically induced from the \((1, 1)\)-tensor field \(R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\). For any \(X, Y, Z, W \in \Gamma(TM)\), we have
\[
(R_{XY}\Phi)(Z, W) = R_{XY}(g(Z, \varphi W)) - \Phi(R_{XY}Z, W) - \Phi(Z, R_{XY}W)
= -g(R_{XY}Z, \varphi W) - g(R_{XY}\varphi Z, W)
= -g(R(X, Y)Z, \varphi W) - g(R(X, Y)\varphi Z, W)
\]
Let us compute again the term \((R_{XY}\Phi)(Z, W)\), using (12). One has
\[
(\nabla_X \nabla_Y \Phi)(Z, W) = X(\nabla_Y \Phi(Z, W)) - \nabla_Y \Phi(\nabla_X Z, W) - \nabla_Y \Phi(Z, \nabla_X W)
= X(g(Z, (\nabla_Y \varphi)(W))) - g(\nabla_X Z, (\nabla_Y \varphi)(W))
+ g((\nabla_Y \varphi)(Z), \nabla_X W)
= r\varepsilon(X(\eta(Z)g(Y, W)) - X(\eta(W)g(Z, Y))
- \eta(\nabla_X Z)g(Y, W) + \eta(W)g(\nabla_X Z, Y)
+ \eta(\nabla_X W)g(Y, Z) - \eta(Z)g(Y, \nabla_X W)).
\]
Switching \(X\) and \(Y\), we have
\[
(\nabla_Y \nabla_X \Phi)(Z, W) = r\varepsilon(Y(\eta(Z)g(X, W)) - Y(\eta(W)g(Z, X))
- \eta(\nabla_Y Z)g(X, W) + \eta(W)g(\nabla_Y Z, X)
+ \eta(\nabla_Y W)g(X, Z) - \eta(Z)g(X, \nabla_Y W)).
\]
Finally,
\[
(\nabla_{[X,Y]} \Phi)(Z, W) = g(Z, (\nabla_{[X,Y]} \varphi)(W))
= r\varepsilon(\eta(Z)g([X,Y], W) - \eta(W)g(Z, [X,Y])).
\]
It follows that
\[
(R_{XY}\Phi)(Z, W) = r\varepsilon((\nabla_X \eta)(Z)g(Y, W) - (\nabla_X \eta)(W)g(Z, Y)
- (\nabla_Y \eta)(Z)g(X, W) + (\nabla_Y \eta)(W)g(Z, X)).
\]
Since $\nabla_X \xi = -\varepsilon \varphi(X)$ and $\Phi = d\eta$, $(\nabla_X \eta)(Y) = d\eta(X, Y)$, we have
\begin{equation}
(R_{XY}\Phi)(Z, W) = r\varepsilon (d\eta(X, Z)g(Y, W) - d\eta(X, W)g(Z, Y)) \nonumber
- d\eta(Y, Z)g(X, W) + d\eta(Y, W)g(Z, X)) = r\varepsilon P(X, Y, Z, W).
\end{equation}
Now, (14) and (15) imply (13). \hfill \blacksquare

It is easy to prove the following lemma.

**Lemma 3.2.** Let $M$ be a manifold endowed with an almost (para)contact metric structure $(\varphi, \xi, \eta, g)$. Then, for any $X_1, X_2, X_3, X_4 \in \Gamma(TM)$,

(i) $P(X_1, X_2, X_3, X_4) = -P(X_2, X_1, X_3, X_4)$;
(ii) $P(X_1, X_2, X_3, X_4) = -P(X_1, X_2, X_4, X_3)$;
(iii) $P(X_1, X_2, X_3, X_4) = -P(X_3, X_4, X_1, X_2)$;
(iv) $P(X_1, X_2, X_3, X_4) = P(X_4, X_3, X_2, X_1)$.

**Proposition 3.3.** Let $M^{2n+1}$ be a manifold with an $[r]$-Sasakian structure $(\varphi, \xi, \eta, g)$. Then
\begin{equation}
\rho(X, \xi) = 2nr\eta(X)
\end{equation}
for any $X \in \Gamma(TM)$.

