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Criteria for univalence, starlikeness and convexity

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Abstract. Let $\mathcal{A}$ denote the class of all normalized analytic functions f ($f(0) = 0 = f'(0) - 1$) in the open unit disc Δ. For $0 < \lambda \le 1$, define

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda,\ z \in \Delta \right\}$$

and

$$\mathcal{P}(2\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)'' \right| < 2\lambda,\ z \in \Delta \right\}.$$

Recently, the problem of finding the starlikeness of these classes has been considered by Obradović and Ponnusamy, and later by Obradović *et al.* In this paper, the authors consider the problem of finding the order of starlikeness and of convexity of $\mathcal{U}(\lambda)$ and $\mathcal{P}(2\lambda)$, respectively. In particular, for $f \in \mathcal{A}$ with $f''(0) = 0$, we find conditions on λ, $\beta^*(\lambda)$ and $\beta(\lambda)$ so that $\mathcal{U}(\lambda) \subsetneq \mathcal{S}^*(\beta^*(\lambda))$ and $\mathcal{P}(2\lambda) \subsetneq \mathcal{K}(\beta(\lambda))$. Here, $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ ($\beta < 1$) denote the classes of functions in $\mathcal{A}$ that are starlike of order β and convex of order β, respectively. In addition to these results, we also provide a coefficient condition for functions to be in $\mathcal{K}(\beta)$. Finally, we propose a conjecture that each function $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$ is convex at least when $0 < \lambda \le 3 - 2\sqrt{2}$.

1. Introduction and main results. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Several subclasses of univalent functions play a prominent role in the theory of univalent functions [1, 2]. Among them are the class of all convex functions of order β, $\beta < 1$, given by

$$\mathcal{K}(\beta) = \{ f \in \mathcal{A} : \mathrm{Re}(zf''(z)/f'(z) + 1) > \beta,\ z \in \Delta \}$$

and the class of all starlike functions of order β, $\beta < 1$, described by

$$\mathcal{S}^*(\beta) = \{ f \in \mathcal{A} : \mathrm{Re}(zf'(z)/f(z)) > \beta,\ z \in \Delta \}.$$

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The class of all convex functions denoted by $\mathcal{K} := \mathcal{K}(0)$ consists of all $f \in \mathcal{S}$ for which the range $f(\Delta)$ is convex. Similarly, the class of starlike functions denoted by $\mathcal{S}^* := \mathcal{S}^*(0)$ consists of all $f \in \mathcal{S}$ for which $f(\Delta)$ is starlike (with respect to the origin). An important member of the class $\mathcal{S}$ is the Koebe function $k(z) = z/(1-z)^2$ together with its rotations. An interesting subclass of $\mathcal{S}$ containing the Koebe function is the class $\mathcal{U} := \mathcal{U}(1)$, where $\mathcal{U}(\lambda)$ $(0 < \lambda \le 1)$ is defined by

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \lambda,\ z \in \Delta \right\}.$$

This class has been studied by Obradović and Ponnusamy [3] together with the class

$$\mathcal{P}(2\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)}\right)'' \right| < 2\lambda,\ z \in \Delta \right\}.$$

There are many relationships between various subclasses of $\mathcal{S}$. However, the classes $\mathcal{U}(1)$ and its direct generalization $\mathcal{U}(\lambda)$ have not been looked at until recently. According to a result due to Ozaki and Nunokawa [6], we have the inclusion

$$\mathcal{U}(\lambda) \subset \mathcal{S} \quad \text{for } 0 < \lambda \le 1,$$

and from [3], we also have the inclusion $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda)$. In [4], the authors have shown that certain results obtained in [3] also hold if $\mathcal{P}(2\lambda)$ is replaced by $\mathcal{U}(\lambda)$. In this connection, we recall the following result from [4].

Theorem 1.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$, *then* $f \in \mathcal{S}^*$ *for* $0 < \lambda \le \lambda^*$, *where*

$$\lambda^* = \frac{-a + \sqrt{2 - a^2}}{2} \quad \textit{with} \quad a = |f''(0)|/2.$$

This result was originally stated as a conjecture in [3] and was proved in [4]. In this article, we discuss the relationship between $\mathcal{U}(\lambda)$ and $\mathcal{S}^*(\delta)$, as well as between $\mathcal{P}(2\lambda)$ and $\mathcal{K}(\delta)$. As a consequence, we improve certain coefficient results due to Reade, Silverman and Todorov [7].

