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Parabolic initial-boundary value problems in Orlicz spaces

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Abstract. We prove some time mollification properties and imbedding results in inhomogeneous Orlicz–Sobolev spaces which allow us to solve a second order parabolic equation in Orlicz spaces.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. In this paper we deal with the following parabolic initial-boundary value problem:

$$\begin{cases} \partial u/\partial t + A(u) = f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where A is an elliptic second order operator of divergence form

$$A(u) = -\operatorname{div}(a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u) \tag{2}$$

with the coefficients a and a_0 satisfying the classical Leray–Lions conditions.

Consider, first, the case where a and a_0 have polynomial growth with respect to u and ∇u. Then A is a bounded operator from $L^p(0,T;W_0^{1,p}(\Omega))$, $1 < p < \infty$, into its dual. In this setting, problems of the form (1) were solved by J.-L. Lions [16] and Brézis–Browder [5] in the case where $p \geq 2$ and by Landes [14] and Landes–Mustonen [15] when $1 < p < 2$. See also [3] and [4] for related topics.

In the case where a and a_0 satisfy a more general growth condition with respect to u and ∇u, it is shown in [6] that the appropriate space in which (1) can be studied is the inhomogeneous Orlicz–Sobolev space $W^{1,x}L_M(Q)$, where the N-function M is related to the actual growth of a and a_0. The solvability of (1) in this setting is proved by Donaldson [6] and Robert [18] by assuming that A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2 condition, and by Elmahi [7] when M satisfies a Δ' condition and $M(t) \ll t^{N/(N-1)}$.

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It is our purpose in this paper to prove the existence of solutions for problem (1) in the setting of Orlicz spaces without assuming any growth restriction on M. We use a Galerkin method due to Landes and Mustonen [14], [15].

Note that, as in [14], we can include a perturbation term of the form $g(x,t,u)$ without difficulties, but we prefer to do this, with a more general term $g(x,t,u,\nabla u)$ having natural growth with respect to $|\nabla u|$, in [8] by using some compactness results.

2. Preliminaries

2.1. Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-*function*, i.e. M is continuous, convex, $M(t)>0$ for $t>0$, $M(t)/t\to 0$ as $t\to 0$ and $M(t)/t\to\infty$ as $t\to\infty$.

Equivalently, M admits the representation $M(t)=\int_0^t m(\tau)\,d\tau$, where $m:\mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing, right continuous, with $m(0)=0$, $m(t)>0$ for $t>0$ and $m(t)\to\infty$ as $t\to\infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t)=\int_0^t \overline{m}(\tau)\,d\tau$, where $\overline{m}:\mathbb{R}^+\to\mathbb{R}^+$ is given by $\overline{m}(t)=\sup\{s: m(s)\le t\}$ (see [1], [12] and [13]).

We will extend these N-functions to even functions on all $\mathbb{R}$.

The N-function M is said to satisfy the Δ_2 *condition* if, for some $k>0$,

$$M(2t)\le kM(t)\qquad \forall t\ge 0. \tag{3}$$

When (3) holds only for $t\ge$ (some) $t_0>0$ then M is said to satisfy the Δ_2 *condition near infinity.*

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The *Orlicz class* $\mathcal{L}_M(\Omega)$ (resp. the *Orlicz space* $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))\,dx<\infty$ (resp. $\int_\Omega M(u(x)/\lambda)\,dx<\infty$ for some $\lambda>0$).

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega}=\inf\Big\{\lambda>0:\int_\Omega M(u(x)/\lambda)\,dx\le 1\Big\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact supports in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega)=L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

Two N-functions M and P are said to be *equivalent* (resp. *equivalent near infinity*) if there exist real numbers $k_1, k_2 > 0$ such that

$$P(k_1 t) \le M(t) \le P(k_2 t) \quad \text{for all } t \ge 0 \qquad (\text{resp. for } t \ge t_0 > 0).$$

$P \ll M$ denotes that P *grows essentially less rapidly than* M, meaning that $P(\varepsilon t)/M(t) \to 0$ as $t \to \infty$, for each $\varepsilon > 0$. This is the case if and only if

$$M^{-1}(t)/P^{-1}(t) \to 0 \qquad \text{as } t \to \infty,$$

therefore, we have the continuous imbedding $L_M(\Omega) \subset E_P(\Omega)$ when Ω has finite measure.

2.3. We now turn to the Orlicz–Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$, and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n *converges to u for the modular convergence* in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M((D^\alpha u_n - D^\alpha u)/\lambda)\,dx \to 0$ for all $|\alpha| \le 1$. This implies convergence with respect to $\sigma(\prod L_M, \prod L_{\overline{M}})$.

If M satisfies the Δ_2 condition on $\mathbb{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

Recall that Ω is said to have the *segment property* if there exist an open covering $\{U_i\}$ of $\overline{\Omega}$ and corresponding vectors $\{y_i \in \mathbb{R}^N\}$ such that, for $x \in \overline{\Omega} \cap U_i$ and $0 < t < 1$, $x - ty_i \in \Omega$. It was proved in [10] that if the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$. Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and $Q = \Omega \times (0,T)$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The *inhomogeneous Orlicz–Sobolev spaces* of order 1 are defined as follows:

$$W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q),\ \forall |\alpha| \le 1\}$$

and

$$W^{1,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q),\ \forall |\alpha| \le 1\}.$$

The latter space is a subspace of the former. Both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \le 1} \|D_x^\alpha u\|_{M,Q}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\prod L_M(Q)$ which has $N+1$ factors. We shall also consider the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$ then the function $t \mapsto u(t) = u(\cdot,t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$ then $u(\cdot,t)$ is $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore, we have the continuous imbedding $W^{1,x}E_M(Q) \subset L^1([0,T], W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable and if $u \in W^{1,x}L_M(Q)$, we cannot conclude that $u(t)$ is measurable from $(0,T)$ into $W^1L_M(\Omega)$. However, the scalar function $t \mapsto \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \le 1$.

2.6. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure of $\mathcal{D}(Q)$ in $W^{1,x}E_M(Q)$. We can easily show as in [10] that, when Ω has the segment property, each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak $*$ topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q)$, of some sequence $(u_n) \subset \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that, for all $|\alpha| \le 1$, $\int_Q M((D_x^\alpha u_n - D_x^\alpha u)/\lambda)\,dx\,dt \to 0$ as $n \to \infty$. This implies that (u_n) converges to u in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\prod L_M, \prod L_{\overline{M}})$. Consequently,

$$\overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod L_{\overline{M}})},$$

and this space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \prod E_M$.

Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$ and there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha| \le 1} \|D_x^\alpha u\|_{M,Q} \le C \sum_{|\alpha| = 1} \|D_x^\alpha u\|_{M,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$.

