ANNALES
POLONICI MATHEMATICI
85.2 (2005)

Period function's convexity for Hamiltonian centers with separable variables

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Abstract. A convexity theorem for the period function T of Hamiltonian systems with separable variables is proved. We are interested in systems with non-monotone T. This result is applied to proving the uniqueness of critical orbits for second order ODE's.

1. Introduction. Let us consider a planar Hamiltonian system with separated variables,

$$x' = F'(y), \quad y' = -G'(x), \tag{1}$$

defined on an open connected set $\Omega \subset \mathbb{R}^2$. If its Hamiltonian $H(x,y) = F(y) + G(x)$ has an isolated extremum at the origin O, then O has a punctured neighbourhood covered with non-trivial cycles. We denote by N_O the largest connected punctured neighbourhood of O covered with non-trivial cycles. We define the *period function* $T : N_O \to \mathbb{R}$ of (1) as the function assigning to every point $(x,y) \in N_O$ the minimal period of the cycle passing through (x,y). We say that the period function T is *increasing* if, for every couple of cycles γ_1, γ_2, with γ_1 enclosed by γ_2, one has $T(\gamma_1) \le T(\gamma_2)$. When T is constant, we say that O is *isochronous*. Let $\delta(s)$, $s \in (\sigma_*, \sigma^*)$, be a curve of class C^1 meeting transversally the cycles of N_O. Assume that $\lim_{s\to\sigma_*^+} \delta(s) = O$. We can consider the function $T(s) \equiv T(\delta(s))$. Then T is increasing if and only if $T(s)$ is a one-variable increasing function. Let $\gamma_{\bar s}$ be the unique cycle met by δ at the point $\delta(\bar s)$. We say that T has an extremum at $\gamma_{\bar s}$ if $T(s)$ has an extremum at $s = \bar s$. We say that γ is a *critical cycle* if $\left[\frac{d}{ds}T(s)\right]_{s=\bar s} = 0$. One can prove that this definition does not depend on the particular transversal curve δ chosen.

2000 *Mathematics Subject Classification*: 34C05, 34C07.

Key words and phrases: center, period function, convexity, critical orbit.

This work has been partially supported by the COFIN group "Equazioni differenziali ordinarie e applicazioni", and by the intergroup project "Dinamica anolonoma, perturbazioni e orbite periodiche".

Studying the period function is essential in some stability, bifurcation, and boundary value problems related to Hamiltonian systems, or to systems reducible to Hamiltonian ones, as Lotka–Volterra systems. The period function's monotonicity for systems of type (1) was studied by several authors ([1], [6]–[9], [11]–[15]); here we do not mention papers devoted to isochronicity. In some cases the monotonicity was proved together with a convexity property related to T ([14]), not implying T's convexity. Systems with a non-monotone period function, hence with critical orbits, were studied in [2], [4], [5], [17]. In [12] it was proved that a system of type (1), with $F'(y) = y$, can have at most a simple critical point for every central region, if $G(x)$ is a polynomial of degree four. In [3] it was proved that critical orbits of an analytic center do not accumulate on a compact set.

The monotonicity ensures that a typical boundary value problem, $x(0) = x(T)$, has a unique solution for T belonging to some interval. Similarly, when $F'(y) = y$, that is, when the system takes the form

$$x' = y, \qquad y' = -G'(x),$$

the uniqueness of Neumann-like problems, $x'(0) = x'(T)$, may be reduced to the study of T's monotonicity, as in [1].

A different situation has to be taken into account when looking for multiple solutions of boundary value problems. If $x(0) = x(T)$ has more than a single solution, then $T(s)$ has different monotonicity properties in distinct intervals. Such intervals, corresponding to distinct subsets of N_O, are separated by values of s where T reaches a local extremum. The problem of counting the exact number of solutions to $x(0) = x(T)$ is related to the problem of counting such local extrema. The simplest way to estimate the number of such extrema is to study the convexity of $T(s)$, which ensures the uniqueness of the extremum. If $T(s)$ is convex, there exists an interval (T_1, T_2) such that the BVP $x(0) = x(T)$ has exactly two solutions for $T \in (T_1, T_2)$.