We now state our first result which gives a condition for functions in $\mathcal{U}(\lambda)$ to be starlike of order $\delta(\lambda)$.

Theorem 1.2. *If* $f \in \mathcal{U}(\lambda)$ *and* $a = |f''(0)|/2 \le 1$, *then* $f \in \mathcal{S}^*(\delta)$ *whenever* $0 < \lambda \le \lambda(\delta)$, *where*

$$\lambda(\delta) = \begin{cases} \dfrac{\sqrt{(1-2\delta)(2-a^2-2\delta)} - a(1-2\delta)}{2(1-\delta)} & \textit{if } 0 \le \delta < \dfrac{1+a}{3+a}, \\[2ex] \dfrac{1-\delta(1+a)}{1+\delta} & \textit{if } \dfrac{1+a}{3+a} \le \delta < \dfrac{1}{1+a}. \end{cases}$$

We observe that if we choose $\delta = 0$ in Theorem 1.2, then Theorem 1.1 follows. Also, we note that

$$\lambda\left(\frac{1-a^2}{2}\right) = \begin{cases} \dfrac{1-a+a^2+a^3}{3-a^2} & \text{if } 0 \le a \le \sqrt{2}-1, \\ \dfrac{a(1-a^2)}{1+a^2} & \text{if } \sqrt{2}-1 \le a < 1. \end{cases}$$

Therefore, Theorem 1.2 is an extension of Theorem 1.1. Further, we believe that the order of starlikeness given above for functions in $\mathcal{U}(\lambda)$ is sharp although at present we do not have a concrete proof. However, from Theorem 1.2, one can obtain a number of new results. For example, if $f \in \mathcal{U}(\lambda)$ and $a = |f''(0)/2| < 1$ then $f \in \mathcal{S}^*(1/2)$ whenever $0 < \lambda \le (1-a)/3$.

COROLLARY 1.3. *If $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$, then $f \in \mathcal{S}^*(\delta)$ whenever*

$$0 < \lambda \le \lambda(\delta) = \begin{cases} \sqrt{\dfrac{1-2\delta}{2(1-\delta)}} & \textit{if } 0 \le \delta \le 1/3, \\ \dfrac{1-\delta}{1+\delta} & \textit{if } 1/3 \le \delta < 1, \end{cases}$$

or equivalently

$$\delta := \delta(\lambda) = \begin{cases} \dfrac{1-\lambda}{1+\lambda} & \textit{if } 0 < \lambda \le 1/2, \\ \dfrac{1-2\lambda^2}{2(1-\lambda^2)} & \textit{if } 1/2 \le \lambda \le 1/\sqrt{2}. \end{cases} \tag{1.1}$$

In particular,

- $f \in \mathcal{U}(\lambda)$, $f''(0) = 0$ *and* $0 < \lambda \le 1/\sqrt{2} \;\Rightarrow\; f \in \mathcal{S}^*$,
- $f \in \mathcal{U}(\lambda)$, $f''(0) = 0$ *and* $0 < \lambda \le 1/3 \;\Rightarrow\; f \in \mathcal{S}^*(1/2)$.

Our next result provides an affirmative answer to the following

PROBLEM 1.4. Find conditions on λ and $\beta(\lambda)$ so that $\mathcal{P}(2\lambda) \subset \mathcal{K}(\beta(\lambda))$.

We recall that $\mathcal{P}(2\lambda) \subset \mathcal{S}^*$ if $0 < \lambda \le (-a + \sqrt{2-a^2})/2$, $a = |f''(0)|/2$.

THEOREM 1.5. *Let $f \in \mathcal{A}$ with $f''(0) = 0$ and suppose that*

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)\left(1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \lambda, \qquad z \in \Delta, \tag{1.2}$$

or equivalently,

$$\left|z^2\left(\frac{z}{f(z)}\right)''\right| < 2\lambda, \qquad z \in \Delta,$$

for some $0<\lambda\leq 1/\sqrt{2}$. *Then* $f\in\mathcal{K}(\beta)$, *where*

$$\beta=\beta(\lambda)=\begin{cases}\dfrac{1+\lambda^2-6\lambda}{1-\lambda^2} & \text{for } 0<\lambda\leq 1/2,\\[2mm] \dfrac{-\lambda(2+3\lambda)}{1-\lambda^2} & \text{for } 1/2<\lambda\leq 1/\sqrt{2}.\end{cases}\tag{1.3}$$

In particular, $\mathcal{P}(2\lambda)\subset\mathcal{K}$ *if* $0<\lambda\leq 3-2\sqrt{2}$.