We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_M(Q)$ and $W_0^{1,x}L_M(Q)$ being the dual space of F_0. F is also, up to isomorphism, the quotient of $\prod L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and it will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$; moreover, it is known that

$$W^{-1,x}L_{\overline{M}}(Q) = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q)\Big\}.$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\overline{M},Q},$$

where the inf is taken over all possible decompositions $f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha$, $f_\alpha \in L_{\overline{M}}(Q)$. The space F_0 is then given by $F_0 = \{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

2.7. We will use the following technical lemmas:

LEMMA 1 (see Gossez [10]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let (ϱ_σ) be a mollifier sequence in $\mathbb{R}^N$. Denote by $u_\sigma = \widetilde{u} * \varrho_\sigma$ the mollification of u, where $\widetilde{u}$ is the zero extension of u. If $u \in L_M(\Omega)$ then $u_\sigma \in L_M(\Omega)$, and if $2u \in \mathcal{L}(\Omega)$, we have*

$$\int_\Omega M(u_\sigma - u)\,dx \to 0 \quad \text{as } \sigma \to 0.$$

LEMMA 2 (see Morrey [17]). *If $u \in W_0^{1,1}(\Omega)$ then $\|u_\sigma - u\|_{1,\Omega} \le \sigma \|\nabla u\|_{1,\Omega}$.*

LEMMA 3. *Let M be an N-function and let (u_n) be a bounded sequence in $W_0^{1,x}L_M(Q) \cap L^\infty(0,T;L^1(\Omega))$. If $u_n(t) \rightharpoonup u(t)$ weakly in $L^1(\Omega)$ for a.e. $t \in [0,T]$ then $u_n \to u$ strongly in $L^1(Q)$.*

Proof. For each $v \in W_0^{1,x}L_M(Q)$ define $v_\sigma(x,t) = \int_{\mathbb{R}^N} v(y,t)\varrho_\sigma(x-y)\,dy$, where $v(y,t) = 0$ if $y \notin \Omega$ and where (ϱ_σ) is a mollifier sequence in $\mathbb{R}^N$.

Since $u_n(t) \rightharpoonup u(t)$ weakly in $L^1(\Omega)$ for a.e. $t \in [0,T]$ we have $u_{n\sigma} \to u_\sigma$ a.e. in Q and $u_{n\sigma}(t) \to u_\sigma(t)$ strongly in $L^1(\Omega)$ for a.e. $t \in [0,T]$.

For all n and k and for a.e. $t \in [0,T]$ we have

$$\begin{aligned}\int_\Omega |u_n(t) - u_k(t)|\,dx &\le \int_\Omega |u_n(t) - u_{n\sigma}(t)|\,dx + \int_\Omega |u_{n\sigma}(t) - u_{k\sigma}(t)|\,dx \\ &\quad + \int_\Omega |u_{k\sigma}(t) - u_k(t)|\,dx \\ &\le \sigma\Big(\int_\Omega |\nabla u_n(t)|\,dx + \int_\Omega |\nabla u_k(t)|\,dx\Big) + \|u_{n\sigma}(t) - u_{k\sigma}(t)\|_{1,\Omega}.\end{aligned}$$

Integrating this inequality over $[0,T]$ yields

$$\int_Q |u_n(t)-u_k(t)|\,dx\,dt \le \sigma\Big(\int_Q |\nabla u_n(t)|\,dx\,dt + \int_Q |\nabla u_k(t)|\,dx\,dt\Big) + \int_0^T \|u_{n\sigma}(t)-u_{k\sigma}(t)\|_{1,\Omega}\,dt,$$

which, by the continuous embedding $L_M(Q)\subset L^1(Q)$, gives

$$\int_Q |u_n(t)-u_k(t)|\,dx\,dt \le \sigma C_1(\|\nabla u_n\|_{M,Q} + \|\nabla u_k\|_{M,Q}) + \int_0^T \|u_{n\sigma}(t)-u_{k\sigma}(t)\|_{1,\Omega}\,dt,$$

where C_1 and C_2 are constants, which do not depend on n and k, such that

$$\|\nabla v\|_{1,Q}\le C_1\|\nabla v\|_{M,Q} \quad \forall v\in L_M(Q) \quad \text{and} \quad \|\nabla u_n\|_{M,Q}\le C_2 \quad \forall n.$$

Consequently, we obtain

$$\int_Q |u_n(t)-u_k(t)|\,dx \le 2C_1C_2\sigma + \int_0^T \|u_{n\sigma}(t)-u_{k\sigma}(t)\|_{1,\Omega}\,dt.$$

Since $\|u_{n\sigma}(t)-u_{k\sigma}(t)\|_{1,\Omega}\to 0$ a.e. in $[0,T]$ as $n,k\to\infty$ and $\|u_{n\sigma}(t)\|_{L^1(\Omega)} \le \|u_n(t)\|_{L^1(\Omega)}\le C$ uniformly with respect to n and $t\in[0,T]$ we deduce by using Lebesgue's theorem that

$$\int_0^T \|u_{n\sigma}(t)-u_{k\sigma}(t)\|_{1,\Omega}\,dt \to 0 \quad \text{as } n,k\to\infty,$$

implying by the arbitrariness of σ that $\int_Q |u_n(t)-u_k(t)|\,dx\,dt \to 0$ as $n,k\to\infty$. Hence (u_n) is a Cauchy sequence in $L^1(Q)$ and thus $u_n\to u$ strongly in $L^1(Q)$. ∎

3. Time mollification. For $u\in L_M(Q)$, define for all $\mu>0$ and all $(x,t)\in Q$,

$$u_\mu(x,t)=\mu\int_{-\infty}^{t}\widetilde{u}(x,s)\exp(\mu(s-t))\,ds,$$

where $\widetilde{u}$ is the zero extension of u.

PROPOSITION 1. *If $u\in L_M(Q)$ then u_μ is measurable in Q and $\partial u_\mu/\partial t = \mu(u-u_\mu)$, and if $u\in \mathcal{L}_M(Q)$ then*

$$\int_Q M(u_\mu)\,dx\,dt \le \int_Q M(u)\,dx\,dt.$$

Proof. Since $(x,t,s)\mapsto u(x,s)\exp(\mu(s-t))$ is measurable in $\Omega\times[0,T]\times[0,T]$, by Fubini's theorem we deduce that u_μ is measurable.

