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# On locally biholomorphic mappings from multi-connected onto simply connected domains

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**Abstract.** We continue E. Ligocka's investigations concerning the existence of $m$-valent locally biholomorphic mappings from multi-connected onto simply connected domains. We decrease the constant $m$, and also give the minimum of $m$ in the case of mappings from a wide class of domains onto the complex plane $\mathbb{C}$.

**1. Introduction.** The Riemann theorem guarantees that any two simply connected domains of the plane $\mathbb{C}$, different from $\mathbb{C}$, are biholomorphically equivalent. In view of the Poincaré theorem (see e.g. [Go, Chap. VI, §1]) every simply connected domain $D \subsetneq \mathbb{C}$ can be mapped locally biholomorphically onto every multi-connected domain $G$ with more than two boundary points, but usually in an infinitely valent way. On the other hand, if $G$ is a $k$-connected domain without isolated boundary points and $D \subsetneq \mathbb{C}$ is a simply connected domain, then the Grunsky theorem (see e.g. [Go, Chap. VI, §5]) gives the existence of a mapping $f$ from $G$ onto $D$, which is holomorphic, $k$-valent, but not locally biholomorphic; in this case $k$-valence of $f$ means that for every $w \in D$ the equation $f(z) = w$ has in $G$ exactly $k$ solutions (counting multiplicity). Chapter VI of [Go] is, in particular, devoted to the study of similar problems.

E. Ligocka [Li] has found a class of multi-connected domains which can be mapped onto the unit disc $B(0,1)$ locally biholomorphically and $m$-valently, where $m \le 24$ (in our paper, similarly to [Li], a mapping $f$ from $X$ onto $Y$ is called *$m$-valent*, $m \in \mathbb{N}$, if for each $y \in Y$ the set $\{f^{-1}(y)\}$ has no more than $m$ elements).

E. Ligocka has also proved that every finitely connected domain of $\mathbb{C}$, not biholomorphic to $\mathbb{C} \setminus \{0\}$, can be mapped onto $\mathbb{C}$ locally biholomorphically and $m$-valently, for some $m \in \mathbb{N}$.

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In the present paper we prove that $m = 3$ and $m = 5$ in the case of locally biholomorphic mappings from domains of a much wider class onto $\mathbb{C}$ and onto $B(0,1)$, respectively. We also show that in the first case the constant $m = 3$ is best possible.

**2. Main results.** We will consider domains with an isolated boundary fragment.

Definition. We shall say that a domain $D \subset \mathbb{C}$ has an *isolated boundary fragment* if one of the following conditions holds:

(I) There exists a continuum $K \subset \partial D$, different from a point, and an open set $U$ such that $K \subset U$ and $(\partial D \setminus K) \cap U = \emptyset$.

(II) There exists a Jordan arc $\Gamma \subset \partial D$ with distinct ends $\xi, \eta$ and an open disc $B$ such that $\xi, \eta \in \partial B$, $\Gamma \setminus \{\xi, \eta\} \subset B$ and $(\partial D \setminus \Gamma) \cap B = \emptyset$.

(III) There exists a point $a \in \partial D$ and an open disc $B(a)$ centred at $a$ such that $(B(a) \setminus \{a\}) \cap \partial D = \emptyset$ ($a$ is an isolated point of the boundary $\partial D$).

We will prove the following theorem:

Theorem 1. *If a domain $D \subset \mathbb{C}$, $D \neq \mathbb{C} \setminus \{a\}$, has an isolated boundary fragment, then there exists a 3-valent locally biholomorphic mapping from D onto $\mathbb{C}$. The constant 3 is best possible.*

In the proof we will consider three cases, according to the cases in the above definition. In the case of a domain with an isolated boundary fragment of type (III) we use the following lemma.

Lemma. *Let $D \subset \mathbb{C}$ be a domain, infinitely connected or $k$-connected, $k > 2$, such that $\partial D = E \cup \infty$, where $E$ is a bounded set. Then there exist two points of $E$ such that $E$ lies on one side of the straight line through those points.*

*Proof.* Let $K$ be the closed convex hull of $E$ and let $l_z$ be a support straight line to the set $K$ at the point $z \in \partial K$.