For our next result, we consider functions f in $\mathcal{A}$ of the form

$$f(z)=\frac{z}{\phi(z)},\tag{1.4}$$

where $\phi(z)=1+\sum_{k=1}^{\infty}b_kz^k$ has no zeros in Δ. Functions of this form have been studied, for example, in [7, 3]. In [3], Obradović and Ponnusamy obtained various coefficient conditions in terms of b_k's for the corresponding f of the above form to be univalent, strongly starlike etc. In particular, the following results are known:

Theorem 1.6. *A function of the form* (1.4) *is in* $\mathcal{K}$ *if any one of the following conditions holds*:

(i) *there exist* $p,q>0$ *with* $1/p+1/q\leq 1$ *such that*

$$(2p+1)|b_1|+\max\left\{\sum_{k=1}^{\infty}(2kp+1)|b_k|,\sum_{k=1}^{\infty}(k-1)(kq+1)|b_k|\right\}\leq 1.$$

(ii) $\displaystyle\sum_{k=2}^{\infty}(k-1)k|b_k|\leq 2\lambda,\quad \textit{where}\quad \lambda=\frac{7+|b_1|-\sqrt{33+30|b_1|+|b_1|^2}}{8}.$

Theorem 1.6(i) is due to [7] while Theorem 1.6(ii) has been obtained recently by Obradović *et al.* [4]. Our next result improves Theorem 1.6.

Theorem 1.7. *Let* $0<\lambda\leq 1/\sqrt{2}$. *If* $f\in\mathcal{A}$ *is of the form* (1.4) *and satisfies the coefficient condition*

$$\sum_{k=2}^{\infty}(k-1)k|b_k|\leq 2\lambda,\tag{1.5}$$

then $f\in\mathcal{K}(\beta)$, *where* $\beta=\beta(\lambda)$ *is defined by* (1.3).

A comparison of the λ-values of Theorems 1.7 and 1.6 shows that Theorem 1.7 improves Theorem 1.6. Indeed, for the case $b_1=0$, it suffices to note that $(7-\sqrt{33})/8<3-2\sqrt{2}$.

It would be an interesting problem to find the largest value of λ so that (1.5) implies that f defined by (1.4) is convex in Δ.

We end this section with a result which provides a sufficient condition for a function f to be starlike or univalent in Δ.

THEOREM 1.8. *Let $0 \le \mu \le 1$, $0 < \lambda \le 1$ and $f \in \mathcal{A}$.*

(i) *If f satisfies*

$$\left|1 + \frac{1}{\mu+1}\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{\log(1+\lambda)}{\mu+1}, \qquad z \in \Delta, \tag{1.6}$$

then

$$\left|\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1\right| < \lambda, \qquad z \in \Delta.$$

(ii) *If f is such that $f''(0) = 0$ and satisfies the condition*

$$\left|1 + \frac{1}{\mu+1}\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{2\log(1+\lambda)}{\mu+1}, \qquad z \in \Delta,$$

then

$$\left|\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1\right| < \lambda, \qquad z \in \Delta.$$

In [5, Theorem 2], Theorem 1.8(i) was proved with $\log(1+\lambda)$ in (1.6) replaced by $\lambda/(\lambda+1)$. Note that $\log(1+\lambda) > \lambda/(\lambda+1)$ for all $\lambda \in (0,1]$, so Theorem 1.8 improves the result of Obradović and Tuneski [5]. In particular, for each $0 < \lambda \le 1$ and $f \in \mathcal{A}$, one has

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \log(1+\lambda) \;\Rightarrow\; \left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda, \qquad z \in \Delta,$$

$$\left|\frac{zf''(z)}{f'(z)} + 2\left(1 - \frac{zf'(z)}{f(z)}\right)\right| < \log(1+\lambda) \;\Rightarrow\; \left|\left(\frac{z}{f(z)}\right)^{2} f'(z) - 1\right| < \lambda, \qquad z \in \Delta.$$

We observe that if $f''(0) = 0$ then the last two implications hold with $2\log(1+\lambda)$ in place of $\log(1+\lambda)$.