By Jensen's integral inequality and the equality $\int_{-\infty}^0 \mu\exp(\mu s)\,ds=1$, we have

$$M\Big(\int_{-\infty}^{t}\mu\widetilde{u}(x,s)\exp(\mu(s-t))\,ds\Big)=M\Big(\int_{-\infty}^{0}\mu\exp(\mu s)\widetilde{u}(x,s+t)\,ds\Big)\le\int_{-\infty}^{0}\mu\exp(\mu s)M(\widetilde{u}(x,s+t))\,ds,$$

which implies

$$\begin{aligned}\int_Q M(u_\mu(x,t))\,dx\,dt&\le\int_{\Omega\times\mathbb{R}}\Big(\int_{-\infty}^{0}\mu\exp(\mu s)M(\widetilde{u}(x,s+t))\,ds\Big)\,dx\,dt\\&\le\int_{-\infty}^{0}\mu\exp(\mu s)\Big(\int_{\Omega\times\mathbb{R}}M(\widetilde{u}(x,s+t))\,dx\,dt\Big)\,ds\\&\le\int_{-\infty}^{0}\mu\exp(\mu s)\Big(\int_Q M(u(x,t))\,dx\,dt\Big)\,ds\le\int_Q M(u)\,dx\,dt.\end{aligned}$$

Furthermore, for a.e. $(x,t)\in Q$,

$$\begin{aligned}\frac{\partial u_\mu}{\partial t}(x,t)&=\lim_{\theta\to0}\frac{1}{\theta}\,(e^{-\mu\theta}-1)u_\mu(x,t)+\lim_{\theta\to0}\frac{1}{\theta}\int_t^{t+\theta}u(x,s)e^{\mu(s-(t+\theta))}\,ds\\&=-\mu u_\mu(x,t)+\mu u(x,t).\ \blacksquare\end{aligned}$$

PROPOSITION 2. (i) *If $u\in L_M(Q)$ then $u_\mu\to u$ as $\mu\to\infty$ in $L_M(Q)$ for the modular convergence.*

(ii) *If $u\in W^{1,x}L_M(Q)$ then $u_\mu\to u$ as $\mu\to\infty$ in $W^{1,x}L_M(Q)$ for the modular convergence.*

Proof. (i) Let $(\varphi_k)\subset\mathcal{D}(Q)$ be such that $\varphi_k\to u$ in $L_M(Q)$ for the modular convergence. Let $\lambda>0$ be large enough such that

$$\frac{u}{\lambda}\in\mathcal{L}_M(Q)\quad\text{and}\quad\int_Q M\Big(\frac{\varphi_k-u}{\lambda}\Big)\,dx\,dt\to0\quad\text{as }k\to\infty.$$

For a.e. $(x,t)\in Q$ we have

$$|(\varphi_k)_\mu(x,t)-\varphi_k(x,t)|=\frac{1}{\mu}\left|\frac{\partial\varphi_k}{\partial t}(x,t)\right|\le\frac{1}{\mu}\left\|\frac{\partial\varphi_k}{\partial t}\right\|_\infty.$$

On the other hand,

$$\begin{aligned}\int_Q M\left(\frac{u_\mu - u}{3\lambda}\right) dx\,dt &\le \frac{1}{3}\int_Q M\left(\frac{u_\mu - (\varphi_k)_\mu}{\lambda}\right) dx\,dt + \frac{1}{3}\int_Q M\left(\frac{(\varphi_k)_\mu - \varphi_k}{\lambda}\right) dx\,dt \\ &\quad + \frac{1}{3}\int_Q M\left(\frac{\varphi_k - u}{\lambda}\right) dx\,dt \\ &\le \frac{1}{3}\int_Q M\left(\frac{(\varphi_k - u)_\mu}{\lambda}\right) dx\,dt + \frac{1}{3}\int_Q M\left(\frac{(\varphi_k)_\mu - \varphi_k}{\lambda}\right) dx\,dt \\ &\quad + \frac{1}{3}\int_Q M\left(\frac{\varphi_k - u}{\lambda}\right) dx\,dt.\end{aligned}$$

This implies that

$$\int_Q M\left(\frac{u_\mu - u}{3\lambda}\right) dx\,dt \le \frac{2}{3}\int_Q M\left(\frac{\varphi_k - u}{\lambda}\right) dx\,dt + \frac{1}{3} M\left(\frac{1}{\mu\lambda}\left\|\frac{\partial \varphi_k}{\partial t}\right\|_\infty\right) \operatorname{meas}(Q).$$

Let $\varepsilon > 0$. There exist k and μ_0 such that

$$\int_Q M\left(\frac{\varphi_k - u}{\lambda}\right) dx\,dt \le \varepsilon$$

and

$$M\left(\frac{1}{\mu\lambda}\left\|\frac{\partial \varphi_k}{\partial t}\right\|_\infty\right) \operatorname{meas}(Q) \le \varepsilon \quad \text{for all } \mu \ge \mu_0.$$

Hence

$$\int_Q M\left(\frac{u_\mu - u}{3\lambda}\right) dx\,dt \le \varepsilon \quad \text{for all } \mu \ge \mu_0.$$

(ii) Since for any α with $|\alpha| \le 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, the first part above applied to each $D_x^\alpha u$ gives the result. ∎

Remark 1. If $u \in E_M(Q)$, we can choose λ arbitrarily small since $\mathcal{D}(Q)$ is (norm) dense in $E_M(Q)$. Thus, for all $\lambda > 0$,

$$\int_Q M\left(\frac{u_\mu - u}{\lambda}\right) dx\,dt \to 0 \quad \text{as } \mu \to \infty$$

and $u_\mu \to u$ strongly in $E_M(Q)$. The same remark is true if one replaces $E_M(Q)$ with $W^{1,x}E_M(Q)$.

Proposition 3. *If $u_n \to u$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence) then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence).*

Proof. It suffices to prove the proposition for the zero order derivative. For all (resp. some) $\lambda > 0$,

$$\int_Q M\bigg(\frac{(u_n)_\mu - u_\mu}{\lambda}\bigg)\,dx\,dt \le \int_Q M\bigg(\frac{u_n - u}{\lambda}\bigg)\,dx\,dt \to 0 \quad \text{as } n \to \infty,$$

so $(u_n)_\mu \to u_\mu$ in $L_M(Q)$ strongly (resp. for the modular convergence). ∎

4. Existence result. Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property, $T > 0$, and set $Q = \Omega \times (0,T)$. Let M and P be two N-functions such that $P \ll M$.

Consider a second order operator $A : D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ of the form

$$A(u) = -\operatorname{div}(a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u)$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying for a.e. $(x,t) \in \Omega \times [0,T]$ and all $s \in \mathbb{R}$, $\xi \ne \xi^* \in \mathbb{R}^N$:

$$|a(x,t,s,\xi)| \le \beta(c(x,t) + \overline{M}^{-1}P(\gamma(|s|)) + \overline{M}^{-1}M(\gamma|\xi|)), \tag{4}$$

$$|a_0(x,t,s,\xi)| \le \beta(c(x,t) + \overline{M}^{-1}P(\gamma(|s|)) + \overline{M}^{-1}P(\gamma|\xi|)), \tag{5}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{6}$$

$$a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \ge \alpha M(|\xi|/\lambda) - d(x,t), \tag{7}$$

where $c(x,t) \in E_{\overline{M}}(Q), c \ge 0$; $d(x,t) \in L^1(Q)$; $\alpha, \beta, \gamma > 0$.

