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Pluriharmonic extension in proper image domains

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Abstract. Let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$ and set $D = D_1 \times \cdots \times D_s$, $j = 1, \ldots, s$, $s \geq 3$. Also let Ω_π be the image of D under the proper holomorphic map π. We characterize those continuous functions $f : \partial\Omega_\pi \to \mathbb{R}$ that can be extended to a real-valued pluriharmonic function in Ω_π.

1. Introduction. For each $j = 1, \ldots, s$, $s \geq 3$, let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$, $n_j \geq 1$. Recall that a bounded domain $\Omega \subseteq \mathbb{C}^n$, is called *hyperconvex* if there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. A bounded hyperconvex domain Ω in $\mathbb{C}^n$, viewed as a domain in $\mathbb{R}^{2n}$, is always regular with respect to the Dirichlet problem for the Laplace operator (see e.g. [4]). Set

$$D = D_1 \times \cdots \times D_s, \quad j = 1, \ldots, s,\ s \geq 3. \tag{1.1}$$

Then $D \subseteq \mathbb{C}^n$, $n = n_1 + \cdots + n_s$, is a hyperconvex domain (see e.g. [2, Proposition 2.1]). Let U be an open neighborhood of $\overline{D}$ and $\pi : U \to \mathbb{C}^n$, $n = n_1 + \cdots + n_s$, be a proper holomorphic map. Set $\Omega_\pi = \pi(D)$. Then $\pi(\partial D) = \partial\Omega_\pi$, since π is a proper map (see e.g. [6]). Furthermore, Ω_π is hyperconvex (Proposition 2.1).

Let $f : \partial\Omega_\pi \to \mathbb{R}$, $s \geq 3$, be a continuous function. Our main goal in Section 2 is to characterize those continuous functions f that can be extended to real-valued pluriharmonic functions in Ω_π. We prove:

Theorem A. *Let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$, $n_j \geq 1$. Set $D = D_1 \times \cdots \times D_s$, $j = 1, \ldots, s$, $s \geq 3$. Moreover, let U be an open neighborhood of $\overline{D}$, let $\pi : U \to \mathbb{C}^n$, $n = n_1 + \cdots + n_s$, be a proper holomorphic map and let $\Omega_\pi = \pi(D)$. If $f : \partial D \to \mathbb{R}$, $n \geq 3$, is a continuous function, then the following assertions are equivalent*:

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(1) *there exists a function h that is pluriharmonic on Ω_π, continuous on $\bar{\Omega}_\pi$ and $h|_{\partial\Omega_\pi} = f$,*
(2) *f is pluriharmonic on $\partial\Omega_\pi$ in the sense of Definition 2.2,*
(3) *the Perron–Bremermann envelope PB_f is pluriharmonic on Ω_π, i.e.,*

$$\mathrm{PB}_{-f} = -\mathrm{PB}_f,$$

(*see Section 2 for the definition of the Perron–Bremermann envelope*),
(4) *for every $z_0 \in \partial\Omega_\pi$ and every Jensen measure $\mu \in \mathcal{J}^c_{z_0}$ with barycenter z_0 we have*

$$f(z_0) = \int\limits_{\partial\Omega_\pi} f\,d\mu.$$

Furthermore, if we assume that Ω_π has the approximation property, then the above conditions are equivalent to:

(5) *the function $\mathrm{PB}_{-f} + \mathrm{PB}_f$ has smallest maximal plurisubharmonic majorant identically zero,*
(6) $\limsup_{\Omega\ni z\to\xi}(\mathrm{PB}_f + \mathrm{PB}_{-f})(z) = 0$ for every $\xi \in \partial\Omega_\pi$.

(*see Section 2 for the definition of the approximation property*).

An elementary example of a continuous function $f : \partial\Omega_\pi \to \mathbb{R}$ satisfying the above conditions is the following: Let u be holomorphic in a neighborhood of $\bar{\Omega}_\pi$ and let $f = \mathrm{Re}(u)$ be defined on $\partial\Omega_\pi$. Then $h = \mathrm{Re}(u)$ satisfies (1).

Let $s = n_1 = 1$. Then property (2) in Theorem A does not make sense and properties (1), (3)–(6) are true for every continuous function $f : \Omega_\pi \to \mathbb{R}$. If $s = 2$, then it is in general not true that (2) implies (1) ([2, Example 3.4]), and if $s \geq 3$ and π is the identity map or is such that Ω_π is the symmetrized polydisc (see e.g. [5] for the definition), then Theorem A was obtained in [2]. The equivalence between (1) and (3) was proved for an arbitrary hyperconvex domain in [1].

This article is organized as follows. The equivalence between assertions (1)–(4) is proved in Section 3 and the final part is proved in Section 4. In Section 5 we study plurisubharmonic boundary values in terms of analytic discs, in the case when $D = D_1 \times \cdots \times D_n$ is a hyperconvex domain in $\mathbb{C}^n$.