2. Proofs of the main theorems. For the proof of Theorem 1.5, we need a special case of the following lemma. However, from the proof of Theorem 1.5, we note that Lemma 2.1 may be used to state a more general result.

LEMMA 2.1. *Let $0 < \lambda < 1$, $\alpha > -2$ and let $g \in \mathcal{H}$, the class of all analytic functions in the unit disc Δ, satisfy the condition $g(z) \prec 1 + \lambda z$ for $z \in \Delta$ with $g(0) = 1$. Suppose that $\operatorname{Re}\phi(z) \ge \delta$ in Δ. If $p \in \mathcal{H}$, $p(0) = 1$ and*

$$|g(z)(\beta + (1-\beta)p(z) + 1 - \alpha - 2\phi(z)) - \alpha| < \lambda(\alpha+2), \qquad z \in \Delta, \tag{2.1}$$

where

$$\beta = \begin{cases} 2\delta + 2\alpha - \dfrac{1+\lambda}{1-\lambda} & \textit{for } -2 < \alpha \le 2\lambda \textit{ with } 0 < \lambda < 1, \\[2mm] 2\delta - 1 + \dfrac{2(\alpha-\lambda)}{1+\lambda} & \textit{for } \alpha \ge \dfrac{2\lambda}{1-2\lambda} \textit{ with } 0 < \lambda < 1/2, \\[2mm] 2\delta + \dfrac{3\alpha-2}{2} - \dfrac{(\alpha+2)^2\lambda^2}{2\alpha(1-\lambda^2)} & \\ \qquad\qquad \textit{for } 2\lambda \le \alpha \le \dfrac{2\lambda}{1-2\lambda} \textit{ with } 0 < \lambda < 1/2, \end{cases} \tag{2.2}$$

then $\operatorname{Re} p(z) > 0$ *for* $z \in \Delta$.

Proof. In order to prove our result we notice that, in view of (2.1), it suffices to find $\inf_{|z|<1,\, \eta\in\mathbb{R}} \operatorname{Re} Q(z)$, where

$$Q(z) = \frac{\alpha + \lambda(\alpha+2)z}{1+\lambda e^{i\eta} z}, \qquad z \in \Delta. \tag{2.3}$$

From (2.3) one can easily verify that

$$\left| Q(z) - \frac{\alpha - r^2\lambda^2(\alpha+2)e^{-i\eta}}{1-r^2\lambda^2} \right| \le \frac{r\lambda|\alpha - (\alpha+2)e^{i\eta}|}{1-r^2\lambda^2} \qquad \text{for } |z| \le r$$

so that for $z \in \Delta$ we have

$$\left| Q(z) - \frac{\alpha - \lambda^2(\alpha+2)e^{-i\eta}}{1-\lambda^2} \right| \le \frac{\lambda|\alpha - (\alpha+2)e^{i\eta}|}{1-\lambda^2}.$$

Therefore,

$$\operatorname{Re} Q(z) \ge \frac{\alpha - \lambda^2(\alpha+2)\cos\eta - \lambda\sqrt{\alpha^2 - 2\alpha(\alpha+2)\cos\eta + (\alpha+2)^2}}{1-\lambda^2}.$$

Define $\psi(\eta) = \alpha - \lambda^2(\alpha+2)\cos\eta - \lambda\sqrt{\alpha^2 - 2\alpha(\alpha+2)\cos\eta + (\alpha+2)^2}$.

Case (i): If $-2 < \alpha \le 0$, then it is easy to see that ψ is an increasing function on $[0,\pi]$ and therefore

$$\psi(\eta) \ge \psi(0) = (1+\lambda)(\alpha - (\alpha+2)\lambda),$$

which means that

$$\operatorname{Re} Q(z) \ge \frac{\alpha - (\alpha+2)\lambda}{1-\lambda}, \qquad z \in \Delta.$$

Case (ii): If $\alpha > 0$ and $0 < \lambda \le \alpha/2(\alpha+1)$, then we observe that ψ is a decreasing function on $[0,\pi]$, which gives

$$\psi(\eta) \ge \psi(\pi) = (\alpha - (\alpha+2)\lambda)(1-\lambda)$$

and therefore,

$$\operatorname{Re} Q(z) \ge \frac{\alpha - (\alpha+2)\lambda}{1+\lambda}, \qquad z \in \Delta.$$