Furthermore let

$$f \in W^{-1,x}E_{\overline{M}}(Q). \tag{8}$$

Consider the following parabolic initial-boundary value problem:

$$\begin{cases} \partial u/\partial t + A(u) = f & \text{in } Q,\\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T),\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{9}$$

where u_0 is a given function in $L^2(\Omega)$.

We shall prove the following existence theorem:

THEOREM 1. *Assume that* (4)–(8) *hold true. Then there exists at least one weak solution* $u \in W_0^{1,x}L_M(Q) \cap C([0,T], L^2(\Omega))$ *of* (9) *in the following sense*:

$$-\int_Q u\,\frac{\partial\varphi}{\partial t}\,dx\,dt + \Big[\int_\Omega u(t)\varphi(t)\,dx\Big]_0^T + \int_Q a(x,t,u,\nabla u)\cdot\nabla\varphi\,dx\,dt + \int_Q a_0(x,t,u,\nabla u)\varphi\,dx\,dt = \langle f,\varphi\rangle$$

for all $\varphi \in C^1([0,T], L^2(\Omega))$.

Remark 2. As in the elliptic case (see [9] and [11]), the introduction of P instead of M in (4) and (5) is done only to guarantee the boundedness in $L_{\overline{M}}(Q)$ of $\overline{M}^{-1}P(\gamma|u_n|)$ and $\overline{M}^{-1}P(\gamma|\nabla u_n|)$ whenever u_n is bounded in $W^{1,x}L_M(Q)$. In the elliptic case, one usually takes $P = M$ in the term $\overline{M}^{-1}P(\gamma|u_n|)$ since u_n is bounded in a smaller space $L_R(\Omega)$ with $M \ll R$ (see [9]).

In the parabolic case, we cannot however deduce the same boundedness. Nevertheless, we can take $P = M$ if one of the following assertions holds:

1) M satisfies the Δ_2 condition near infinity.
2) A is monotone, i.e. $\langle A(u) - A(v), u - v\rangle \geq 0$ for all $u, v \in D(A) \cap W_0^{1,x}L_M(Q)$.
3) M grows essentially less rapidly than the N-function $\overline{M} \circ M$.

Indeed, suppose first that M satisfies the Δ_2 condition. Then (4) and (5), with now $P = M$, imply that for all $\varepsilon > 0$,

$$|a(x,t,s,\xi)| \leq \beta_\varepsilon(c_\varepsilon(x,t) + \overline{M}^{-1}M(\varepsilon|s|) + \overline{M}^{-1}M(\varepsilon|\xi|)),$$
$$|a_0(x,t,s,\xi)| \leq \beta_\varepsilon(c_\varepsilon(x,t) + \overline{M}^{-1}M(\varepsilon|s|) + \overline{M}^{-1}M(\varepsilon|\xi|)),$$

which allows us to deduce the boundedness in $L_{\overline{M}}(Q)$ of $a(x,t,u_n,\nabla u_n)$ and $a_0(x,t,u_n,\nabla u_n)$.

In the case where A is monotone, for all $\varphi \in W_0^{1,x}E_M(Q)$ we have

$$\langle A(u_n) - A(\varphi), u_n - \varphi\rangle \geq 0,$$

where $\langle\,,\rangle$ is the pairing between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$. This gives

$$\langle A(u_n), \varphi\rangle \leq \langle A(u_n), u_n\rangle - \langle A(\varphi), u_n - \varphi\rangle,$$

implying that, since u_n is bounded in $W_0^{1,x}L_M(Q)$ and $\langle A(u_n), u_n\rangle \leq C_1$ thanks to the a priori estimates,

$$\langle A(u_n), \varphi\rangle \leq C_\varphi \quad \text{for all } \varphi \in W_0^{1,x}E_M(Q).$$

Therefore, the Banach–Steinhaus theorem yields the boundedness of $A(u_n)$ in $W^{-1,x}L_{\overline{M}}(Q)$.

Assume, finally, that $M \ll \overline{M} \circ M$. Then for all $\varepsilon > 0$ there is $t_\varepsilon \geq 0$ such that

$$M(\gamma t) \leq \overline{M}(M(\varepsilon^2 t)) \quad \text{for all } t \geq t_\varepsilon$$

implies that

$$|a(x,t,s,\xi)| \leq \beta(c_\varepsilon(x,t) + \varepsilon M(\varepsilon|s|) + \varepsilon M(\varepsilon|\xi|)),$$
$$|a_0(x,t,s,\xi)| \leq \beta(c_\varepsilon(x,t) + \varepsilon M(\varepsilon|s|) + \varepsilon M(\varepsilon|\xi|)),$$

which gives the boundedness and the weak convergence in $L^1(Q)$ of $a(x,t,u_n,\nabla u_n)$ and $a_0(x,t,u_n,\nabla u_n)$. This leads to

$$u_n(t) \to u(t) \quad \text{a.e. in } \Omega \quad \text{and then} \quad u_n \to u \quad \text{in } L^1(Q).$$

Hence, the proof below can be adapted to this situation by proving the existence of an entropy solution of (9).

Note that there are N-functions M for which M and $\overline{M} \circ M$ are equivalent. Indeed, take $M(t) = \exp(t)$ near infinity. We have $\overline{M}(t) = t \log t$ near infinity and so $(\overline{M} \circ M)(t) = t \exp(t)$ is equivalent to $M(t)$ since $M(t) \le (\overline{M} \circ M)(t) \le M(2t)$ for t large enough.

Proof of Theorem 1. For convenience we suppose that $u_0 = 0$. The general case can be handled similarly.

We will use a Galerkin method due to Landes and Mustonen [15]. For the Galerkin method we choose the sequence $\{w_1, w_2, \ldots\}$ in $\mathcal{D}(\Omega)$ such that $\bigcup_{n=1}^{\infty} V_n$ with

$$V_n = \operatorname{span}\{w_1, \ldots, w_n\}$$

is dense in $W_0^j L_M(\Omega)$ for the modular convergence, where $j > q(M, N)$ is taken such that $W_0^j L_M(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$.

For any $v \in W_0^j L_M(\Omega)$ there exists a sequence $(v_k) \subset \bigcup_{n=1}^{\infty} V_n$ such that $v_k \to v$ in $W_0^j L_M(\Omega)$ for the modular convergence.

We set further $\mathcal{V}_n = C([0,T], V_n)$. It is easy to see that the closure of $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ with respect to the norm

$$\|v\|_{C^{1,0}(Q)} = \sup_{|\alpha| \le 1} \{|D^\alpha v(x,t)| : (x,t) \in Q\}$$

contains $\mathcal{D}(Q)$. This implies that for any $f \in W^{-1,x} E_{\overline{M}}(Q)$ there exists a sequence $(f_k) \subset \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $f_k \to f$ strongly in $W^{-1,x} E_{\overline{M}}(Q)$.