**Proof.** We choose a local orthonormal frame $(E_i)_{1 \leq i \leq 2n+1}$ on $M$. Putting $\alpha_i := g(E_i, E_i)$, using (11), (13) and the definition of $P$, since $I = -r\varphi^2 + \eta \otimes \xi$ and $d\eta(X, Y) = \Phi(X, Y) = g(X, \varphi Y)$, one has, for any $X \in \Gamma(TM)$,
\begin{align*}
\rho(X, \xi) &= \sum_{i=1}^{2n+1} \alpha_i R(X, E_i, \xi, E_i) = -r \sum_{i=1}^{2n+1} \alpha_i R(X, E_i, \xi, \varphi^2 E_i) \\
&= -r \left( \sum_{i=1}^{2n+1} \alpha_i g(R(X, E_i) \varphi(\xi), \varphi E_i) + \varepsilon r \sum_{i=1}^{2n+1} \alpha_i P(X, E_i, \xi, \varphi E_i) \right) \\
&= -\varepsilon \sum_{i=1}^{2n+1} \alpha_i (g(\varphi X, \varphi E_i) g(\xi, E_i) - g(\varphi E_i, \varphi E_i) g(X, \xi)) \\
&= -r \varepsilon \sum_{i=1}^{2n+1} \alpha_i (g(X, E_i) g(\xi, E_i) - g(E_i, E_i) g(X, \xi)) \\
&= -r \varepsilon \left\{ g(X, \xi) - \sum_{i=1}^{2n+1} \alpha_i^2 g(X, \xi) \right\} = 2nr\eta(X). \hfill \blacksquare
\end{align*}

4. **Mixed 3-Sasakian structures and Ricci curvature.** As stated in [8], a split three Sasakian manifold is Einstein. We give here a direct proof and examine some consequences.
Theorem 4.1. Any mixed 3-Sasakian manifold $(M^{4n+3}, \varphi, \xi, \eta, g)$ is Einstein. More precisely, for any $X, Y \in \Gamma(TM)$, one has

$$\rho(X, Y) = -\sigma(4n + 2)g(X, Y),$$

where $\sigma = \pm 1$, according as the 3-structure is positive or negative.

Proof. Let us put, for any $X, Y \in \Gamma(TM)$,

$$(17) \quad Q(X, Y) := \rho(X, \varphi_3 Y) - \rho(Y, \varphi_3 X) + 2\sigma(4n + 1)g(X, \varphi_3 Y).$$

We are going to prove that

$$(18) \quad Q(X, Y) = \sum_{i=1}^{4n+3} \varepsilon_i g(R(X, Y)e_i, \varphi_3(e_i)),$$

where $(e_i)_{1 \leq i \leq 4n+3}$ is an arbitrary orthonormal local frame on $M$, and $\varepsilon_i := g(e_i, e_i)$. Since the structure $(\varphi_3, \xi, \eta, g)$ is indefinite Sasakian, from (13), with $r = 1$ and $\varepsilon = g(\xi, \xi) = \mp 1 = -\sigma$ according as the 3-structure is positive or negative, we have

$$(19) \quad g(R(X, Y)Z, \varphi_3 W) = -g(R(X, Y)\varphi_3 Z, W) + \sigma P_3(X, Y, Z, W)$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Using Bianchi’s First Identity, (19) and Lemma 3.2, the right hand side of (18) becomes

$$(20) \quad \sum_{i=1}^{4n+3} \varepsilon_i g(R(X, Y)e_i, \varphi_3(e_i))$$

$$= -\sum_{i=1}^{4n+3} \varepsilon_i \{g(R(Y, e_i)X, \varphi_3(e_i)) + g(R(e_i, X)Y, \varphi_3(e_i))\}$$

$$= \sum_{i=1}^{4n+3} \varepsilon_i \{g(R(Y, e_i)\varphi_3 X, e_i) - \sigma P_3(Y, e_i, X, e_i)$$

$$+ g(R(e_i, X)\varphi_3 Y, e_i) - \sigma P_3(e_i, X, Y, e_i)\}$$

$$= -\rho(Y, \varphi_3 X) + \rho(X, \varphi_3 Y) - 2\sigma \sum_{i=1}^{4n+3} \varepsilon_i P_3(Y, e_i, X, e_i).$$