CASE (iii): Similarly, we find that

$$\psi(\eta) \geq \psi(\eta_1) = \frac{\alpha^2 - (\alpha+2)^2\lambda^2 - \lambda^2\alpha^2}{2\alpha}$$

whenever $\lambda^2(\alpha^2 - 2\alpha(\alpha+2)\cos\eta_1 + (\alpha+2)^2) = \alpha^2$. This shows that

$$\operatorname{Re} Q(z) \geq \frac{\alpha^2 - (\alpha+2)^2\lambda^2 - \lambda^2\alpha^2}{2\alpha(1-\lambda^2)} \quad \text{if} \quad \frac{\alpha}{2(\alpha+1)} \leq \lambda \leq \frac{\alpha}{2}.$$

CASE (iv): If $\lambda \geq \alpha/2$, then we can easily see that ψ is an increasing function on $[0,\pi]$ so that

$$\psi(\eta) \geq \psi(0) = (\alpha - (\alpha+2)\lambda)(1+\lambda),$$

which implies that

$$\operatorname{Re} Q(z) \geq \frac{\alpha - (\alpha+2)\lambda}{1-\lambda}, \qquad z \in \Delta.$$

Finally, it follows that $\operatorname{Re} Q(z) > \beta_0(\alpha,\lambda)$, $z \in \Delta$, where

$$\beta_0(\alpha,\lambda) = \begin{cases} \dfrac{\alpha - (\alpha+2)\lambda}{1-\lambda} & \text{for } -2 < \alpha \leq 2\lambda \text{ with } 0 < \lambda < 1, \\ \dfrac{\alpha - (\alpha+2)\lambda}{1+\lambda} & \text{for } \alpha \geq \dfrac{2\lambda}{1-2\lambda} \text{ with } 0 < \lambda < 1/2, \\ \dfrac{\alpha^2(1-\lambda^2) - (\alpha+2)^2\lambda^2}{2\alpha(1-\lambda^2)} & \text{for } 2\lambda \leq \alpha \leq \dfrac{2\lambda}{1-2\lambda} \text{ with } 0 < \lambda < 1/2. \end{cases}$$

A simple computation shows that $\beta = \beta_0(\alpha,\lambda) + 2\delta - (1-\alpha)$, where β is given by (2.2). Therefore, $\operatorname{Re} Q(z) > \beta_0(\alpha,\lambda)$ is equivalent to

$$\operatorname{Re} p(z) > \frac{\beta_0(\alpha,\lambda) + 2\delta - (1-\alpha) - \beta}{1-\beta} = 0, \qquad z \in \Delta,$$

and the desired conclusion follows. ■

Proof of Theorem 1.2. Suppose that $f \in \mathcal{U}(\lambda)$. Then we can write

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + \lambda w(z), \tag{2.4}$$

where w is the Schwarz function with an additional condition $w'(0) = 0$. We observe from the Schwarz lemma that $|w(z)| \leq |z|^2$. As usual it follows that

$$\frac{z}{f(z)} = 1 - a_2 z - \lambda \int_0^1 \frac{w(tz)}{t^2}\,dt, \qquad a_2 = \frac{f''(0)}{2!},$$

and therefore, by (2.4), we see that

$$\frac{zf'(z)}{f(z)} = \frac{1 + \lambda w(z)}{1 - a_2 z - \lambda \int_0^1 \frac{w(tz)}{t^2}\,dt}.$$

Thus,

$$\frac{1}{1-\delta}\left(\frac{zf'(z)}{f(z)}-\delta\right)=\frac{1+\dfrac{\lambda w(z)}{1-\delta}+\dfrac{\delta}{1-\delta}\left[a_2z+\lambda\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt\right]}{1-a_2z-\lambda\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt}.$$

Now, $\mathrm{Re}(zf'(z)/f(z))>\delta$ is equivalent to the condition

$$\frac{1+\dfrac{\lambda w(z)}{1-\delta}+\dfrac{\delta}{1-\delta}\left[a_2z+\lambda\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt\right]}{1-a_2z-\lambda\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt}\neq -iT \quad \text{for all } T\in\mathbb{R} \text{ and } z\in\Delta,$$

which is equivalent to

$$\lambda\left[\frac{w(z)+(\delta-i(1-\delta)T)\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt}{(1-\delta)(1+iT)+a_2z(\delta-iT(1-\delta))}\right]\neq -1 \quad \text{for all } T\in\mathbb{R} \text{ and } z\in\Delta.$$