First we will show that there exists $z \in \partial K$ such that $l_z \cap K \neq \{z\}$. Suppose, on the contrary, that $l_z \cap K = \{z\}$ for every $z \in \partial K$. Fix $z \in \partial K$ and denote by $l_n, n \in \mathbb{N}$, the straight lines parallel to $l_z$, at distance $1/n$ from $l_z$ and lying on the same side of $l_z$ as $K$. Taking, for every $n \in \mathbb{N}$, a point $z_n \in E$ lying between $l_n$ and $l_z$, we obtain a sequence $(z_n)$ which has a subsequence convergent to a point $a \in E$, because $E$ is a compact set. Of course, $a \in l_z$. Since $l_z \cap K = \{z\}$, we have $a = z$. Thus, $z \in E$ and so $\partial K \subset E$, by the arbitrariness of $z$.

Since $\partial K$ is a curve which bounds a convex compact set, we deduce that $\partial D = \partial K \cup \infty$. Thus, $D$ is a two-connected domain, which contradicts the assumptions of the lemma.

Now, let $z \in \partial K$ be such that $l_z \cap K \neq \{z\}$. Then in $l_z \cap K$, there exist points different from $z$, lying on one side of $z$. Among them there exists a point $z'$ most distant from $z$. Similarly, we choose a point $z''$ most distant from $z$ among all points lying on the other side of $z$ (if such points do not exist, then we take $z'' = z$). From the choice of $z', z''$ and from the properties of the closed convex hull $K$ of $E$ it follows that $z', z'' \in E$.

Hence, the assertion of the lemma holds. ∎

Now, we give the proof of Theorem 1.

*Proof.* 1. First, assume that $D$ has an isolated boundary fragment of type (III). Then, by a homography, we map the isolated point $a$ of $\partial D$ onto $\infty$, and the rest of $\partial D$ onto a compact set $E$.

In view of the lemma there exist two different points in $E$ such that $E$ lies on one side of the straight line going through those points. Take two points of the intersection of $E$ and this line such that their distance is maximum possible, and send them onto $-1, 1$ by a linear mapping. We can assume that then the set $E$ transforms onto a compact set included in the upper half-plane $P^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. As a result, $D$ is mapped biholomorphically onto a domain $D_1$ which includes the lower half-plane $P^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, the rays $(-\infty, -1), (1, \infty)$, but does not include the points $-1, 1$.

The polynomial $Q(z) = z^3 - 3z$ is locally biholomorphic in $D_1$. We will show that $Q(D_1) = \mathbb{C}$.

Indeed, since $S = (-\infty, -1) \cup (1, \infty) \subset D_1$ and $Q(S) = \mathbb{R}$, it is sufficient to observe that the equation $Q(z) = w$, $w \in \mathbb{C} \setminus \mathbb{R}$, has a solution $z \in P^-$. This follows from the fact that the roots $\zeta_1$, $\zeta_2$, $\zeta_3$ of this equation belong to $\mathbb{C} \setminus \mathbb{R}$ and $\zeta_1 + \zeta_2 + \zeta_3 = 0$.

2. Now consider $D$ with an isolated boundary fragment of type (II). We can assume that $(\partial D \setminus \Gamma) \cap \partial B = \emptyset$.

Let $G$ be a subdomain of $D$ which is included between the Jordan arc $\Gamma$ and an open arc $\gamma$ of the circle $\partial B$, joining the ends of $\Gamma$ (see the definition).

Denote by $K$ the connected component of the boundary $\partial D$ such that $\Gamma \subset K$ and by $\widehat{D}$ the simply connected domain such that $D \subset \widehat{D}$, $\partial \widehat{D} = K$. In view of the Riemann theorem there exists a biholomorphic mapping from $\widehat{D}$ onto the half-plane $P^+$. Denote by $D_*, G_*, \gamma_*$ the images of the sets $D, G, \gamma$, respectively under the above mapping.

Since $\partial G = \gamma \cup \Gamma$ is a Jordan curve, all points of $\partial G$ are attainable from $G$. All points of $\gamma_*$ are attainable from $G_*$, and $\gamma_*$ is the homeomorphic image of $\gamma$. Since the set of points which correspond to points of $\partial G_*$ attainable from $G_*$ is everywhere dense in $\gamma \cup \Gamma$ (see [Go, Chap. II, §3]), there exist at least two points $\zeta_1, \zeta_2 \in \partial G_* \setminus \gamma_*$ attainable from $G_*$. From this it follows that $\zeta_1, \zeta_2 \in \mathbb{R}$.