Computing the last term, by the definition of $P_3$, one has

$$(21) \quad \sum_{i=1}^{4n+3} \varepsilon_i P_3(Y, e_i, X, e_i) = \sum_{i=1}^{4n+3} \varepsilon_i \{d\eta^3(Y, X)g(e_i, e_i) - d\eta^3(Y, e_i)g(e_i, X)$$

$$- d\eta^3(e_i, X)g(Y, e_i) - d\eta^3(e_i, e_i)g(Y, X)\}$$

$$= (4n + 3)g(\varphi_3 X, Y) + g(X, \varphi_3 Y) - g(\varphi_3 X, Y).$$
From (20) and (21), we obtain (17).

Now, let us choose a local orthonormal frame adapted to the 3-structure

\[(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i, \xi_1, \xi_2, \xi_3)_{1 \leq i \leq n}.\]

For any \(i \in \{1, \ldots, n\}\), we put \(e_i := E_i\), \(e_{n+i} := \varphi_1 E_i\), \(e_{2n+i} := \varphi_2 E_i\) and \(e_{3n+i} := \varphi_3 E_i\), and

\[
\alpha_i := g(E_i, E_i) = -g(\varphi_1 E_i, \varphi_1 E_i) = -g(\varphi_2 E_i, \varphi_2 E_i) = g(\varphi_3 E_i, \varphi_3 E_i);
\]

for any \(a \in \{1, 2, 3\}\), we put also \(e_{4n+a} := \xi_a\), and \(\alpha_{4n+a} := g(\xi_a, \xi_a) = \varepsilon_a\).

We get

\[
Q(X, Y) = \sum_{i=1}^{n} \alpha_i \{ g(R(X, Y)E_i, \varphi_3 E_i) - g(R(X, Y)\varphi_1 E_i, \varphi_3 \varphi_1 E_i) \\
- g(R(X, Y)\varphi_2 E_i, \varphi_3 \varphi_2 E_i) + g(R(X, Y)\varphi_3 E_i, \varphi_3^2 E_i) \}
+ \varepsilon_1 g(R(X, Y)\xi_1, \varphi_3 \xi_1) + \varepsilon_2 g(R(X, Y)\xi_2, \varphi_3 \xi_2)
= \sum_{i=1}^{n} \alpha_i \{ g(R(X, Y)E_i, \varphi_3 E_i) + g(R(X, Y)\varphi_1 E_i, \varphi_1 \varphi_3 E_i) \\
+ g(R(X, Y)\varphi_2 E_i, \varphi_2 \varphi_3 E_i) + g(R(X, Y)E_i, \varphi_3 E_i) \}
+ \varepsilon_1 g(R(X, Y)\xi_1, \varphi_1 \xi_3) + \varepsilon_2 g(R(X, Y)\xi_2, \varphi_2 \xi_3).
\]

Since the structures \((\varphi_1, \xi_1, \eta^1, g)\) and \((\varphi_2, \xi_2, \eta^2, g)\) are both paraskasik, using (13) with \(r = -1\), one has

\[
Q(X, Y) = \sum_{i=1}^{n} \alpha_i \{ g(R(X, Y)E_i, \varphi_3 E_i) - g(R(X, Y)\varphi_1^2 E_i, \varphi_3 E_i) \\
+ \varepsilon_1 P_1(X, Y, \varphi_1 E_i, \varphi_3 E_i) - g(R(X, Y)\varphi_2^2 E_i, \varphi_3 E_i) \\
+ \varepsilon_2 P_2(X, Y, \varphi_2 E_i, \varphi_3 E_i) + g(R(X, Y)E_i, \varphi_3 E_i) \}
+ P_1(X, Y, \xi_1, \xi_3) + P_2(X, Y, \xi_2, \xi_3)
= \sum_{i=1}^{n} \alpha_i \{ \varepsilon_1 P_1(X, Y, \varphi_1 E_i, \varphi_3 E_i) + \varepsilon_2 P_2(X, Y, \varphi_2 E_i, \varphi_3 E_i) \}
+ P_1(X, Y, \xi_1, \xi_3) + P_2(X, Y, \xi_2, \xi_3).
\]