If we let

$$M=\sup_{z\in\Delta,\,w\in\mathcal{B},\,T\in\mathbb{R}}\left|\frac{w(z)+(\delta-i(1-\delta)T)\displaystyle\int_0^1\frac{w(tz)}{t^2}\,dt}{(1-\delta)(1+iT)+a_2z(\delta-iT(1-\delta))}\right|,$$

then, in view of the rotation invariance of the space $\mathcal{B}$, we obtain

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\delta \quad \text{if } \lambda M\leq 1.$$

This observation shows that it suffices to find M. First we notice that

$$M\leq\sup_{T\in\mathbb{R}}\left\{\frac{1+\sqrt{\delta^2+(1-\delta)^2T^2}}{|(1-\delta)\sqrt{1+T^2}-a\sqrt{\delta^2+(1-\delta)^2T^2}|}\right\},$$

where, for convenience, we have set $a=|a_2|$. Define $\phi:[0,\infty)\to\mathbb{R}$ by

$$\phi(x)=\frac{1+\sqrt{\delta^2+(1-\delta)^2x}}{(1-\delta)\sqrt{1+x}-a\sqrt{\delta^2+(1-\delta)^2x}}. \tag{2.5}$$

Observe that the denominator in the expression of $\phi(x)$ is positive for all $x\in[0,\infty)$ provided $0\leq\delta<1/(1+a)$ and $0\leq a\leq 1$. Further, it is a simple exercise to see that

$$\phi'(x)=\frac{(1-\delta)N(x)}{2[(1-\delta)\sqrt{1+x}-a\sqrt{\delta^2+(1-\delta)^2x}]^2\sqrt{1+x}\,\sqrt{\delta^2+(1-\delta)^2x}},$$

where $N(x)=1-2\delta-\sqrt{\delta^2+(1-\delta)^2x}+a(1-\delta)\sqrt{1+x}$.

Case (I): Let $a=0$. Then

$$\phi'(x)=\frac{1-2\delta-\sqrt{\delta^2+(1-\delta)^2x}}{2(1-\delta)\sqrt{(1+x)^3}\,\sqrt{\delta^2+(1-\delta^2)^2x}}.$$

For $\delta \geq 1/3$, we note that $\phi'(x) \leq 0$ for all $x \geq 0$ and therefore

$$\phi(x) \leq \phi(0) = \frac{1+\delta}{1-\delta}.$$

If $0 \leq \delta < 1/3$, then $x_0 = (1-3\delta)/(1-\delta)$ is the only critical point and $\phi''(x_0) < 0$. This shows that for $0 < \delta < 1/3$, ϕ attains its maximum value at x_0 so that

$$\phi(x_0) = \sqrt{\frac{2(1-\delta)}{1-2\delta}}.$$

This gives essentially a direct proof for Corollary 1.3.

Case (II): Now we consider the case $a \neq 0$. In this case, we have several subcases. Firstly, we let $1/2 \leq \delta < 1/(1+a)$. It follows that

$$N(x) \leq 1 - 2\delta \leq 0,$$

because $a(1-\delta)\sqrt{1+x} \leq \sqrt{\delta^2 + (1-\delta)^2 x}$. Indeed, the last inequality follows from the fact that $a \leq 1$,

$$0 \geq (1-\delta)^2 - \delta^2 = 1 - 2\delta \geq a^2(1-\delta)^2 - \delta^2$$

and

$$x(1-\delta)^2(1-a^2) \geq a^2(1-\delta)^2 - \delta^2.$$

Thus, $\phi'(x) \leq 0$ for all $x \geq 0$ whenever $1/2 \leq \delta < 1/(1+a)$. Next, we consider the case

$$\frac{1+a}{3+a} \leq \delta < 1/2.$$

In this case, it suffices to compute

$$N'(x) = -\frac{(1-\delta)^2}{2\sqrt{\delta^2 + (1-\delta)^2 x}} + \frac{a(1-\delta)}{2\sqrt{1+x}}$$

and note that $N'(x) \leq 0$ for $x \geq 0$ if and only if

$$x(1-\delta)^2(1-a^2) \geq a^2\delta^2 - (1-\delta)^2.$$

Since $\delta < 1/2$ implies that $0 > 2\delta - 1 = \delta^2 - (1-\delta)^2 \geq a^2\delta^2 - (1-\delta)^2$, the function $N(x)$ is decreasing for $x \geq 0$. Therefore, for $\frac{1+a}{3+a} \leq \delta < 1/2$, we have

$$N(x) \leq N(0) = 1 - 2\delta - \delta + a(1-\delta) \leq 0 \quad \text{ for } x \geq 0.$$