Let $l_1, l_2 \subset G_*$ be the two Jordan arcs which end in $\zeta_1$, $\zeta_2$, and let $\gamma_1, \gamma_2$ be the preimages in $G$ of $l_1, l_2$, respectively. Denote by $\gamma_3 \subset G$ a Jordan arc which joins the arcs $\gamma_1, \gamma_2$, and by $l_3 \subset G_*$ the image of $\gamma_3$. Of course, we can choose $\gamma_3$ in such a way that the set $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \Gamma$ bounds a simply connected domain $G' \subset G$ and $\partial G'$ is a Jordan curve. Then the closure of $G'$ is homeomorphic to the closure of the domain $G'_* \subset G_*$, bounded by the curve $l_1 \cup l_2 \cup l_3 \cup \mathbb{R}$.

Therefore, in $G'_*$ there exists a semi-disc with centre $z_0 \in \mathbb{R}$ and with diameter included in $\mathbb{R}$.

Applying the function $(z_0 - z)^{-1}$ we map $D_*$ biholomorphically onto a domain $D_1 \subset P^+$ such that $\partial D_1 \setminus \mathbb{R}$ is included in a disc centred at zero, of a sufficiently large radius.

By a translation we transform the upper half-plane $P^+$ onto itself in such a way that $\partial D_1 \setminus \mathbb{R}$ is included in the sector $\{z : \operatorname{Arg} z \in (0, \pi/4)\}$.

To this translation image of $D_1$ we now apply the mapping $1 + e^{i\pi\alpha} z^{1-\alpha}$, where $\alpha \in (0, 1/4)$ and $z^{1-\alpha}$ is the main branch of the power. Then $D_1$ maps onto a domain $D_2 \subset \{z : \operatorname{Arg}(z-1) \in (\pi\alpha, \pi)\}$, whose boundary $\partial D_2$ consists of rays $(-\infty, 1], \{1 + te^{i\pi\alpha} : t \geq 0\}$ and maybe of other components lying in the sector $\{z : \operatorname{Arg}(z-1) \in (\pi\alpha, \frac{\pi}{4}(1+3\alpha))\}$.

Next, by the mapping $\varphi(z) = z^2$, we transform $D_2$ biholomorphically onto $D_3 = \varphi(D_2)$. The boundary $\partial D_3$ consists of the ray $[0, \infty)$, of the arc $L = \{(1 + te^{i\pi\alpha})^2 : t \geq 0\}$ and maybe of other components, which are included in $P^+$. Observe that $L$ lies in the sector $\{z : \operatorname{Arg} z \in (0, 2\pi\alpha)\}$ and is asymptotic to the ray $\{te^{i2\pi\alpha} : t \geq 0\}$ as $t \to \infty$.

Let $\psi(z) = (z-1)^{1+\alpha/2} = \exp\{(1+\alpha/2)[\ln|z-1| + i \operatorname{Arg}(z-1)]\}$, where $\operatorname{Arg}(z-1)$ varies in $(0, 2\pi)$, and let $D_4$ be the biholomorphic image of $D_3$ under $\psi$.

Rotation through $-\pi\alpha/2$ sends $D_4$ onto a domain $D_5$ such that $D_5 \supset P^-$ and $\partial D_5 \supset [-1, 0] \cup \{z : \operatorname{Arg} z = \pi\alpha/2\} \cup L'$, where $L'$ is the image of the arc $L$ under the mapping $e^{-i\pi\alpha/2}\psi(z)$; the boundary $\partial D_5$ may also include some points of $P^+$. Therefore, applying successively the mappings $1 + 2z$ and $Q(z)$ to $D_5$, we obtain a locally biholomorphic and not more than 3-valent mapping from $D_5$ onto $\mathbb{C}$, because $Q(P^-) \supset \mathbb{C} \setminus \mathbb{R}$ and $Q((-\infty, -1) \cup (1, \infty)) = \mathbb{R}$, as shown in the first part of the proof.

As a result, we have a locally biholomorphic 3-valent mapping from $D$ onto $\mathbb{C}$.

3. Now let $D$ have an isolated boundary fragment of type (I). Then $K$ is the boundary of a simply connected domain $D_0$ such that $D \subset D_0$ and $\partial D \setminus K \subset K_0$, where $K_0 \subset D_0$ is a compact set in $\overline{\mathbb{C}}$. Map $D_0$ biholomorphically onto the upper half-plane $P^+$. Then $K_0$ transforms onto a compact $K^0 \subset P^+$. The image of $D$ under this mapping will be denoted by $D_1$.