Recalling the definition of the tensor field \(P\), since \(d\eta^1 = \Phi_1, d\eta^2 = \Phi_2,\)
\(\varepsilon_1 = \varepsilon_2 = \sigma = -\varepsilon_3\) and \(\sigma \varepsilon_1 = \sigma \varepsilon_2 = 1\), using (1), (3) and (4), one has

\[
Q(X, Y) = -2\sigma \left\{ \sum_{i=1}^{n} \alpha_i((g(X, E_i)g(\varphi_3 Y, E_i) - g(X, \varphi_2 E_i)g(\varphi_3 Y, \varphi_2 E_i) + g(\varphi_3 Y, \varphi_3 E_i)g(X, \varphi_3 E_i) - g(\varphi_3 Y, \varphi_1 E_i)g(X, \varphi_1 E_i)) + \varepsilon_1 g(X, \xi_1)g(\varphi_3 Y, \xi_1) + \varepsilon_2 g(X, \xi_2)g(\varphi_3 Y, \xi_2) \right\} = -2\sigma g(X, \varphi_3 Y).
\]

From (17) and (22), it follows that

\[
\rho(X, \varphi_3 Y) - \rho(\varphi_3 X, Y) = -2\sigma(4n + 2)g(X, \varphi_3 Y).
\]

Since the structure \((\varphi_3, \xi_3, \eta^{\alpha}, g)\) is indefinite Sasakian, one has \(\rho(X, \varphi_3 Y) = -\rho(\varphi_3 X, Y)\) for any \(X, Y\) orthogonal to \(\xi_3\) (cf. [3] for the Riemannian case). From (23) it follows that \(\rho(X, \varphi_3 Y) = -\sigma(4n + 2)g(X, \varphi_3 Y)\) for any \(X, Y\) orthogonal to \(\xi_3\). Replacing \(Y\) with \(\varphi_3 Y\), since \(Y\) is orthogonal to \(\xi_3\), one has

\[
\rho(X, Y) = -\sigma(4n + 2)g(X, Y), \quad X, Y \perp \xi_3.
\]

Using (16), we have

\[
\rho(X, \xi_3) = -\sigma(4n + 2)g(X, \xi_3), \quad X \in \Gamma(TM),
\]

hence, putting \(X = \xi_3\),

\[
\rho(\xi_3, \xi_3) = -\sigma(4n + 2)g(\xi_3, \xi_3).
\]

Finally, if \(X, Y \in \Gamma(TM)\), writing \(X = X_0 + \lambda \xi_3\) and \(Y = Y_0 + \mu \xi_3\) with \(X_0, Y_0\) orthogonal to \(\xi_3\), and \(\lambda, \mu \in \mathbb{R}(M)\), using (24)–(26), one gets \(\rho(X, Y) = -\sigma(4n + 2)g(X, Y)\) for any \(X, Y \in \Gamma(TM)\), concluding the proof. \(\blacksquare\)

As an obvious consequence of the above result, we have

**Proposition 4.2.** Any mixed 3-Sasakian manifold \((M^{4n+3}, \varphi_a, \xi_a, \eta^{\alpha}, g)\) has constant scalar curvature

\[
\text{Sc} = -\sigma(4n + 2)(4n + 3),
\]

therefore negative or positive according as the 3-structure is positive or negative.

**Proposition 4.3.** Let \((M^{4n+3}, \varphi_a, \xi_a, \eta^{\alpha}, g)\) be a mixed 3-Sasakian manifold. Then \(M\) has (pointwise) constant sectional curvature \(k\) if and only if \(k = \pm 1\) according as the 3-structure is positive or negative.