The above observation shows that $\phi(x)$ defined by (2.5) is a decreasing function on $[0, \infty)$ whenever $\frac{1+a}{3+a} \leq \delta < \frac{1}{1+a}$. In particular,

$$\phi(x) \leq \phi(0) = \frac{1+\delta}{1-\delta-a\delta} \quad \text{for} \quad \frac{1+a}{3+a} \leq \delta < \frac{1}{1+a}.$$

Case (III): Assume $a \neq 0$ and $0 \leq \delta < \frac{1+a}{3+a}$. We make the substitution

$$t = \frac{1}{\sqrt{\delta^2 + (1-\delta)^2 x}}$$

and note that $\sup_{x\in[0,\infty)} \phi(x) = \sup_{t\in(0,1/\delta]} \psi(t)$, where $\phi(x)$ becomes

$$\psi(t) = \frac{1+t}{\sqrt{1+(1-2\delta)t^2} - a}$$

with the above substitution. Now we compute

$$\psi'(t) = \frac{R(t)}{[\sqrt{1+(1-2\delta)t^2} - a]^2\sqrt{1+(1-2\delta)t^2}},$$

where $R(t) = 1-(1-2\delta)t - a\sqrt{1+(1-2\delta)t^2}$. Since $R(t)$ decreases,

$$R(0) = 1-a \geq 0 > R(1/\delta) = \frac{3+a}{\delta}\left[\delta - \frac{1+a}{3+a}\right],$$

and $R(t) \neq 0$ for $t > 1/(1-2\delta)$, we get the estimate

$$M \leq \{\psi(t) : 0 \leq t \leq 1/(1-2\delta),\ R(t) = 0\} = \psi(s),$$

where

$$s = \frac{2\delta - 1 + a\sqrt{(1-2\delta)(2-a^2-2\delta)}}{(1-2\delta)(a^2-1+2\delta)}.$$

A simple calculation shows that

$$\frac{1}{\psi(s)} = \frac{\sqrt{(1-2\delta)(2-a^2-2\delta)} - a(1-2\delta)}{2(1-\delta)},$$

a desired result. ■

Recall the following lemma from [3] which is required for proving Theorem 1.7.

LEMMA 2.2. *Let $\phi(z) = 1+\sum_{n=1}^{\infty} b_n z^n$ be a nonvanishing analytic function in Δ and $f(z) = z/\phi(z)$ and $0 < \lambda \leq 1$. If any one of the following coefficient conditions holds*:

(i) $\displaystyle\sum_{n=2}^{\infty}(n-1)|b_n| \leq \lambda$,

(ii) $\displaystyle\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2\lambda$,

then $f \in \mathcal{U}(\lambda)$.

Proof of Theorem 1.5. Assume that $f \in \mathcal{A}$ with $f''(0) = 0$ and satisfies the condition (1.2), i.e.

$$(2.6) \qquad \left|2\left(\frac{z}{f(z)}\right)^2 f'(z)\left(1+\frac{zf''(z)}{2f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \equiv \left|z^2\left(\frac{z}{f(z)}\right)''\right| < 2\lambda, \quad z \in \Delta.$$

We know that (2.6) implies $f \in \mathcal{U}(\lambda)$, which in turn, by Lemma 2.1, gives that $f \in \mathcal{S}^*(\delta)$, where δ is as in Corollary 1.3. Now, the proof may be completed by applying Lemma 2.1. To do this, we let

$$g(z)=\left(\frac{z}{f(z)}\right)^2 f'(z),\qquad \phi(z)=\frac{zf'(z)}{f(z)},\qquad p(z)=\left(\frac{zf''(z)}{f'(z)}+1-\beta\right)\frac{1}{1-\beta}.$$