2. Definitions, basic facts and notations. Let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$, $n_j \geq 1$, and set

$$D = D_1 \times \cdots \times D_s \subseteq \mathbb{C}^n,$$

where $n = n_1 + \cdots + n_s$. For an open neighborhood U of $\bar{D}$ and a proper holomorphic map $\pi : U \to \mathbb{C}^n$ we use the notation $\Omega_\pi = \pi(D)$.

Let $I_k = (j_1, \ldots, j_k)$ be an increasing multi-index of length k: $1 \le j_1 < \cdots < j_k \le s$, where $1 \le k \le s$. Define

$$\Lambda^{I_k} = D_1 \times \cdots \times \overbrace{\partial D_{j_1}}^{j_1} \times \cdots \times \overbrace{\partial D_{j_k}}^{j_k} \times \cdots \times D_s \quad \text{and} \quad \Lambda^{I_k}_\pi = \pi(\Lambda^{I_k}).$$

Hence,

$$\partial D = \bigcup_{I_k} \Lambda^{I_k} \quad \text{and} \quad \partial \Omega_\pi = \pi(\partial D) = \bigcup_{I_k} \Lambda^{I_k}_\pi.$$

Finally, denote by ∂D^+ the distinguished boundary of D, i.e.

$$\partial D^+ = \partial D_1 \times \cdots \times \partial D_s.$$

Proposition 2.1. *The domain Ω_π is hyperconvex.*

Proof. For every j, let φ_j be an exhaustion function for D_j. Then

$$u(\zeta_1, \ldots, \zeta_s) = \max\{\varphi_1(\zeta_1), \ldots, \varphi_s(\zeta_s)\}$$

is a plurisubharmonic exhaustion function for D in $\mathbb{C}^n$. Now define

$$\varphi(w) = \max\{u(z) : z \in \pi^{-1}(w)\}.$$

From [7] it follows that φ is a plurisubharmonic exhaustion function for Ω_π. Thus, Ω_π is hyperconvex. ∎

Definition 2.2. An upper semicontinuous function $u : \partial\Omega_\pi \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* if u is plurisubharmonic on every $\Lambda^{I_k}_\pi$ for every increasing multi-index $I_k = (j_1, \ldots, j_k)$ of length $k < s$, i.e., for every $w_{j_1} \in \partial D_{j_1}, \ldots, w_{j_k} \in \partial D_{j_k}$, the function defined by

$$u_{w_{j_1}}, \ldots, w_{j_k} : (z_1, \ldots, z_{s-k}) \mapsto u \circ \pi(z_1, \ldots, w_{j_1}, \ldots, w_{j_k}, \ldots, z_{s-k}) \tag{2.1}$$

is plurisubharmonic on

$$D_{I_k} = D_1 \times \cdots \times \widehat{\partial D_{j_1}} \times \cdots \times \widehat{\partial D_{j_k}} \times \cdots \times D_s.$$

The identically $-\infty$ function is by fiat not considered as plurisubharmonic. In a similar manner a continuous function $u : \partial\Omega_\pi \to \mathbb{R}$ is *pluriharmonic* if u is pluriharmonic on every $\Lambda^{I_k}_\pi$ for every increasing multi-index I_k of length $k < s$.

Remark. Note that, if we take $\pi = \mathrm{id}_D$ in Definition 2.2, then an upper semicontinuous function u is plurisubharmonic on ∂D if for every increasing multi-index I_k the restriction of u to Λ^{I_k} is plurisubharmonic.

The following definition comes from [11].

Definition 2.3. Let $\Omega \subseteq \mathbb{C}^n$. We say that Ω has the *approximation property* if for all upper bounded plurisubharmonic functions u in Ω there exists a decreasing sequence $u_j \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $u_j \to u^*$ on $\overline{\Omega}$.

Wikström proved that B-regular domains and polydiscs have the approximation property (see [11]).

Definition 2.4. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let μ be a nonnegative, regular Borel measure on $\overline{\Omega}$. The measure μ is a *Jensen measure with barycenter at* $z \in \overline{\Omega}$ *for continuous plurisubharmonic functions* if

$$u(z) \leq \int_{\overline{\Omega}} u \, d\mu$$

for every continuous function $u \in \mathcal{PSH}(\Omega)$. The set of all such measures will be denoted by $\mathcal{J}_z^c$.

Similarly the measure μ is a *Jensen measure with barycenter at* $z \in \overline{\Omega}$ *for upper bounded plurisubharmonic functions* if

$$u^*(z) \leq \int_{\overline{\Omega}} u^* \, d\mu$$

for every upper bounded function $u \in \mathcal{PSH}(\Omega)$. The set of all such measures will be denoted by $\mathcal{J}_z$.

It is clear that $\{\delta_z\} \subset \mathcal{J}_z \subset \mathcal{J}_z^c$, where δ_z denotes the Dirac measure at z. If Ω is a hyperconvex domain, then $\operatorname{supp} \mu \subset \partial\Omega$ for all $z \in \partial\Omega$ and all $\mu \in \mathcal{J}_z^c$ (see [3]). Moreover, for a bounded hyperconvex domain $\Omega \subseteq \mathbb{C}^n$ it was proved in [11] that Ω has the approximation property if, and only if, $\mathcal{J}_z^c = \mathcal{J}_z$ for all $z \in \overline{\Omega}$.