Now it is sufficient to transform $D_1$ as in the second part of the proof. Consequently, we obtain a locally biholomorphic and 3-valent mapping $f$ from $D$ onto $\mathbb{C}$.

4. Finally, we will prove that in our class of domains with an isolated boundary fragment, there exist domains which cannot be mapped onto $\mathbb{C}$ locally biholomorphically and 2-valently.

Set $D = \mathbb{C} \setminus \{a_1, \ldots, a_M\}$, where $M > 1$ is an integer, and suppose that there exists a locally biholomorphic 2-valent mapping $f$ from $D$ onto $\mathbb{C}$.

Of course, the points $a_1, \ldots, a_M$ and $\infty$ are isolated singularities of $f$. The Picard theorem and our supposition that $f$ is 2-valent imply that none of the above points can be an essential singularity of $f$. Thus $f$ is rational as a meromorphic function on $\overline{\mathbb{C}}$. From this and our supposition that $f$ is 2-valent, it follows that $f$ has one pole of the second order at $\infty$, or one pole of the first order at $\infty$ and one pole of the first order at a point $\alpha \in \mathbb{C}$, or poles of the first order at two points $\alpha, \beta \in \mathbb{C}, \alpha \neq \beta$, or one pole of the second order at a point $\alpha \in \mathbb{C}$. Equivalently, $f$ is defined by one of the following formulas:

(i) $f(z) = Az^2 + Bz + C$,

(ii) $f(z) = Az + B + \frac{C}{z-\alpha}$,

(iii) $f(z) = A + \frac{Bz+C}{(z-\alpha)(z-\beta)}$.

In case (i), if $A = 0$, then $f$ has the form $f(z) = Bz + C$ and it cannot map $D$ onto $\mathbb{C}$. Assume that $A \neq 0$; we can put $A = 1$. If we write $w_0 = f(z_0)$, where $z_0 = -B/2$, then $z_0 \notin D$, because $f$ is locally biholomorphic in $D$ and $f'(z_0) = 0$. Since $f(z) = (z - z_0)^2 + w_0$, the equation $f(z) = w_0$ has only the double root $z_0$. The assumption that $f$ is 2-valent and the fact that $z_0 \notin D$ show that $w_0 \notin f(D)$. Hence $f(D) \neq \mathbb{C}$.

In case (ii) we can rewrite $f$ in the form $f(z) = A(z - \alpha) + C/(z - \alpha)$. If $A = 0$ or $C = 0$, then $f$ is a homography and consequently $f(D) \neq \mathbb{C}$. Suppose $AC \neq 0$ and let $d, -d$ be the square roots of $C/A$. Then the points $z_1 = \alpha + d$ and $z_2 = \alpha - d$ do not belong to $D$, because $f$ is locally biholomorphic in $D$ and $f'(z_j) = 0$, $j = 1, 2$. Set $w_j = f(z_j)$, $j = 1, 2$, and consider the equation $f(z) = w_1$. By the change of variable $\zeta = (z - \alpha)/d$ we get the equation $\zeta + 1/\zeta = 2$, which has only the double root $\zeta = 1$. Thus the equation $f(z) = w_1$ has the double root $z_1 = \alpha + d$. Similarly, the equation $f(z) = w_2$ has the double root $z_2 = \alpha - d$. This and the assumption that $f$ is 2-valent shows that $w_j \notin f(D)$, $j = 1, 2$. Hence $f(D) \neq \mathbb{C}$.

In case (iii) it suffices to consider the function $f(z) = \frac{Bz+C}{(z-\alpha)(z-\beta)}$. Since $f'(z) \neq 0$ in $D$, our supposition that $f$ is 2-valent implies that $g(z) = 1/f(z)$ is locally biholomorphic in $D$ and maps $D$ onto $\overline{\mathbb{C}} \setminus \{0\}$ 2-valently. However, this is impossible, as shown in the previous cases. ∎

Since finitely connected domains, different from $\mathbb{C} \setminus \{0\}$, are domains with an isolated boundary fragment, we obtain the following generalization of Ligocka's result cited in the introduction.