Then (2.6) is equivalent to $|g(z)(\beta+(1-\beta)p(z)+1-2\phi(z))|<\lambda$, which is the same as (2.1) with $\alpha=0$. By Lemma 2.1, we have $\operatorname{Re} p(z)>0$ for $z\in\Delta$ with

$$\beta=2\delta-1-\frac{2\lambda}{1-\lambda}.$$

Substituting the values of δ from (1.1) we get the desired conclusion. ∎

Proof of Theorem 1.7. Let $f\in\mathcal{A}$ be of the form

$$f(z)=\frac{z}{1+\sum_{k=1}^{\infty} b_k z^k},$$

where the denominator is nonvanishing on Δ. Then a simple calculation shows that

$$\begin{aligned}2\left(\frac{z}{f(z)}\right)^2 f'(z)\left[1+\frac{1}{2}\,\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]&=-z\left\{z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right\}'\\&=-\sum_{n=2}^{\infty} n(n-1)b_n z^n\end{aligned}$$

and therefore, the given coefficient condition (1.5) implies that

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)\left(1+\frac{1}{2}\,\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<\lambda,\qquad z\in\Delta. \tag{2.7}$$

Further, by Lemma 2.2(ii) and (2.7), the coefficient condition (1.5) implies that $f\in\mathcal{U}(\lambda)$. The desired conclusion follows from Theorem 1.5. ∎

Proof of Theorem 1.8. Define

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)=p(z)=1+a_2(1-\mu)z+\frac{(2-\mu)(2a_3-(\mu+1)a_2^2)z^2}{2}+\cdots.$$

Then p is analytic in Δ, $p(0)=1$ and $p(z)\neq 0$ in Δ. Logarithmic derivative of the last equation shows that

$$(\mu+1)\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)},\qquad z\in\Delta,$$

and therefore, the given condition (1.6) is equivalent to

$$\left|\frac{zp'(z)}{p(z)}\right|<\log(1+\lambda),\qquad z\in\Delta.$$

We can write

$$\frac{zp'(z)}{p(z)}=[\log(1+\lambda)]\,w(z),$$

where $w \in \mathcal{B} := \{w \in \mathcal{H} : w(0) = 0,\ |w(z)| < 1 \text{ for } z \in \Delta\}$. Therefore

$$\int_0^z \frac{p'(t)}{p(t)}\,dt = \log(1+\lambda)\int_0^z \frac{w(t)}{t}\,dt, \qquad z \in \Delta,$$

from which we get

$$\log p(z) = \log(1+\lambda)\int_0^1 \frac{w(tz)}{t}\,dt$$

so that

$$p(z) = \exp\left[\log(1+\lambda)\int_0^1 \frac{w(tz)}{t}\,dt\right].$$

Thus,

$$|p(z)-1| \le \exp\left(\log(1+\lambda)\left|\int_0^1 \frac{w(tz)}{t}\,dt\right|\right) - 1 < \lambda, \qquad z \in \Delta.$$

The desired conclusion follows.

For the proof of (ii), because $f''(0) = 0$, it suffices to observe that $w \in \mathcal{B}$ with $w'(0) = 0$ so that, by the Schwarz lemma, $|w(z)| \le |z|^2$ for $z \in \Delta$. The desired conclusion follows if we apply this inequality to the last inequality. ■

3. Two conjectures. The results of this paper (e.g. Theorem 1.5) motivate the following

Conjecture 1. *If $0 < \lambda \le 3 - 2\sqrt{2}$ then each $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$ is convex in Δ.*

We recall that

- $f \in \mathcal{U}(\lambda)$, $f''(0) = 0$ and $0 < \lambda \le 1/3 \ \Rightarrow\ f \in \mathcal{S}^*(1/2)$,
- $f \in \mathcal{P}(2\lambda)$, $f''(0) = 0$ and $0 < \lambda \le 3 - 2\sqrt{2} \ \Rightarrow\ f \in \mathcal{K}$.

We also observe $1/3 > 3 - 2\sqrt{2}$ which is expected, since $\mathcal{K} \subsetneq \mathcal{S}^*(1/2)$ and $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda)$. Further, the method of proof of Theorem 1.2 suggests the following which we are unable to settle at present.

Conjecture 2. *The $\lambda(\delta)$ given in Theorem 1.2 is sharp for $f \in \mathcal{U}(\lambda)$ to be starlike of order δ.*

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