We say that there exists a *strong plurisubharmonic barrier at* $z \in \partial\Omega$ if there exists $u \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $u(z) = 0$ and $u < 0$ in $\overline{\Omega} \setminus \{z\}$. A bounded domain Ω in $\mathbb{C}^n$ is called *B-regular* (see [9]) if for each $z \in \partial\Omega$ there exists a strong plurisubharmonic barrier at z.

Proposition 2.5. *Let $\Omega \subset \mathbb{C}^n$ be bounded domain and let $z \in \partial\Omega$ be such that there exists a strong plurisubharmonic barrier at z. Then $\mathcal{J}_z^c = \{\delta_z\}$.*

Proof. Let $z \in \Omega$. Assume that there exists $u \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $u(z) = 0$ and $u < 0$ in $\overline{\Omega} \setminus \{z\}$. Then we get

$$0 = u(z) \leq \int_{\partial\Omega} u \, d\mu \leq 0.$$

Therefore $\operatorname{supp} \mu \subset \{z\}$, so $\mathcal{J}_z^c = \{\delta_z\}$. ∎

Proposition 2.6. *Let D_j be a bounded B-regular domain in $\mathbb{C}^{n_j}$, and let $D = D_1 \times \cdots \times D_s \subset \mathbb{C}^n$, where $n = n_1 + \cdots + n_s$. Fix $k \in \{1, \ldots, s-1\}$, $1 \leq j_1 < \cdots < j_k \leq s$, $z_{j_l} \in \partial D_{j_l}$ for $l = 1, \ldots, k$ and $z_m \in D_m$ for $m \notin \{j_1, \ldots, j_k\}$. Let $z = (z_1, \ldots, z_{j_1}, \ldots, z_{j_k}, \ldots, z_s) \in \partial D$. If $\mu \in \mathcal{J}_z^c$, then*

$$\operatorname{supp} \mu \subset \overline{D}_1 \times \cdots \times \{z_{j_1}\} \times \cdots \times \{z_{j_k}\} \times \cdots \times \overline{D}_s.$$

If $z \in \partial D^+$, then $\mathcal{J}_z^c = \{\delta_z\}$.

Proof. Let $z_0 \in \partial D \setminus \partial D^+$, e.g. $z_0 = (z_1, \ldots, w_{j_1}, \ldots, w_{j_k}, \ldots, z_s)$. where $z_j \in D_j$, $w_{j_k} \in \partial D_{j_k}$. Let h_{j_k} be a strong plurisubharmonic barrier at w_{j_k} for D_{j_k}. Define $u_{j_k}(z) = h_{j_k}(z_{j_k})$, and let $\mu \in \mathcal{J}^c_{z_0}$. Then

$$0 = u_{j_k}(z) \le \int\limits_{\partial D} u_{j_k}\, d\mu,$$

which implies that

$$\operatorname{supp} \mu \subset \overline{D}_1 \times \cdots \times \{w_{j_k}\} \times \cdots \times \overline{D}_s.$$

Hence,

$$\begin{aligned}\operatorname{supp} \mu &\subset \bigcap_{j_k} \overline{D}_1 \times \cdots \times \{w_{j_k}\} \times \cdots \times \overline{D}_s \\ &= \overline{D}_1 \times \cdots \times \{w_{j_1}\} \times \cdots \times \{w_{j_k}\} \times \cdots \times \overline{D}_s.\end{aligned}$$

To prove the second part of the proposition, we will show that for each $z \in \partial D^+$ there exists a strong plurisubharmonic barrier at z; then Proposition 2.5 will finish the proof. Let $(w_1, \ldots, w_s) \in \partial D^+$. Then for each j there exists $\varphi_j \in \mathcal{PSH}(D_j) \cap C(\overline{D}_j)$ such that $\varphi_j(w_j) = 0$ and $\varphi_j < 0$ in $\overline{D}_j \setminus \{w_j\}$. Now define

$$u(z_1, \ldots, z_s) = \sum_{j=1}^{s} \varphi_j(z_j).$$

Then $u \in \mathcal{PSH}(D) \cap C(\overline{D})$, $u(w) = 0$ and $u < 0$ in $\overline{D} \setminus \{w\}$. ∎

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. The *Perron–Bremermann envelope* is defined by

$$\mathrm{PB}_f(z) = \sup\{w(z) : w \in \mathcal{PSH}(\Omega),\ \limsup_{\Omega \ni \zeta \to \xi} w(\zeta) \le f(\xi)\ \forall \xi \in \partial\Omega\}.$$

Hence PB_f is always plurisubharmonic, but not necessarily continuous. In [10] Walsh proved that if

$$\liminf_{\Omega \ni z \to \xi} \mathrm{PB}_f(z) = \limsup_{\Omega \ni z \to \xi} \mathrm{PB}_f(z) = f(\xi)$$

for every $\xi \in \partial\Omega$, then $\mathrm{PB}_f \in C(\overline{\Omega})$. We will refer to this result as *Walsh's theorem.*