COROLLARY 1. *If $D \subset \mathbb{C}$ is a finitely connected domain, different from $\mathbb{C} \setminus \{0\}$, then there exists a locally biholomorphic and 3-valent mapping from $D$ onto $\mathbb{C}$. The constant 3 is best possible.*

In [Li] it is also proved that there exists a function which maps, locally biholomorphically and 24-valently, a domain $D$ with an isolated component of $\overline{\mathbb{C}} \setminus D$ onto the open unit disc $B(0,1)$.

We will prove the following theorem.

THEOREM 2. *If $D \subset \mathbb{C}$ is a multi-connected domain with an isolated boundary fragment of type* (I) *or type* (II), *then there exists a* 5*-valent, locally biholomorphic mapping from $D$ onto the disc $B(0,1)$.*

*Proof.* 1. Asume that $D$ has an isolated boundary fragment of type (I) and let $K$ be the relevant continuum. Then $K$ is the boundary of a simply connected domain $D_0 \subset \overline{\mathbb{C}}$ such that $D \cap D_0 \neq \emptyset$. From the definition it follows that $\partial D \setminus K \subset K_0$, where $K_0 \subset D_0$ is a compact set in $\overline{\mathbb{C}}$.

Let $\zeta = \varphi(z)$ be a biholomorphic mapping from $D_0$ onto $B(0,1)$ and let $B'$ be a circular open neighbourhood of the point $\zeta = -1$. We can assume that the compact set $K^0 = \varphi(K_0)$ is included in a simply connected domain $E_1 \subset B(0,1)$ such that $E_1 \cap B' \neq \emptyset$.

Let $L = P^+ \cap Q^{-1}((-\infty,-2))$, where $Q(\zeta) = \zeta^3 - 3\zeta$. It is clear that $L$ is a simple and analytic arc which joins the points $\zeta = 1$, $\zeta = \infty$. Fix $A \in L$ and denote by $D'$ the simply connected domain which is obtained from $P^+$ by removing the part of $L$ joining $A$ and $\infty$. Let $\eta = \psi(w)$ be a biholomorphic mapping from $D'$ onto $B(0,1)$. Set $E_2 = \psi(D'')$, where $D'' \subset D'$ is the domain bounded by the arc $L$ and the ray $[1,+\infty)$. The simply connected domain $E_2$ is bounded by an open arc of the circle $\partial B(0,1)$ and by a Jordan curve $\Gamma'' \subset \overline{B(0,1)}$. We can assume that the above arc of $\partial B(0,1)$ includes $\zeta = 1$. Then there exists a circular open neighbourhood $B_1$ of $\zeta = 1$ such that $B_1 \cap B(0,1) = B_1 \cap E_2$.

The mapping $\frac{\zeta+a}{1+a\zeta}$, with properly chosen $a \in (-1,1)$, transforms $B(0,1) \setminus \overline{B'}$ into $B(0,1) \cap B_1$. Then

$$w = \psi^{-1}\left(\frac{\varphi(\zeta)+a}{1+a\varphi(\zeta)}\right)$$

maps $D$ biholomorphically onto a domain $D_2 \subset D'$ such that $\partial D_2 \setminus \partial D' \subset \psi^{-1}(E_2) = D''$.

Observe that $Q(P^+) \supset \mathbb{C} \setminus \mathbb{R}$ (just as for $P^-$ in part 2 of the proof of Theorem 1). Moreover, $Q$ maps the set $(-\infty,-2) \cup (2,\infty)$ biholomorphically onto itself and for every $w \in [-2,2]$ the equation $Q(z) = w$ has three roots (counting multiplicity). Thus $Q(P^+) = \mathbb{C} \setminus [-2,2]$ and $Q(D') = \mathbb{C} \setminus ([-2,2] \cup (-\infty, A'])$, where $A' = Q(A) < -2$. Set $D_3 = Q(D')$.

We will show that the mapping $Q$ in the upper half-plane $P^+$ gives a 2-valent covering of $P^+$. To do this take $w \in P^+$ and $R$ so large that all roots $z \in P^+$ of the equation $Q(z) = w$ belong to the interior of the closed curve $\Gamma_R$ consisting of a semi-circle $C_R$ and the segment $[-R, R]$; denote by $N_w$ the number of these roots, counted with multiplicity. Using the argument principle, we have

$$\begin{aligned} 2\pi N_w &= \Delta_{\Gamma_R} \arg(Q(z) - w) \\ &= \Delta_{C_R} \arg(z^3) + \Delta_{C_R} \arg\left(1 - \frac{3}{z^2} - \frac{w}{z^3}\right) + \Delta_{[-R,R]} \arg(z^3 - 3z - w), \end{aligned}$$

hence, for $R \to \infty$,

$$2\pi N_w = 3\pi + \pi + o(1).$$

Thus $N_w = 2$.