**Proof.** Since the 3-structure \((\varphi_a, \xi_a, \eta^{\alpha}, g)\) is mixed 3-Sasakian, (13) holds for any \(a \in \{1, 2, 3\}\). Using the constant \(\sigma = \pm 1\) according as the 3-structure is positive or negative, and recalling that \(\tau_a \varepsilon_a = -\sigma\), we have, for any
\(a \in \{1, 2, 3\}\) and \(X, Y, Z, W \in \Gamma(TM)\),
\[
g(R(X, Y)Z, \varphi_a W) + g(R(X, Y)\varphi_a Z, W) = \sigma P_a(X, Y, Z, W).
\]

Supposing that \(M\) has pointwise constant sectional curvature \(k \in \mathfrak{S}(M)\), i.e. \(R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}\), we have
\[
\sigma P_a(X, Y, Z, W) = g(R(X, Y)Z, \varphi_a W) + g(R(X, Y)\varphi_a Z, W)
\]
\[
= k\{g(Y, Z)g(X, \varphi_a W) - g(X, Z)g(Y, \varphi_a W)
+ g(Y, \varphi_a Z)g(X, W) - g(X, \varphi_a Z)g(Y, W)\}
\]
\[
= k\{d\eta^a(X, W)g(Y, Z) - d\eta^a(Y, W)g(X, Z)
+ d\eta^a(Y, Z)g(X, W) - d\eta^a(X, Z)g(Y, W)\}
\]
\[
= -kP_a(X, Y, Z, W),
\]
hence, for any \(a \in \{1, 2, 3\}\) and any \(X, Y, Z, W \in \Gamma(TM)\), it follows that
\[(k+\sigma)P_a(X, Y, Z, W) = 0,\]
and so \(k = -\sigma = \mp 1\) according as the 3-structure is positive or negative. Namely, choosing a vector field \(Y\) orthogonal to \(\xi_1, \xi_2, \xi_3\) such that \(g(Y, Y) \neq 0\), by the definition of \(P_a\) given in Lemma 3.1, we get \(P_a(\xi_a, Y, \xi_a, \varphi_a Y) = -\varepsilon_3 g(Y, Y) \neq 0\).

5. Mixed metric 3-contact and mixed 3-Sasakian structures. In this section we shall be concerned with some properties of particular classes of mixed metric 3-structures, namely the class of mixed metric 3-contact structures, which reflect analogous properties of classical metric 3-structures (see [3] for more details).

**Definition 5.1.** Let \(M\) be a manifold with a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\). The structure is said to be a mixed metric 3-contact structure if \(d\eta^a = \Phi_a\) for each \(a \in \{1, 2, 3\}\), where \(\Phi_a\) is the fundamental 2-form defined by \(\Phi_a(X, Y) := g(X, \varphi_a Y)\). Then \((M, \varphi_a, \xi_a, \eta^a, g)\) will be called a mixed metric 3-contact manifold.

Our intent here is to prove that any mixed metric 3-contact manifold is in fact a mixed 3-Sasakian manifold.

Let \(M\) be a manifold with a mixed metric 3-structure \((\varphi_a, \xi_a, \eta^a, g)\). Setting \(\tilde{M} = M \times \mathbb{R}\), and denoting by \(t\) the coordinate on \(\mathbb{R}\), define three \((1,1)\)-type tensor fields \(J_a\), \(a = 1, 2, 3\), by putting, for any \(\tilde{X} = (X, f \frac{d}{dt}) \in \Gamma(TM)\), with \(X \in \Gamma(TM)\) and \(f \in \mathfrak{S}(\tilde{M})\),
\[
J_a(\tilde{X}) = J_a\left(X, f \frac{d}{dt}\right) := \left(\varphi_a X - \tau_a f \xi_a, \eta^a(X) \frac{d}{dt}\right),
\]
where \(\tau_1 = \tau_2 = -1 = -\tau_3\). Furthermore, define the \((0, 2)\)-type tensor field \(G\), by putting, for any \(\tilde{X} = (X, f \frac{d}{dt})\) and \(\tilde{Y} = (Y, h \frac{d}{dt})\) in \(\Gamma(TM)\), with
\(X, Y \in \Gamma(TM)\) and \(f, h \in \mathcal{F}(\tilde{M})\),
\[G(\tilde{X}, \tilde{Y}) := g(X, Y) - \sigma fh,\]
where \(\sigma = \pm 1\) according as the 3-structure is positive or negative.