3. Proof of the equivalence (1)–(4) in Theorem A

LEMMA 3.1. *Let U be an open neighborhood of $\overline{D}$ defined in Section 2 and let $\pi : U \to \mathbb{C}^n$ be a proper holomorphic map. Let $\Omega_\pi := \pi(D)$ and let $f : \partial\Omega_\pi \to \mathbb{R}$ be a continuous function. If there exists $u \in \mathcal{PSH}(\Omega_\pi) \cap C(\overline{\Omega}_\pi)$ such that $u|_{\partial\Omega_\pi} = f$, then f is plurisubharmonic in the sense of Definition 2.2.*

Proof. Let I_k be an increasing multi-index of length $k < s$, and $w_{j_1} \in \partial D_{j_1}, \ldots, w_{j_k} \in \partial D_{j_k}$. Let $f_{w_{j_1},\ldots,w_{j_k}} : D_{I_k} \to \mathbb{R} \cup \{-\infty\}$ be defined as in (2.1). We need to prove that this function is plurisubharmonic under the assumption that there exists $u \in \mathcal{PSH}(\Omega_\pi) \cap C(\overline{\Omega}_\pi)$ such that $u|_{\partial\Omega_\pi} = f$. Take a sequence $[(w_{j_1}^m, \ldots, w_{j_k}^m)]_{m=1}^\infty$ in $D_{j_1} \times \cdots \times D_{j_k}$ which converges to $(w_{j_1}, \ldots, w_{j_k})$ as $m \to \infty$. Moreover, let $[u_m]$ be the sequence of real-valued functions on D_{I_k} defined by

$$u_m(z_1, \ldots, z_{s-k}) = u \circ \pi(z_1, \ldots, w_{j_1}^m, \ldots, w_{j_k}^m, \ldots, z_{s-k}).$$

This construction implies that u_m is plurisubharmonic on D_{I_k} and continuous up to the boundary. The sequence $[u_m]$ converges uniformly to $f_{w_{j_1},\ldots,w_{j_k}}$ on D_{I_k} as $m \to \infty$, and therefore f is plurisubharmonic in the sense of Definition 2.2. ∎

Next we prove a characterization of those continuous boundary values which can be extended to continuous plurisubharmonic functions inside the domain.

PROPOSITION 3.2. *Let $D = D_1 \times \cdots \times D_s$, where D_j is a B-regular domain in $\mathbb{C}^{n_j}$, $j = 1, \ldots s$, and let $f : \partial D \to \mathbb{R}$ be a continuous function. The following conditions are then equivalent*:

(1) *there exists $u \in \mathcal{PSH}(D) \cap C(\overline{D})$ such that $u|_D = f$*,

(2) *f is plurisubharmonic in the sense of Definition* 2.2 (*with* $\pi = \mathrm{id}_D$).

Proof. (2)⇒(1): If $z \in \partial D^+$, then $\mathcal{J}_z^c = \{\delta_z\}$ by Proposition 2.6. If $z \in \partial D \setminus \partial D^+$ then there exist $k < s$ and $1 \le j_1 < \cdots < j_k \le s$ such that $z_{j_l} \in \partial D_{j_l}$ for $l = 1, \ldots, k$ and $z_m \in D_m$ for $m \notin \{j_1, \ldots, j_k\}$. If $\mu \in \mathcal{J}_z^c$, then

$$\operatorname{supp} \mu \subset \overline{D}_1 \times \cdots \times \{z_{j_1}\} \times \cdots \times \{z_{j_k}\} \times \cdots \times \overline{D}_s,$$

by Proposition 2.6. By our assumption f is plurisubharmonic on $D_1 \times \cdots \times \{z_{j_1}\} \times \cdots \times \{z_{j_k}\} \times \cdots \times D_s$ and then by definition of Jensen measures we get

$$f(z) \le \int\limits_{\overline{D}_1 \times \cdots \times \{z_{j_1}\} \times \cdots \times \{z_{j_k}\} \times \cdots \times \overline{D}_s} f \, d\mu.$$

Taking $\mu = \delta_z$ we find that for all $z \in \partial D$,

$$f(z) = \inf\Big\{ \int\limits_{\partial D} f \, d\mu : \mu \in \mathcal{J}_z^c \Big\}.$$

Then from Theorem 3.5 in [11] there exists $u \in \mathcal{PSH}(D) \cap C(\overline{D})$ such that $u|_D = f$.

(1)⇒(2): Follows from Lemma 3.1. ∎

LEMMA 3.3. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$, and let U be an open neighborhood of $\overline{D}$. Let $\pi : U \to \mathbb{C}^n$ be a proper holomorphic* *map and let* $\Omega_\pi = \pi(D)$. *If* $f : \partial\Omega_\pi \to \mathbb{R}$ *is a continuous function such that* $\mathrm{PB}_{f\circ\pi} \in \mathcal{PSH}(D) \cap C(\overline{D})$ *and* $\mathrm{PB}_{f\circ\pi} = f \circ \pi$ *on* ∂D, *then*

$$\mathrm{PB}_{f\circ\pi} = \mathrm{PB}_f \circ \pi.$$

Furthermore, $\mathrm{PB}_{f\circ\pi}$ *is pluriharmonic in* D *if*, *and only if*, PB_f *is pluriharmonic on* Ω_π.