Now observe that $Q(D'') = P^+$. Therefore $Q(D_2) = D_3$ and the mapping $Q$ in $D_2$ is locally biholomorphic and 2-valent.

In this way we obtain a locally biholomorphic and 2-valent mapping from $D$ onto $D_3$, but only points from $P^+$ can be covered twice.

2. Now we will show that in the case of $D$ with an isolated boundary fragment of type (II), there also exists a locally biholomorphic 2-valent mapping from $D$ onto $D_3$.

For this purpose, we map $D$ onto $D_1$ locally biholomorphically and 2-valently (notation from part 2 of the proof of Theorem 1). Next, as in part 1 of the proof of Theorem 2, we map $P^+$ onto $B(0,1)$ by a homography such that the inner components of the image of $D_1$ belong to the domain $E_1 \subset B(0,1)$. Now, to obtain a locally biholomorphic 2-valent mapping from $D$ onto $D_3$, it is sufficient to repeat the same steps as in part 1 of the proof of Theorem 2.

3. Now we construct a locally biholomorphic 3-valent mapping from $D_3$ onto $B(0,1)$.

Let $0 < c < d$ and transform $D_3$ onto $\mathbb{C} \setminus ((-\infty, 0] \cup [c,d])$ linearly; then, of course, the upper half-plane $P^+$ maps onto itself. Next applying the function $-i\frac{\sqrt{z}-1}{\sqrt{z}+1}$ ($\sqrt{z}$ is the main branch of the square root) we transform $\mathbb{C} \setminus ((-\infty, 0] \cup [c,d])$ onto $B(0,1)$ with a segment $[ai, bi]$ removed (in view of the arbitrariness of $c, d$, we can assume that $a, b$ are arbitrary numbers such that $-1 < a < b < 1$). This mapping sends $P^+$ onto the right semi-disc.

Denote by $D_4$ the image of $B(0,1) \setminus [ai, bi]$ under the mapping $i\sqrt[4]{3}\frac{1+z}{1-z}$. Then $D_4 = P^+ \setminus l$, where $l$ is a closed arc included in the upper semi-circle $\{z \in P^+ : |z| = \sqrt[4]{3}\}$. Moreover, the right semi-disc of $B(0,1)$ will be transformed onto $D_4' = \{z \in P^+ : |z| > \sqrt[4]{3}\}$. In this way we obtain a locally biholomorphic and 2-valent mapping from $D$ onto $D_4$, but in $D_4$ only points from $D_4'$ can be covered twice.

In view of the arbitrariness of $a, b$ we can assume that the arc $l$ starts from the point $\sqrt[4]{3}e^{i\pi/4}$ and finishes at $\sqrt[4]{3}e^{3i\pi/4}$.

Now consider the image of $D_4$ under the mapping

$$f(z) = \frac{z^3 - 3z}{z^2 - 1}.$$

Since $f(z) = z + \frac{z}{z+1} + \frac{z}{z-1}$ and every summand of the right-hand side transforms $P^+$ onto itself and $f(\mathbb{R})$ is the real axis, covered three times, it follows that $f(P^+)$ is the upper half-plane $P^+$ covered three times. In the remaining part of the proof we will treat the image $f(X)$ of a domain $X \subset D_4$ as a part of the Riemann surface of the inverse function.

Observe that $f'$ vanishes in $P^+$ only at the points $z_1 = \sqrt[4]{3}e^{i\pi/4}$, $z_2 = \sqrt[4]{3}e^{3i\pi/4}$. They are the branch points of the Riemann surface $f(P^+)$. It is obvious that $f(-\sqrt[4]{3}) = \sqrt[4]{3^3}$, $f(\sqrt[4]{3}) = -\sqrt[4]{3^3}$ and

$$f : \{\sqrt[4]{3}e^{it} : t \in [0, \pi]\} \to \{\sqrt[4]{3^3}e^{i\tau} : \tau \in [\pi, 0]\}.$$