**Proposition 5.2.** \((\tilde{M}, J_a, G)_{a=1,2,3}\) is an almost hyper parahermitian manifold.

**Proof.** Let \(a \in \{1, 2, 3\}\) and \(\tilde{X} \in \Gamma(\tilde{M})\) with \(\tilde{X} = (X, f \frac{d}{dt})\). Since by definition \(\varphi_a^2 = -\tau_a (I - \eta^a \otimes \xi_a)\), we have
\[
(J_a)^2(\tilde{X}) = \left((\varphi_a)^2 X - \tau_a \eta^a (X) \xi_a, -\tau_a f \frac{d}{dt}\right) = -\tau_a \tilde{X},
\]
hence \((J_a)^2 = -\tau_a I\). Let now \((a, b, c)\) be a cyclic permutation of \((1, 2, 3)\). Using (1)–(3), one has, for any \(\tilde{X} \in \Gamma(\tilde{M})\) with \(\tilde{X} = (X, f \frac{d}{dt})\),
\[
J_a J_b(\tilde{X}) = \left(\varphi_a \varphi_b X - \tau_b f \varphi_a \xi_b - \tau_a \eta^b(X) \xi_a, (\eta^a(\varphi_b X) - \tau_b \eta^a \xi_b) \frac{d}{dt}\right)
\]
\[
= \left(\tau_c \varphi_c X - f \xi_c, \tau_c \eta^c(X) \frac{d}{dt}\right) = \tau_c J_c(\tilde{X}),
\]
hence \(J_a J_b = \tau_c J_c\). Analogously, \(J_b J_a = -\tau_c J_c\), and this proves that \((J_a)_{a=1,2,3}\) is an almost hyper paracomplex structure on \(\tilde{M}\). Let now \(a \in \{1, 2, 3\}\), \(\tilde{X} = (X, f \frac{d}{dt})\) and \(\tilde{Y} = (Y, h \frac{d}{dt})\). Since, by (4), \(g(\varphi_a X, Y) = -g(X, \varphi_a Y)\), using the identity \(\tau_a \xi_a = -\sigma\), by standard calculations we have \(G(\tilde{X}, J_a \tilde{Y}) = -G(J_a(\tilde{X}), \tilde{Y})\), and by Definition 2.2 it follows that \((\tilde{M}, J_a, G), a \in \{1, 2, 3\}\), is an almost hyper parahermitian manifold. \(\blacksquare\)

**Remark 5.3.** It is clear that the tensor fields \(J_a\) constructed on \(\tilde{M}\) are almost product structures for \(a = 1, 2\), and an almost complex structure for \(a = 3\). The three structures \((\varphi_a, \xi_a, \eta^a, g)\) are normal if and only if the manifold \((\tilde{M}, J_a, G), a \in \{1, 2, 3\}\), is hyper parahermitian.

Thus, we may state:

**Proposition 5.4.** Let \(M\) be a manifold endowed with a mixed 3-structure \((\varphi_a, \xi_a, \eta^a)\). Then the structures are normal if and only if at least two of them are normal.

We shall see in a moment that the manifold \((\tilde{M}, J_a, G), a \in \{1, 2, 3\}\), is indeed hyper parahermitian if the 3-structure is a mixed metric 3-contact structure. To this end, let us prove the following preliminary results.

**Lemma 5.5.** Let \(M\) be a manifold endowed with a mixed metric 3-contact structure. Denoting, for any \(a \in \{1, 2, 3\}\), by \(\Omega_a\) the fundamental 2-form associated with the structure \((J_a, G)\) defined by \(\Omega_a(\tilde{X}, \tilde{Y}) := G(\tilde{X}, J_a \tilde{Y})\), we have
\[
d\Omega_a = 2\sigma dt \wedge \Omega_a\]
for any \( a \in \{1, 2, 3\} \), where \( \sigma = \pm 1 \) according as the 3-structure is positive or negative.