Proof. Define $g = f \circ \pi : \partial D \to \mathbb{R}$. By our assumption, PB_g is plurisubharmonic on D and continuous on $\overline{D}$. Set

$$\varphi(w) = \max\{\mathrm{PB}_g(z) : z \in \pi^{-1}(w)\}.$$

Then $\varphi \in \mathcal{PSH}(\Omega_\pi) \cap C(\Omega_\pi)$ (see [7]). We prove that $\varphi|_{\partial\Omega_\pi} = f$. Let $\Omega_\pi \ni w_j \to w \in \partial\Omega_\pi$. Then there exist finitely many $z_j^1, \ldots, z_j^{k_j} \in \pi^{-1}(w_j)$. Take $z_j^{l_j}$ such that $\varphi(w_j) = \mathrm{PB}_g(z_j^{l_j})$. Since π is a proper map, we have $z_j^{l_j} \to z_0 \in \partial D$ and then $\varphi(w_j) = \mathrm{PB}_g(z_j^{l_j}) \to \mathrm{PB}_g(z_0) = g(z_0) = f(\pi(z_0)) = f(w_0)$.

Hence $\varphi \le \mathrm{PB}_f \in \mathcal{PSH}(\Omega_\pi) \cap C(\overline{\Omega}_\pi)$, by Walsh's theorem. Therefore $\mathrm{PB}_f \circ \pi \in \mathcal{PSH}(D)$ and $(\mathrm{PB}_f \circ \pi)|_{\partial D} = g$. Thus, for $z \in \pi^{-1}(w)$ we get $(\mathrm{PB}_f \circ \pi)(z) \le \mathrm{PB}_g(z)$ and so $\mathrm{PB}_f(w) \le \varphi(w)$, which implies that $\varphi = \mathrm{PB}_f$. Therefore

$$\mathrm{PB}_{f\circ\pi} = \mathrm{PB}_f \circ \pi,$$

since both functions are maximal with the same boundary values g.

Now we prove the second part of the lemma. From the first part it is clear that if PB_f is pluriharmonic on Ω_π, then $\mathrm{PB}_{f\circ\pi}$ is pluriharmonic on D.

Now assume that PB_g is pluriharmonic on D. Note that $\mathrm{PB}_g = -\mathrm{PB}_{-g}$, since PB_g pluriharmonic on D and continuous on $\overline{D}$. Hence by the first part of the proof,

$$\begin{aligned}\mathrm{PB}_f(w) &= \max\{\mathrm{PB}_g(z) : z \in \pi^{-1}(w)\} = \max\{-\mathrm{PB}_{-g}(z) : z \in \pi^{-1}(w)\}\\ &= -\min\{\mathrm{PB}_{-g}(z) : z \in \pi^{-1}(w)\}.\end{aligned}$$

Similarly, $\mathrm{PB}_{-f}(w) = \max\{\mathrm{PB}_{-g}(z) : z \in \pi^{-1}(w)\}$. Combining these two representations we obtain

$$\begin{aligned}0 \ge \mathrm{PB}_f + \mathrm{PB}_{-f} = {}& \max\{\mathrm{PB}_{-g}(z) : z \in \pi^{-1}(w)\}\\ &- \min\{\mathrm{PB}_{-g}(z) : z \in \pi^{-1}(w)\} \ge 0,\end{aligned}$$

so $\mathrm{PB}_f = -\mathrm{PB}_{-f}$, which means that PB_f is pluriharmonic. ∎

We are now in a position to prove the first part of Theorem A.

Proof of Theorem A. (1)⇒(2): Follows immediately from Lemma 3.1.
(3)⇒(1): Obvious.
(3)⇒(4): Follows from Theorem 2.4 in [2].

(4)⇒(2): Let I_k be an increasing multi-index $1 \le j_1 < \cdots < j_k \le s$ of length $k < s$, and $z_0 \in \Lambda^{I_k}$. Take any complex line l through z_0 and $r > 0$ such that $z_0 + r\mathbb{E} \subset l \cap \Lambda^{I_k}$, where $\mathbb{E}$ is in $\mathbb{C}$. Since the Lebesgue measure λ on the unit disc $\mathbb{E}$ is a Jensen measure at z_0, the measure $\mu_\pi(A) = \lambda(\pi^{-1}(A))$, where $A \subset z_0 + r\mathbb{E}$, is a Jensen measure at $\pi(z_0)$. By assumption,

$$f(\pi(z_0)) = \int\limits_{\pi(z_0+r\mathbb{E})} f\, d\mu_\pi = \int\limits_{z_0+r\mathbb{E}} f \circ \pi\, d\lambda,$$

which implies that f is harmonic on $\pi(z_0 + r\mathbb{E})$ and therefore pluriharmonic on $\partial\Omega_\pi$.