We will examine the image $f(\partial D_4')$. We have:

$$f : (-\infty, -\sqrt[4]{3}] \to (-\infty, \sqrt[4]{3^3}],$$
$$f : \{\sqrt[4]{3}e^{it} : t \in [\pi, 3\pi/4]\} \to \{\sqrt[4]{3^3}e^{i\tau} : \tau \in [0, 7\pi/12]\},$$

and since $z_1$ and $z_2$ are simple zeros of $f'(z)$, we also have:

$$f : \{\sqrt[4]{3}e^{it} : t \in [3\pi/4, \pi/4]\} \to \{\sqrt[4]{3^3}e^{i\tau} : \tau \in [7\pi/12, 5\pi/12]\},$$
$$f : \{\sqrt[4]{3}e^{it} : t \in [\pi/4, 0]\} \to \{\sqrt[4]{3^3}e^{i\tau} : \tau \in [5\pi/12, \pi]\},$$
$$f : (\sqrt[4]{3}, +\infty] \to [\sqrt[4]{3^3}, +\infty].$$

From this it follows that $f(D_4')$ is a Riemann surface $\mathfrak{R}_1$ which consists of two copies of the semi-disc $B^+ = \{z \in P^+ : |z| < \sqrt[4]{3^3}\}$ and one copy of the domain $\widehat{B^+} = \{z \in P^+ : |z| > \sqrt[4]{3^3}\}$. Moreover, $\widehat{B^+}$ is stuck to one copy of $B^+$ along the arc $\{\sqrt[4]{3^3}e^{i\tau} : \tau \in [7\pi/12, \pi]\}$, but to the other copy of $B^+$ along the arc $\{\sqrt[4]{3^3}e^{i\tau} : \tau \in [0, 5\pi/12]\}$. Accordingly, the image of the semi-disc $\{z \in P^+ : |z| < \sqrt[4]{3}\}$ is a Riemann surface $\mathfrak{R}_2$ which consists of two copies of $\widehat{B^+}$ and one copy of $B^+$, stuck along arcs of the semi-circle $\{z \in P^+ : |z| = \sqrt[4]{3^3}\}$.

Therefore, the Riemann surface $\mathfrak{R} = f(D_4)$ covers the whole upper half-plane $P^+$, because $f(l) = \{\sqrt[4]{3^3}e^{i\tau} : \tau \in [5\pi/12, 7\pi/12]\} \subset f(\{\sqrt[4]{3}e^{it} : t \in [0, \pi/4)\}) \cup f(\{\sqrt[4]{3}e^{it} : t \in (3\pi/4, \pi]\})$. Moreover, the mapping $f$ is not more than 2-valent in $D_4'$ and in $D_4 \setminus D_4'$.

However, the points from $D_4 \setminus D'_4$ were univalently covered by the locally biholomorphic image of the domain $D$, and the points $D'_4$ were covered not more than twice. Thus, the points of the Riemann surfaces $\mathfrak{R}_2$ are covered not more than twice by the locally biholomorphic image of $D$, and the points of $\mathfrak{R} \setminus \mathfrak{R}_2$ are covered univalently.

From this it follows that the constructed mapping on $D$ covers the upper half-plane $P^+$ not more than fivefold.

Now it is sufficient to map $P^+$ biholomorphically onto $B(0,1)$. ∎

CONJECTURE. *The constant* 5 *from Theorem* 2 *can be replaced by* 3.

E. Ligocka [Li] has remarked that theorems about locally biholomorphic mappings from multi-connected domains onto a disc can be applied in the investigation of mappings from domains of the space $\mathbb{C}^n, n \geq 1$, onto $n$-dimensional complex manifolds.

Observe that if we combine our Theorem 2 with Fornæss and Stout's result from [FS], we obtain the following corollaries:

COROLLARY 2. *If a multi-connected domain $D \subset \mathbb{C}$ has an isolated boundary fragment of type* (I) *or type* (II) *and $X$ is a connected Riemann surface* (*compact or open*), *then there exists an $m$-valent locally biholomorphic mapping $f$ from $D$ onto $X$ with $m \leq 5 \cdot 14 = 70$.*

COROLLARY 3. *If $X = D_1 \times \cdots \times D_n$, where the domains $D_j$, $j = 1, \ldots, n$, fulfil the assumptions of Corollary* 2, *and $Y$ is a connected paracompact $n$-dimensional complex manifold*, *then there exists a locally biholomorphic and $m$-valent mapping $f$ from $X$ onto $Y$ with $m \leq 5^n[(2n+1)4^n+2]$.*

## References

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