For any $u_0 \in L^2(\Omega)$ there is a sequence $(u_{0k}) \subset \bigcup_{n=1}^{\infty} V_n$ such that $u_{0k} \to u_0$ in $L^2(\Omega)$.

We divide the proof into three steps.

Step 1. *A priori estimates.* As in [15], by using Lemma 1 of [14], there exists a Galerkin solution u_n of (9) in the following sense:

$$\begin{cases} u_n \in \mathcal{V}_n, \dfrac{\partial u_n}{\partial t} \in L^1(0,T; V_n), u_n(0) = u_{0n}, \text{ and for all } \varphi \in \mathcal{V}_n, \\ \displaystyle\int_{Q_\tau} \frac{\partial u_n}{\partial t} \varphi \, dx \, dt + \int_{Q_\tau} a(x,t,u_n,\nabla u_n) \nabla \varphi \, dx \, dt \\ \displaystyle\qquad + \int_{Q_\tau} a_0(x,t,u_n,\nabla u_n) \varphi \, dx \, dt = \int_{Q_\tau} f_n \varphi \, dx \, dt \end{cases} \tag{10}$$

for all $\tau \in (0,T)$, where $Q_\tau = \Omega \times (0,\tau)$.

Letting $\varphi = u_n$ in (10) and using (4) and (7) yields

$$\|u_n\|_{W_0^{1,x} L_M(Q)} \le C, \qquad \|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le C,$$
$$\int_Q [a(x,t,u_n,\nabla u_n) \nabla u_n + a_0(x,t,u_n,\nabla u_n) u_n] \, dx \, dt \le C;$$

here and below, C is a constant not depending on n.

Using (5) and the fact that $P \ll M$, it is easy to see that $a_0(x,t,u_n,\nabla u_n)$ is bounded in $L_{\overline{M}}(Q)$. This implies that

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le C.$$

To prove that $a(x,t,u_n,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$, let $\varphi \in (E_M(Q))^N$ and $\|\varphi\|_{M,Q} = 1$. By (6), we have

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\varphi)][\nabla u_n - \varphi]\,dx\,dt \ge 0,$$

which gives

$$\int_Q a(x,t,u_n,\nabla u_n)\varphi\;dx\,dt \le \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt - \int_Q a(x,t,u_n,\varphi)[\nabla u_n - \varphi]\,dx\,dt.$$

Since $a(x,t,u_n,\varphi)$ is uniformly bounded in $(L_{\overline{M}}(Q))^N$, thanks to (4), we deduce that

$$\int_Q a(x,t,u_n,\nabla u_n)\varphi\,dx\,dt \le C \quad \text{for all } \varphi \in (E_M(Q))^N,\ \|\varphi\|_{M,Q} = 1.$$

Using the dual norm of $(L_{\overline{M}}(Q))^N$ we conclude that $a(x,t,u_n,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$.

Hence, for a subsequence,

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}) \text{ and weakly in } L^2(Q), \\ a_0(x,t,u_n,\nabla u_n) \rightharpoonup h_0, a(x,t,u_n,\nabla u_n) \rightharpoonup h \text{ in } L_{\overline{M}}(Q) \text{ for } \sigma(\prod L_{\overline{M}}, \prod E_M) \end{cases}$$

for some $h_0 \in L_{\overline{M}}(Q)$ and some $h \in (L_{\overline{M}}(Q))^N$.

As in [15], by using Lemma 3 we deduce that $u_n \to u$ strongly in $L^1(Q)$ and for some subsequence $u_n(x,t) \to u(x,t)$ a.e. in Q.

Step 2. *Almost everywhere convergence of the gradients.* For all $\varphi \in C^1([0,T],\mathcal{D}(\Omega))$, from (10) we get

$$(11)\qquad -\int_Q u\,\frac{\partial\varphi}{\partial t}\,dx\,dt + \Big[\int_\Omega u(t)\varphi(t)\,dx\Big]_0^T + \int_Q h\cdot\nabla\varphi\,dx\,dt + \int_Q h_0\varphi\,dx\,dt = \langle f,\varphi\rangle.$$

Let $(\phi_j) \subset \mathcal{D}(Q)$ be such that $\phi_j \to u$ in $L^2(Q)$ and in $W_0^{1,x}L_M(Q)$ for the modular convergence. For $\mu \in \mathbb{N}$, let

$$(T_l(\phi_j))_\mu(x,t) = \mu\int_{-\infty}^{t} T_l(\phi_j)(x,s)\exp(\mu(s-t))\,ds,$$

where T_l is the usual truncation at height l defined by

$$T_l(s) = \begin{cases} s & \text{if } |s| \le l, \\ T_l(s) = ls/|s| & \text{if } |s| > l. \end{cases}$$

Then $(T_l(\phi_j))_\mu \to T_l(\phi_j)$ in $W_0^{1,x}L_M(Q)$ strongly as $\mu \to \infty$ and

$$\frac{\partial}{\partial t}(T_l(\phi_j))_\mu = \mu(T_l(\phi_j) - (T_l(\phi_j))_\mu).$$

Take the mollification with respect to the space variable, $[(T_l(\phi_j))_\mu]_\sigma$ for $\sigma > 0$. It is obvious that this sequence is in $C^1([0,T],\mathcal{D}(\Omega))$. Finally, choose v_k as a diagonal sequence of $[(T_l(\phi_j))_\mu]_\sigma$ such that $v_k \to u$ in $W_0^{1,x}L_M(Q)$ for the modular convergence.

Indeed, let $\lambda > 0$ be such that

$$\text{(12)} \quad \frac{1}{\lambda}D_x^\alpha u \in \mathcal{L}_M(Q), \qquad \int_Q M\left(\frac{D_x^\alpha \phi_j - D_x^\alpha u}{\lambda}\right) dx\,dt \to 0, \ \forall |\alpha| \le 1.$$

We have

$$\begin{aligned} \int_Q M\left(\frac{D_x^\alpha v_k - D_x^\alpha u}{4\lambda}\right) dx\,dt \le \int_Q M\left(\frac{D_x^\alpha[(T_l(\phi_j))_\mu]_\sigma - D_x^\alpha(T_l(\phi_j))_\mu}{\lambda}\right) dx\,dt \\ + \int_Q M\left(\frac{D_x^\alpha(T_l(\phi_j))_\mu - D_x^\alpha T_l(\phi_j)}{\lambda}\right) dx\,dt \\ + \int_Q M\left(\frac{D_x^\alpha T_l(\phi_j) - D_x^\alpha \phi_j}{\lambda}\right) dx\,dt \\ + \int_Q M\left(\frac{D_x^\alpha \phi_j - D_x^\alpha u}{\lambda}\right) dx\,dt. \end{aligned}$$

The first three integrals of the right side go to 0 since $D_x^\alpha[(T_l(\phi_j))_\mu]_\sigma$, $D_x^\alpha(T_l(\phi_j))_\mu$ and $D_x^\alpha T_l(\phi_j)$ are strongly convergent in $W_0^{1,x}E_M(Q)$ respectively as $\sigma \to 0$, $\mu \to \infty$ and $l \to \infty$ by using the facts that $(T_l(\phi_j))_\mu$, $T_l(\phi_j)$ and ϕ_j are in $W_0^{1,x}E_M(Q)$ (see Lemma 5 of [10]).