(2)⇒(3): Let $g = f \circ \pi : \partial D \to \mathbb{R}$. Then g is pluriharmonic on ∂D and therefore Theorem 3.3 in [2] implies that PB_g is pluriharmonic on D, continuous on $\overline{D}$ and $\mathrm{PB}_g = g$ on ∂D. Therefore Lemma 3.3 finishes the proof. ■

4. The final part of Theorem A. We prove the following theorem.

THEOREM 4.1. *Assume that $\Omega \subseteq \mathbb{C}^n$ is a bounded hyperconvex domain having the approximation property and that $f : \partial\Omega \to \mathbb{R}$ is a continuous function. The following assertions are then equivalent*:

(1) *for every $\xi \in \partial\Omega$,*

$$\limsup_{\Omega \ni z \to \xi} (\mathrm{PB}_f + \mathrm{PB}_{-f})(z) = 0, \tag{4.1}$$

(2) *for every $z_0 \in \partial\Omega$ and every $\mu \in \mathcal{J}_{z_0}$,*

$$f(z_0) = \int\limits_{\partial\Omega} f\, d\mu,$$

(3) $\mathrm{PB}_f, \mathrm{PB}_{-f} \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega)$, $\mathrm{PB}_f = f$ *and* $\mathrm{PB}_{-f} = -f$ *on* $\partial\Omega$.

Proof. (3)⇒(1): Obvious.

(2)⇒(3): Lemma 3.3 in [11] implies that there exist $u, v \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ such that

$$\lim_{\Omega \ni z \to \zeta} u(z) = f(\zeta) \quad \text{and} \quad \lim_{\Omega \ni z \to \xi} v(z) = -f(\xi)$$

for all $\zeta, \xi \in \partial\Omega$, hence

$$\lim_{\Omega \ni z \to \zeta} \mathrm{PB}_f(z) = f(\zeta) \quad \text{and} \quad \lim_{\Omega \ni z \to \xi} \mathrm{PB}_{-f}(z) = -f(\xi),$$

and by [10] we get $\mathrm{PB}_f, \mathrm{PB}_{-f} \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega)$.

(1)⇒(2): First we will prove that assumption (4.1) implies that

$$\limsup_{\Omega \ni z \to \zeta} \mathrm{PB}_f(z) = f(\zeta) \quad \text{and} \quad \limsup_{\Omega \ni z \to \xi} \mathrm{PB}_{-f}(z) = -f(\xi)$$

for all $\zeta, \xi \in \partial\Omega$. Assume that this is not the case, for example there exists a $\xi \in \partial\Omega$ such that $\limsup_{z\to\xi} \mathrm{PB}_f(z) < f(\xi)$. Then

$$\begin{aligned}0 &= \limsup_{\Omega\ni z\to\xi} (\mathrm{PB}_f + \mathrm{PB}_{-f})(z) \le \limsup_{\Omega\ni z\to\xi} \mathrm{PB}_f(z) + \limsup_{\Omega\ni z\to\xi} \mathrm{PB}_{-f}(z)\\ &< f(\xi) - f(\xi) = 0,\end{aligned}$$

a contradiction, hence $\limsup \mathrm{PB}_f = f$ and $\limsup \mathrm{PB}_{-f} = -f$ on $\partial\Omega$.

Fix $z_0 \in \partial\Omega$ and take $\mu \in \mathcal{J}_{z_0} = \mathcal{J}^c_{z_0}$. Then $\operatorname{supp}\mu \subset \partial\Omega$ and

$$f(z_0) = \mathrm{PB}^*_f(z_0) \le \int_{\overline{\Omega}} \mathrm{PB}^*_f\, d\mu = \int_{\partial\Omega} \mathrm{PB}^*_f\, d\mu = \int_{\partial\Omega} f\, d\mu.$$

Thus

$$f(z_0) \le \inf\Big\{ \int_{\partial\Omega} f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.$$

If $\mu = \delta_{z_0}$, then we obtain

$$f(z_0) = \inf\Big\{ \int_{\partial\Omega} f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.$$

A similar formula can be obtained for $-f$ and therefore

$$\begin{aligned}\inf\Big\{ \int_{\partial\Omega} -f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\} &= -f(z_0) = -\inf\Big\{ \int_{\partial\Omega} f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}\\ &= \sup\Big\{ \int_{\partial\Omega} -f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.\end{aligned}$$

Thus, for every $z_0 \in \partial\Omega$ and every $\mu \in \mathcal{J}_{z_0}$,

$$f(z_0) = \int_{\partial\Omega} f\, d\mu,$$

and the proof is complete. ∎

Proof of the final part of Theorem A. (5)⇒(6): If a bounded plurisubharmonic function u on a bounded hyperconvex domain Ω has smallest maximal plurisubharmonic majorant identically zero, then $\limsup_{z\to\xi\in\partial\Omega} u(z) = 0$.

(3)⇒(5): Obvious.