**Proof.** Fixing \( a \in \{1, 2, 3\} \), let us compute \( d\Omega_a \) using the formula
\[
(27) \quad 3d\Omega_a(\tilde{X}, \tilde{Y}, \tilde{Z}) = \mathcal{G}(\tilde{X}(\Omega_a(\tilde{Y}, \tilde{Z})), (\tilde{Y}(\Omega_a(\tilde{X}, \tilde{Z}))), (\tilde{Z}(\Omega_a(\tilde{X}, \tilde{Y})))),
\]
for any \( \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(T\tilde{M}) \). Putting \( \tilde{X} = (X, f \frac{d}{dt}), \tilde{Y} = (Y, h \frac{d}{dt}) \) and \( \tilde{Z} = (Z, k \frac{d}{dt}) \) and using \( \tau_a \varepsilon_a = -\sigma \), we have
\[
(28) \quad \Omega_a(\tilde{Y}, \tilde{Z}) = \Phi_a(Y, Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)).
\]
Furthermore, \([\tilde{X}, \tilde{Y}] = ([X, Y], (X(h) - Y(f) + f \frac{dh}{dt} - h \frac{df}{dt}) \frac{d}{dt}) \) and \( \Omega_a([\tilde{X}, \tilde{Y}], \tilde{Z}) = \Phi_a([X, Y], Z) \)
\[
\quad + \sigma \left\{ k\eta^a[X, Y] - \left( X(h) - Y(f) + f \frac{dh}{dt} - h \frac{df}{dt} \right) \eta^a(Z) \right\}.
\]
Finally, from (28),
\[
\tilde{X}(\Omega_a(\tilde{Y}, \tilde{Z})) = X(\Phi_a(Y, Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)))
\]
\[
\quad + f \frac{d}{dt}(\Phi_a(Y, Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)))
\]
\[
\quad = X(\Phi_a(Y, Z)) + \sigma(X(k)\eta^a(Y) + kX(\eta^a(Y))
\]
\[
\quad - X(h)\eta^a(Z) - hX(\eta^a(Z))) + \sigma \left( f \frac{dk}{dt} \eta^a(Y) - f \frac{dh}{dt} \eta^a(Z) \right).
\]
From (27), using the above identities and \( d\Phi_a = 0 \), one gets
\[
3d\Omega_a(\tilde{X}, \tilde{Y}, \tilde{Z}) = 2\sigma(\Phi_a(X, Y)k + \Phi_a(Y, Z)f + \Phi_a(Z, X)h).
\]
Finally, using (28), it follows that
\[
3d\Omega_a(\tilde{X}, \tilde{Y}, \tilde{Z}) = 2\sigma(f\Omega_a(\tilde{Y}, \tilde{Z}) - \sigma(fk\eta^a(Y) - fh\eta^a(Z))
\]
\[
\quad + h\Omega_a(\tilde{Z}, \tilde{X}) - \sigma(hf\eta^a(Z) - h\eta^a(X))
\]
\[
\quad + k\Omega_a(\tilde{X}, \tilde{Y}) - \sigma(kh\eta^a(X) - k\eta^a(Y))
\]
\[
\quad = 2\sigma(f\Omega_a(\tilde{Y}, \tilde{Z}) + h\Omega_a(\tilde{Z}, \tilde{X}) + k\Omega_a(\tilde{X}, \tilde{Y}))
\]
\[
\quad = 6\sigma(dt \wedge \Omega_a)(\tilde{X}, \tilde{Y}, \tilde{Z}),
\]
hence \( d\Omega_a = 2\sigma dt \wedge \Omega_a \).

**Lemma 5.6.** Let \((M, J_a, g), a \in \{1, 2, 3\}\), be an almost hyper parahermitian manifold such that, denoting by \( \Omega_a \) the fundamental 2-form associated with \( J_a \), there exists a 1-form \( \omega \) satisfying \( d\Omega_a = k\omega \wedge \Omega_a \) for any \( a \in \{1, 2, 3\} \) with \( k \in \mathfrak{X}(M) \). Then each structure \( J_a \) is integrable and the manifold is hyper parahermitian.