Since the last integral goes to 0 by (12), we deduce that $v_k \to u$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and hence, for a subsequence,

$$v_k \to u, \nabla v_k \to \nabla u \text{ a.e. in } Q \text{ and weakly in } L_M(Q) \text{ for } \sigma(\prod L_M, \prod L_{\overline{M}}).$$

On the other hand, setting as in [15], $Q_l = \{(x,t) \in Q : |u(x,t)| \le l\}$, we have

$$T_l(u) = u \text{ in } Q_l, \qquad \operatorname{sgn}(T_l(u) - (T_l(u))_\mu) = \operatorname{sgn}(u - (T_l(u))_\mu) \text{ in } Q \backslash Q_l.$$

Therefore, as in [15],

$$\begin{aligned}\int_Q \frac{\partial v_k}{\partial t}(v_k-u)\,dx\,dt &= \mu\int_Q \{(T_l(\phi_j))_\sigma - [(T_l(\phi_j))_\sigma]_\mu\}\{[(T_l(\phi_j))_\sigma]_\mu - u\}\,dx\,dt\\ &\to \mu\int_Q \{T_l(u)-(T_l(u))_\mu\}\{(T_l(u))_\mu - u\}\,dx\,dt\\ &= -\mu\int_{Q_l}(u-(T_l(u))_\mu)^2\,dx\,dt\\ &\quad+\mu\int_{Q\setminus Q_l}\{T_l(u)-(T_l(u))_\mu\}\{(T_l(u))_\mu-u\}\,dx\,dt \le 0\end{aligned}$$

as $\sigma\to 0$ and $j\to\infty$, for any μ and l. Consequently,

$$\limsup_{k\to\infty}\int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt \le 0$$

and then

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt \le 0, \quad \text{since} \quad \frac{\partial v_k}{\partial t}\in E_{\overline{M}}(Q).$$

This implies that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt \le 0$$

since

$$\begin{aligned}\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt &= -\frac{1}{2}\int_Q \frac{\partial}{\partial t}(u_n(t)-v_k(t))^2\,dx\,dt + \int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt\\ &= -\frac{1}{2}\|u_n(T)-v_k(T)\|^2_{L^2(\Omega)} + \int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt.\end{aligned}$$

From (10) and (11) we have

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\int_Q (a(x,t,u_n,\nabla u_n)\nabla u_n - h\nabla v_k + a_0(x,t,u_n,\nabla u_n)u_n - h_0 v_k)\,dx\,dt\Big)\\ &\quad\le \limsup_{n\to\infty}\langle f_n,u_n\rangle - \langle f,v_k\rangle + \limsup_{n\to\infty}\Big(-\int_Q \frac{\partial u_n}{\partial t}u_n\,dx\,dt\Big) - \int_Q \frac{\partial v_k}{\partial t}u\,dx\,dt\\ &\qquad + \Big[\int_\Omega u(t)v_k(t)\,dx\Big]_0^T\\ &\quad= \langle f,u-v_k\rangle + \limsup_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt,\end{aligned}$$

where we have used the fact that

$$-\int_Q \frac{\partial v_k}{\partial t}\, u\, dx\, dt + \Big[\int_\Omega u(t) v_k(t)\, dx\Big]_0^T = \lim_{n\to\infty}\Big(-\int_Q \frac{\partial v_k}{\partial t}\, u_n\, dx\, dt + \Big[\int_\Omega u_n(t) v_k(t)\, dx\Big]_0^T\Big) = \lim_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}\, v_k\, dx\, dt.$$

We deduce that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\Big(\int_Q (a(x,t,u_n,\nabla u_n)\nabla u_n - h\nabla v_k + a_0(x,t,u_n,\nabla u_n)u_n - h_0 v_k)\, dx\, dt\Big) \le \limsup_{k\to\infty}\langle f, u - v_k\rangle + \limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)\, dx\, dt \le 0,$$

which implies that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\Big(\int_Q (a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla v_k] + a_0(x,t,u_n,\nabla u_n)(u_n - v_k))\, dx\, dt\Big) \le 0 \tag{13}$$

since, as can be easily seen,

$$\lim_{n\to\infty}\int_Q (a(x,t,u_n,\nabla u_n)\nabla v_k + a_0(x,t,u_n,\nabla u_n)v_k)\, dx\, dt = \int_Q (h\nabla v_k + h_0 v_k)\, dx\, dt.$$

For any $r > 0$ and $k \in \mathbb{N}$, we denote by χ_k^r and χ^r the characteristic functions of $\{(x,t) \in Q : |\nabla v_k| \le r\}$ and $\{(x,t) \in Q : |\nabla u| \le r\}$, respectively.

For any $l > 0$, we have

$$\int_{\{|u_n|\le l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u\cdot\chi^s)][\nabla u_n - \nabla u\cdot\chi^s]\, dx\, dt$$
$$- \int_{\{|u_n|\le l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla v_k\cdot\chi_k^s)][\nabla u_n - \nabla v_k\cdot\chi_k^s]\, dx\, dt$$

$$
\begin{aligned}
&= \int\limits_{\{|u_n|\le l\}} a(x,t,u_n,\nabla u\cdot\chi^s)[\nabla u_n-\nabla u\cdot\chi^s]\,dx\,dt\\
&\quad+ \int\limits_{\{|u_n|\le l\}} a(x,t,u_n,\nabla u_n)[\nabla v_k\cdot\chi^s_k-\nabla u\cdot\chi^s]\,dx\,dt\\
&\quad- \int\limits_{\{|u_n|\le l\}} a(x,t,u_n,\nabla v_k\cdot\chi^s_k)[\nabla u_n-\nabla v_k\cdot\chi^s_k]\,dx\,dt\\
&:= I_1+I_2+I_3.
\end{aligned}
$$

We shall go to the limit, first as $n\to\infty$ and next as $k\to\infty$ and finally as $s\to\infty$, in all integrals I_i, for $i=1,2,3$.