(6)⇒(4): Follows from Theorem 4.1. ∎

5. Plurisubharmonicity in terms of analytic discs. Let $\mathbb{E}$ be the open unit disc in $\mathbb{C}$. Let U be an open neighborhood of the closure of a bounded hyperconvex domain $D = D_1 \times \cdots \times D_n$ in $\mathbb{C}^n$, and let $\pi : U \to \mathbb{C}^n$ be a proper holomorphic map. Set $\Omega_\pi := \pi(D)$. Note that since each D_j is a one-dimensional hyperconvex domain, it is also B-regular.

Proposition 5.1. *Let U be an open neighborhood of the closure of a bounded hyperconvex domain $D = D_1 \times \cdots \times D_n$ in $\mathbb{C}^n$, $n \ge 2$, and let π :*

$U \to \mathbb{C}^n$ be a proper holomorphic map. Let $\Omega_\pi := \pi(D)$ and let $f : \partial\Omega_\pi \to \mathbb{R}$ be a continuous function. The following conditions are then equivalent:

1. *there exists $u \in \mathcal{PSH}(\Omega_\pi) \cap C(\overline{\Omega}_\pi)$ such that $u|_{\partial\Omega_\pi} = f$,*
2. *f is plurisubharmonic in the sense of Definition 2.2,*
3. *f is subharmonic on every analytic disc d embedded in $\partial\Omega_\pi$, i.e., $f \circ d$ is subharmonic on $\mathbb{E}$ for every injective, holomorphic function $d : \mathbb{E} \to \overline{\Omega}_\pi$ with $d(\mathbb{E}) \subseteq \partial\Omega_\pi$.*

Proof. Since each D_j is B-regular, the equivalence (1)⇔(2) was proved in Lemma 3.3.

(3)⇒(2): Let I_k be an increasing multi-index of length $k < n$, $w_{j_1} \in \partial D_{j_1}, \ldots, w_{j_k} \in \partial D_{j_k}$. Let $f_{w_{j_1},\ldots,w_{j_k}} : D_{I_k} \to \mathbb{R} \cup \{-\infty\}$ be defined as in (2.1). To prove that this function is plurisubharmonic, take $z_0 \in D_{I_k}$. Let $\tilde{z}_0 = (z_1, \ldots, w_{j_1}, \ldots, w_{j_k}, \ldots, z_n) \in \Lambda^{I_k}$, choose $X \in \mathbb{C}^{n-k}$ and let $\tilde{X} = (X_1, \ldots, X_{j_l}, \ldots, X_n)$, where $X_{j_l} = 0$ for $l = 1, \ldots, k$. Choose $r > 0$ such that $\{\tilde{z}_0 + \zeta r\tilde{X} : \zeta \in \mathbb{E}\} \subseteq \Lambda^{I_k}$. Let $d : \mathbb{E} \to \Lambda^{I_k}$ be an analytic disc embedded in Λ^{I_k} defined by $d(\zeta) = \tilde{z}_0 + \zeta r\tilde{X}$. Then $\pi \circ d$ is an analytic disc imbedded in $\Lambda^{I_k}_\pi$. Thus $f \circ \pi \circ d$ is subharmonic on $\mathbb{E}$ by assumption, hence $f_{w_{j_1},\ldots,w_{j_k}}$ is plurisubharmonic on Λ^{I_k}.

(2)⇒(3): Let $d : \mathbb{E} \to \partial\Omega_\pi$. It is enough to show that there exists an increasing multi-index I_k of length $k < n$ such that $d(\mathbb{E}) \subset \Lambda^{I_k}_\pi$. It is clear that $d(\mathbb{E}) \not\subset \pi((\partial\mathbb{E})^n)$. So there exist $z \in d(\mathbb{E})$ and an increasing multi-index I_k of length $k < n$ such that $z \in \Lambda^{I_k}_\pi$.

We prove that $d(\mathbb{E}) \subset \Lambda^{I_k}_\pi$. Assume that it is not true. Then, since both sets are connected, there exist $\lambda_1, \lambda_2 \in \mathbb{E}$ such that $d(\lambda_1) \in \Lambda^{I_k}_\pi$ and $d(\lambda_1) \in \partial\Lambda^{I_k}_\pi$ (we can treat $\Lambda^{I_k}_\pi$ like a domain in $\mathbb{C}^{n-k}$). Therefore we can assume that there exists an analytic disc $\tilde{d} : \mathbb{E} \to \overline{\Lambda^{I_k}_\pi}$ such that $d(\lambda_1), d(\lambda_2) \in \tilde{d}(\mathbb{E})$. Let

$$\begin{aligned}\psi(z_1, \ldots, \hat{z}_{j_1}, \ldots, \hat{z}_{j_k}, \ldots, z_n) \\ = \max(\phi_1(z_1), \ldots, \widehat{\phi_{j_1}(z_{j_1})}, \ldots, \widehat{\phi_{j_k}(z_{j_k})}, \ldots, \phi_n(z_n))\end{aligned}$$

be an exhaustion function for Λ^{I_k} treated like a subset of $\mathbb{C}^{n-k}$, where ϕ_j is an exhaustion function for D_j. We can consider the restriction $\pi : \mathbb{C}^{n-k} \supset \Lambda^{I_k} \to \Lambda^{I_k}_\pi \subset \mathbb{C}^{n-k}$. Define

$$v(w) = \max\{\psi(z) : z \in \pi^{-1}(w)\}.$$

Then v is an exhaustion function for $\Lambda^{I_k}_\pi$. Define $h(\lambda) = v \circ \tilde{d}(\lambda)$. Then h is a negative subharmonic function on $\mathbb{E}$. But on the other hand, $h(\lambda_1) < 0$ and $h(\lambda_2) = 0$, which is impossible. ∎