**Proof.** Let us prove that \( N_1 = 0 \). It is well known that
\[
N_1(X, Y) = (\nabla_{J_1X}J_1)(Y) - (\nabla_{J_1Y}J_1)(X) - J_1(\nabla_XJ_1)(Y) + J_1(\nabla_YJ_1)(X),
\]
hence, using (i) and (ii) of Definition 2.2, we get
\begin{equation}
J_2N_1(X, Y) = -J_2(\nabla_{J_1}J_1)(X) - J_3(\nabla_YJ_1)(X) \\
+ J_2(\nabla_{J_1}J_1)(Y) + J_3(\nabla_XJ_1)(Y).
\end{equation}
Then, for any \( Z \in \Gamma(TM) \), using (iii) of Definition 2.2, by standard calculations, one has
\[
g(-J_2(\nabla_{J_1}J_1)(X), Z) = -g(J_2\nabla_{J_1}J_1(J_1X), Z) - g(J_3\nabla_{J_1}J_1X, Z) \\
= -g(X, (\nabla_{J_1}J_1)(Z)) - g(J_1X, (\nabla_{J_1}J_1)(Z)) \\
= (\nabla_{J_1}J_1)(Z, X) + (\nabla_{J_1}J_1)(Z, J_1X).
\]
Switching \( X \) and \( Y \) one has
\[
g(J_2(\nabla_{J_1}J_1)(Y), Z) = (\nabla_{J_1}J_1)(Y, Z) + (\nabla_{J_1}J_1)(J_1Y, Z).
\]
Analogously, one obtains
\[
g(-J_3(\nabla_JJ_1)(X), Z) = (\nabla_JJ_1)(Z, X) + (\nabla_JJ_1)(J_1X)
\]
and switching \( X \) and \( Y \) one gets
\[
g(J_3(\nabla_JJ_1)(Y), Z) = (\nabla_JJ_1)(Y, Z) + (\nabla_JJ_1)(J_1Y, Z).
\]
Since \( 3d\Omega(X, Y, Z) = \mathcal{S}_{(X,Y,Z)}(\nabla_X\omega)(Y, Z) \), from (29) we have
\[
g(J_2N_1(X, Y), Z) = 3d\Omega_2(X, Y, Z) + 3d\Omega_3(X, J_1Y, Z) \\
+ 3d\Omega_3(J_1X, Y, Z) + 3d\Omega_2(J_1X, J_1Y, Z).
\]
As \( d\Omega_a = k\omega \wedge \Omega_a \), we get
\[
g(J_2N_1(X, Y), Z) = k\{\omega(X)\Omega_2(Y, Z) + \omega(Y)\Omega_2(Z, X) \\
+ \omega(Z)\Omega_2(X, Y) + \omega(X)\Omega_3(J_1Y, Z) \\
+ \omega(J_1Y)\Omega_3(Z, X) + \omega(Z)\Omega_3(X, J_1Y) \\
+ \omega(J_1X)\Omega_3(Y, Z) + \omega(Y)\Omega_3(Z, J_1X) \\
+ \omega(Z)\Omega_3(J_1X, Y) + \omega(J_1X)\Omega_2(J_1Y, Z) \\
+ \omega(J_1Y)\Omega_2(Z, J_1X) + \omega(Z)\Omega_2(J_1X, J_1Y)\}.
\]
It is easy to check that \( \Omega_3(J_1Y, Z) = -\Omega_2(Y, Z) \), \( \Omega_3(Y, J_1Z) = -\Omega_2(Y, Z) \), \( \Omega_2(Z, J_1X) = -\Omega_3(Z, X) \), \( \Omega_2(J_1Z, X) = -\Omega_3(Z, X) \) and \( \Omega_2(J_1X, J_1Y) = \Omega_2(X, Y) \). Therefore, \( g(J_2N_1(X, Y), Z) = 0 \), hence \( N_1 = 0 \). In an analogous way, one proves that \( N_2 = 0 \) and \( N_3 = 0 \).

As an obvious consequence of Lemmas 5.5 and 5.6, one obtains the following result.

**Theorem 5.7.** Any mixed metric 3-contact structure on a manifold is mixed 3-Sasakian.

Thus, Theorems 4.1–4.3 may be formulated for mixed metric 3-contact manifolds.
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