Since $\chi_{\{|u_n|\le l\}}a(x,t,u_n,\nabla v_k\cdot\chi^s_k)\to\chi_{\{|u|\le l\}}a(x,t,u,\nabla v_k\cdot\chi^s_k)$ strongly in $(E_{\overline{M}}(Q))^N$, by (4) and the fact that $u_n\to u$ a.e. in Q we deduce that

$$
I_1\to \int\limits_{\{|u|\le l\}\cap\{|\nabla u|\ge s\}} a(x,t,u,0)\nabla u\,dx\,dt \quad \text{as } n\to\infty,
$$

which clearly tends to zero as $s\to\infty$.

Observe that I_2 tends to

$$
\int\limits_{\{|u|\le l\}} h[\nabla v_k\cdot\chi^s_k-\nabla u\cdot\chi^s]\,dx\,dt \quad \text{as } n\to\infty,
$$

which tends to 0 as $k\to\infty$ since $\nabla v_k\cdot\chi^s_k-\nabla u\cdot\chi^s\to 0$ strongly in $(E_M(Q))^N$.

For the third term I_3, since $\nabla u_n\rightharpoonup\nabla u$ in $(L_M(Q))^N$, we have

$$
I_3\to-\int\limits_{\{|u|\le l\}} a(x,t,u,\nabla v_k\cdot\chi^s_k)[\nabla u-\nabla v_k\cdot\chi^s_k]\,dx\,dt \quad \text{as } n\to\infty;
$$

since $\chi_{\{|u|\le l\}}a(x,t,u,\nabla v_k\cdot\chi^s_k)\to\chi_{\{|u|\le l\}}a(x,t,u,\nabla u\cdot\chi^s)$ strongly in $(E_{\overline{M}}(Q))^N$ as $k\to\infty$ by Lebesgue's theorem, the above tends to

$$
-\int\limits_{\{|u|\le l\}\cap\{|\nabla u|\ge s\}} a(x,t,u,0)\nabla u\,dx\,dt
$$

as $k\to\infty$, which clearly tends to zero as $s\to\infty$.

We have thus proved that

$$
\begin{aligned}
&\int\limits_{\{|u_n|\le l\}} [a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla u\cdot\chi^s)][\nabla u_n-\nabla u\cdot\chi^s]\,dx\,dt\\
&= \int\limits_{\{|u_n|\le l\}} [a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla v_k\cdot\chi^s_k)][\nabla u_n-\nabla v_k\cdot\chi^s_k]\,dx\,dt+\varepsilon(n,k,s),
\end{aligned}
$$

where $\varepsilon(n,k,s)$ denotes quantities (possibly different) depending on l such that

$$
\lim_{s\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\varepsilon(n,k,s)=0.
$$

For all $s \geq r > 0$ and all $l \geq \bar{l}$, we have

$$
\begin{aligned}
(14)\quad 0 &\leq \int_{\{|u_n|\leq \bar{l},\, |\nabla u|\leq r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt \\
&\leq \int_{\{|u_n|\leq l,\, |\nabla u|\leq s\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt \\
&\leq \int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u\cdot\chi^s)][\nabla u_n - \nabla u\cdot\chi^s]\,dx\,dt \\
&= \int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla v_k\cdot\chi_k^s)][\nabla u_n - \nabla v_k\cdot\chi_k^s]\,dx\,dt \\
&\quad + \varepsilon(n,k,s) \\
&= -\int_{\{|u_n|\leq l\}} a(x,t,u_n,\nabla v_k\cdot\chi_k^s)[\nabla u_n - \nabla v_k\cdot\chi_k^s] dx\,dt \\
&\quad + \int_Q (a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla v_k] + a_0(x,t,u_n,\nabla u_n)(u_n - v_k))\,dx\,dt \\
&\quad - \Big(\int_{\{|u_n|> l\}} a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla v_k]\,dx\,dt \\
&\quad + \int_Q a_0(x,t,u_n,\nabla u_n)(u_n - v_k)\,dx\,dt \Big) \\
&\quad + \int_{\{|u_n|\leq l\}\cap\{|\nabla v_k|> s\}} a(x,t,u_n,\nabla u_n)\nabla v_k\,dx\,dt + \varepsilon(n,k,s) \\
&:= J_1 + J_2 + J_3 + J_4 + \varepsilon(n,k,s).
\end{aligned}
$$

We shall take the limsup first over n and next over k and finally over s in all integrals of the right hand side.

Remark that, by (13),

$$\limsup_{k\to\infty}\limsup_{n\to\infty} J_2 \leq 0.$$

Just as for I_1 above, it is easy to see that

$$\lim_{s\to\infty}\limsup_{k\to\infty}\limsup_{n\to\infty} J_1 = 0.$$

The third term reads

$$
\begin{aligned}
J_3 = &-\int_{\{|u_n|> l\}} [a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla v_k] + a_0(x,t,u_n,\nabla u_n)(u_n - v_k)]\,dx\,dt \\
&- \int_{\{|u_n|\leq l\}} a_0(x,t,u_n,\nabla u_n)(u_n - v_k)\,dx\,dt
\end{aligned}
$$

and, by using (7),

$$J_3 \le \int_{\{|u_n|>l\}} [a(x,t,u_n,\nabla u_n)\nabla v_k + a_0(x,t,u_n,\nabla u_n)v_k]\,dx\,dt + \int_{\{|u_n|>l\}} d(x,t)\,dx\,dt - \int_{\{|u_n|\le l\}} a_0(x,t,u_n,\nabla u_n)(u_n - v_k)\,dx\,dt,$$

which gives

$$\limsup_{n\to\infty} J_3 \le \int_{\{|u|\ge l\}} (h\nabla v_k + h_0 v_k)\,dx\,dt + \int_{\{|u|\ge l\}} d(x,t)\,dx\,dt - \int_{\{|u|\le l\}} h_0(u - v_k)\,dx\,dt,$$

where we have used the strong convergence of $\chi_{\{|u_n|>l\}}|\nabla v_k|$ and $\chi_{\{|u_n|>l\}}v_k$ and $\chi_{\{|u_n|\le l\}}u_n$ in $E_M(Q)$ as $n \to \infty$. This implies that

$$\limsup_{k\to\infty}\limsup_{n\to\infty} J_3 \le \int_{\{|u|\ge l\}} (h\nabla u + h_0 u)\,dx\,dt + \int_{\{|u|\ge l\}} d(x,t)\,dx\,dt,$$

since $v_k \to u$ in $W_0^{1,x}L_M(Q)$ for the modular convergence.