Remark. Let $D = D_1 \times \cdots \times D_s$, where each D_j is a hyperconvex domain in $\mathbb{C}^{n_j}$ with C^1-boundary, $1 \le j \le s$, $s \ge 2$, and let U be an open neighborhood of D. Let $\pi : U \to \mathbb{C}^n$ be a proper holomorphic map, where $n = n_1 + \cdots + n_s$, and let $\Omega_\pi := \pi(D)$. Following the idea from [2] one can show that the conclusion of Proposition 5.1 is also true for the Ω_π described above.

THEOREM 5.2. *Let $f \in \mathcal{C}(\partial\mathbb{E}\times\partial\mathbb{E})$, and let $d\sigma$ be the normalized Lebesgue measure on $\partial\mathbb{E}$. Then the Poisson integral of f defined by*

$$P[f](z_1, z_2) = \int\limits_{\partial\mathbb{E}\times\partial\mathbb{E}} \frac{(1-|z_1|^2)(1-|z_2|^2)}{|w_1 - z_1|^2|w_2 - z_2|^2} f(w_1, w_2)\, d\sigma(w_1)\, d\sigma(w_2)$$

is 2-harmonic (i.e. harmonic in each variable separately) on $\mathbb{E}^2$ and continuous on $\overline{\mathbb{E}^2}$. Furthermore, $P[f]$ is pluriharmonic in $\mathbb{E}^2$ if, and only if,

$$\int\limits_{\partial\mathbb{E}\times\partial\mathbb{E}} w_1^{k_1}\overline{w}_2^{k_2} f(w_1, w_2)\, d\sigma(w_1)\, d\sigma(w_2) = 0$$

for all $k_1, k_2 \in \mathbb{N}$. Moreover, if u is a 2-harmonic function on $\mathbb{E}^2$, continuous on $\overline{\mathbb{E}^2}$, then $u = P[u]$.

Proof. See [8]. ∎

Using Rudin's result we obtain a similar result for $\partial\Omega_\pi$.

PROPOSITION 5.3. *Let U be an open neighborhood of the closure of $\mathbb{E}^2$ in $\mathbb{C}^2$, and let $\pi : U \to \mathbb{C}^2$ be a proper holomorphic map. Let $\Omega_\pi := \pi(\mathbb{E}^2)$, and let $f : \partial\Omega_\pi \to \mathbb{R}$ be a continuous function. The following are then equivalent:*

(1) *there exists a function u which is pluriharmonic on Ω_π, continuous on $\overline{\Omega}_\pi$ and $u|_{\partial\Omega_\pi} = f$,*
(2) *f is harmonic in the sense of Definition 2.2 and*

$$\int\limits_{\partial\mathbb{E}\times\partial\mathbb{E}} w_1^{k_1}\overline{w}_2^{k_2} f(\pi(w_1, w_2))\, d\sigma(w_1)\, d\sigma(w_2) = 0 \tag{5.1}$$

for all $k_1, k_2 \in \mathbb{N}$.

Proof. (1)⇒(2): Similarly to the proof of Lemma 3.1 one can show that f is harmonic in the sense of Definition 2.2. By assumption it also follows that PB_f is pluriharmonic on Ω_π. Therefore by Lemma 3.3, $\mathrm{PB}_{f\circ\pi} = \mathrm{PB}_f \circ \pi$ and $\mathrm{PB}_f \circ \pi$ is pluriharmonic on $\mathbb{E}^2$, continuous on $\overline{\mathbb{E}^2}$, and $\mathrm{PB}_f \circ \pi = f \circ \pi$ on $\partial\mathbb{E}^2$. Then $\mathrm{PB}_{f\circ\pi} = P[f\circ\pi]$ is the Poisson integral of $f\circ\pi$, so (5.1) holds by Theorem 5.2.

(2)⇒(1): From (2) it follows that (5.1) holds for all $k_1, k_2 \in \mathbb{N}$. By Theorem 5.2, $P[f \circ \pi]$ is pluriharmonic on $\mathbb{E}^2$. Since f is harmonic in the sense of Definition 2.2, the function $P[f \circ \pi] = \mathrm{PB}_{f\circ\pi}$ is pluriharmonic on $\mathbb{E}^2$, continuous on $\partial\mathbb{E}^2$, and $\mathrm{PB}_{f\circ\pi} = f \circ \pi$ on $\partial\mathbb{E}^2$. Therefore by Lemma 3.3, PB_f is pluriharmonic on Ω_π, and the proof is complete. ∎

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