For J_4, we have

$$\lim_{n\to\infty} J_4 = \int_{\{|u|\le l\}\cap\{|\nabla v_k|>s\}} h\nabla v_k\,dx\,dt$$

since $\chi_{\{|u_n|\le l,\,|\nabla v_k|>s\}}\nabla v_k \to \chi_{\{|u|\le l,\,|\nabla v_k|>s\}}\nabla v_k$ strongly in $(E_M(Q))^N$ as $n \to \infty$. This implies that

$$\lim_{k\to\infty}\lim_{n\to\infty} J_4 = \int_{\{|u|\le l\}\cap\{|\nabla u|>s\}} h\nabla u\,dx\,dt \le \int_{\{|\nabla u|\ge s\}} |h\nabla u|\,dx\,dt$$

and thus

$$\limsup_{s\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} J_4 \le 0.$$

Combining these estimates with (14) and taking the limsup first over n, then over k and next over s, we deduce that

$$0 \le \limsup_{n\to\infty} \int_{\{|u_n|\le \bar{l},\,|\nabla u|\le r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt \le \int_{\{|u|\ge l\}} (h\nabla u + h_0 u + d(x,t))\,dx\,dt,$$

in which we can let $l \to \infty$ to get

$$\lim_{n\to\infty} \int_{\{|u_n|\le \bar{l},\,|\nabla u|\le r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt = 0$$

and thus, as in the elliptic case (see [2]), we deduce that, for a subsequence still denoted by u_n,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q.$$

This implies that $h = a(x,t,u,\nabla u)$ and $h_0 = a_0(x,t,u,\nabla u)$. Therefore, for all $\varphi \in C^1([0,T],\mathcal{D}(\Omega))$ we get

$$-\int_Q u\,\frac{\partial \varphi}{\partial t}\,dx\,dt + \Big[\int_\Omega u(t)\varphi(t)\,dx\Big]_0^T + \int_Q a(x,t,u,\nabla u)\nabla\varphi\,dx\,dt + \int_Q a_0(x,t,u,\nabla u)\varphi\,dx\,dt = \langle f,\varphi\rangle.$$

STEP 3. *Regularity of the solution.* Note that we may choose v_k such that

$$\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial v_k}{\partial t}\,(v_k-u_n)\,dx\,dt \le \varepsilon_k$$

uniformly in $\tau\in[0,T]$, where $\varepsilon_k\to 0$ as $k\to\infty$.

For all k and all τ in $(0,T)$, from (10) we have

$$\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial u_n}{\partial t}\,v_k\,dx\,dt + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla v_k\,dx\,dt + \int_{Q_\tau} a_0(x,t,u,\nabla u)v_k\,dx\,dt = \langle f,v_k\rangle_{Q_\tau},$$

which implies, by using Fatou's lemma,

$$\begin{aligned}
\limsup_{n\to\infty}\int_{Q_\tau}\frac{\partial u_n}{\partial t}\,(u_n-v_k)\,dx\,dt
&= -\liminf_{n\to\infty}\int_{Q_\tau}(a(x,t,u_n,\nabla u_n)\nabla u_n + a_0(x,t,u_n,\nabla u_n)u_n)\,dx\,dt\\
&\quad+\int_{Q_\tau}(a(x,t,u,\nabla u)\nabla v_k + a_0(x,t,u,\nabla u)v_k)\,dx\,dt\\
&\quad+\lim_{n\to\infty}\int_{Q_\tau} f_n(u_n-v_k)\,dx\,dt\\
&\le -\int_{Q_\tau}(a(x,t,u,\nabla u)\nabla u + a_0(x,t,u,\nabla u)u)\,dx\,dt\\
&\quad+\int_{Q_\tau}(a(x,t,u,\nabla u)\nabla v_k + a_0(x,t,u,\nabla u)v_k)\,dx\,dt + \langle f,u-v_k\rangle_{Q_\tau}
\end{aligned}$$

and hence

$$\limsup_{n\to\infty}\int_{Q_\tau}\Big(\frac{\partial u_n}{\partial t}-\frac{\partial v_k}{\partial t}\Big)(u_n-v_k)\,dx\,dt \le \varepsilon_k+\varepsilon_k'$$

uniformly in $\tau \in [0,T]$. Since

$$\limsup_{n\to\infty} \int_{Q_\tau} \left(\frac{\partial u_n}{\partial t} - \frac{\partial v_k}{\partial t}\right)(u_n - v_k)\,dx\,dt = \limsup_{n\to\infty} \left[\frac{1}{2}\int_{\Omega} (u_n - v_k)^2\,dx\right]_0^\tau$$
$$= \frac{1}{2}\limsup_{n\to\infty} \int_{\Omega} (u_n(\tau) - v_k(\tau))^2\,dx,$$

we deduce the inequality

$$\limsup_{n\to\infty} \int_{\Omega} (u_n(\tau) - v_k(\tau))^2\,dx \le 2\varepsilon_k + 2\varepsilon'_k,$$

implying that (v_k) is a Cauchy sequence in $C([0,T], L^2(\Omega))$ and that $u \in C([0,T], L^2(\Omega))$. ∎

Corollary 1. *The function u can be used as a testing function, i.e.*

$$\frac{1}{2}\left[\int_{\Omega} (u(t))^2\,dx\right]_0^\tau + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla u\,dx\,dt + \int_{Q_\tau} a_0(x,t,u,\nabla u)u\,dx\,dt$$
$$= \langle f, u\rangle_{Q_\tau}.$$

Proof. As in [15], by using Fatou's lemma we have

$$\begin{aligned}
0 &\le \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\tau} (a(x,t,u_n,\nabla u_n)(\nabla u_n - \nabla v_k) \\
&\quad + a_0(x,t,u_n,\nabla u_n)(u_n - v_k))\,dx\,dt \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial u_n}{\partial t}(v_k - u_n)\,dx\,dt + \lim_{k\to\infty}\lim_{n\to\infty} \langle f_n, u_n - v_k\rangle_{Q_\tau} \\
&\le \frac{1}{2}\lim_{k\to\infty}\limsup_{n\to\infty} \left[\int_{\Omega} -(u_n(t) - v_k(t))^2\,dx\right]_0^\tau \\
&\quad + \lim_{k\to\infty}\limsup_{n\to\infty} \int_{Q_\tau} \frac{\partial v_k}{\partial t}(v_k - u_n)\,dx\,dt \le 0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \int_{Q_\tau} (a(x,t,u_n,\nabla u_n)\nabla u_n + a_0(x,t,u_n,\nabla u_n)u_n)\,dx\,dt$$
$$= \int_{Q_\tau} (a(x,t,u,\nabla u)\nabla u + a_0(x,t,u,\nabla u)u)\,dx\,dt$$

and

$$\frac{1}{2}\lim_{n\to\infty} \|u_n(t) - v_k(t)\|^2_{L^2(\Omega)} \le \varepsilon_k + \varepsilon'_k.$$

Since $v_k(\tau) \to u(\tau)$ in $L^2(\Omega)$, we also have $u_n(\tau) \to u(\tau)$ in $L^2(\Omega)$ and then

$$\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial u_n}{\partial t} u_n\,dx\,dt = \frac{1}{2}\left[\int_{\Omega} (u(t))^2\,dx\right]_0^\tau.$$

Therefore, it is easy to pass to the limit in (10) with $\varphi = u_n$ to get the result. ∎

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