In this paper we give sufficient conditions for the existence of a transversal curve $\delta(s)$ such that $T(\delta(s))$ is convex on some interval. The main tool applied is a theorem proved in [9], where T was studied by means of a suitable auxiliary system,

$$x' = \frac{G(x)}{G'(x)}, \qquad y' = \frac{F(y)}{F'(y)}. \tag{2}$$

Such a system is a *normalizer* of (1), that is, its local flow takes orbits of (1) into orbits of (1). If we denote by $V(x, y)$ the vector field of (1), and by $W(x, y)$ the vector field of (2), this is equivalent to saying that there exists a function $\mu : N_O \to \mathbb{R}$ such that

$$[V, W] = \mu V.$$

If $\delta(s)$ is a solution to (2), then, as proved in [9],

$$T'(s) = \frac{d}{ds}T(\delta(s)) = \int_0^{T(s)} \mu(\gamma_s(t))\,dt. \tag{3}$$

In the case of the couple of systems (1) and (2), one has

$$\mu(x,y) = \left(\frac{G(x)}{G'(x)}\right)' + \left(\frac{F(y)}{F'(y)}\right)' - 1.$$

Hence, proving the convexity of $T(s)$ reduces to proving that the integral in (3) has larger values on outer cycles. This can be done, on a suitable subset A of N_O, by adapting a technique used to study the uniqueness of limit cycles in Liénard systems (see [10], [16], [18]).

In Theorem 1 we show that under suitable assumptions on the sign of some functions depending on F, G and their derivatives up to the third order, $T'(s)$ is increasing on A, hence $T(s)$ is convex on A. As a consequence, (1) has at most one critical orbit in A. Conditions for the existence and uniqueness of critical orbits are given for some classes of second order conservative ODE's. It is maybe worth noticing that the function $N(x)$ introduced in [1],

$$N(x) = 6G(x)G''^2(x) - 3G'(x)^2G''(x) - 2G(x)G'(x)G'''(x),$$

plays a role also in the study of convexity. On the other hand, we find an example of degenerate planar center with T strictly decreasing at the origin, such that $N(x) \geq 0$ in a neighbourhood of O. This shows that Theorem A in [1] cannot be extended to degenerate centers.

2. Results. Let $G \in C^3(I,\mathbb{R})$, $F \in C^3(J,\mathbb{R})$, I, J open intervals containing 0, possibly unbounded. We consider the system (1), assuming F and G to have isolated minima at the origin. We do not assume the minima to be non-degenerate, because the results proved in [9] hold under the only assumption that $H(x,y) = G(x)+F(y)$ has a minimum at O. Also, we assume $xG'(x) > 0$ on $I \setminus \{0\}$, and $yF'(y) > 0$ on $J \setminus \{0\}$.

We say that (1) satisfies the *conditions* (L) if there exist $\alpha \in C^0(I,\mathbb{R})$, $\beta \in C^0(J,\mathbb{R})$ and $a,b \in I$, $a \leq 0 \leq b$, $c,d \in J$, $c \leq 0 \leq d$, such that:

(L_1) $\quad \alpha(x) + \beta(y) = \left(\frac{G(x)}{G'(x)}\right)' + \left(\frac{F(y)}{F'(y)}\right)' - 1,$

(L_2) $\quad \alpha(x) \geq 0$ for $x \notin [a,b]$, $\alpha(x)F''(y) \leq 0$ for $x \in [a,b]$, $y \notin [c,d]$;

(L_3) $\quad \beta(y) \geq 0$ for $y \notin [c,d]$, $G''(x)\beta(y) \leq 0$ for $x \notin [a,b]$, $y \in [c,d]$;

(L_4) $\quad \left(\frac{\alpha(x)}{G'(x)}\right)' \geq 0$ for $x \notin [a,b]$,

(L_5) $\left(\dfrac{\beta(y)}{F'(y)}\right)' \geq 0$ for $y \notin [c, d]$.

The above conditions are considered even in the case of intervals reducing to a single point, as it occurs when $a = 0 = b$.

We denote by $\mathcal{O}^e_{abcd}$ the family of cycles contained in N_O and enclosing the rectangle $[a, b] \times [c, d]$, and by $\mathcal{O}^i_{abcd}$ the family of cycles contained in $N_O \cap [a, b] \times [c, d]$. In general, $N_O \neq \mathcal{O}^i_{abcd} \cup \mathcal{O}^e_{abcd}$. If $c = 0 = d$, $a < 0 < b$, we denote by $\mathcal{O}^e_{ab00}$ the family of cycles meeting both the lines $x = a$ and $x = b$, and by $\mathcal{O}^i_{ab00}$ the family of cycles contained in the strip $a < x < b$. Similarly for $a = 0 = b$, $c < 0 < d$.

Convexity is not assumed to be necessarily strict. Since there is one-to-one correspondence between the parameters s and the orbits γ_s, we say equivalently that T is (strictly) convex at s or at γ_s. Similarly, we say that T is (strictly) convex on $\mathcal{O}^e_{abcd}$, or on $\mathcal{O}^i_{abcd}$.

The main result of this paper is the following theorem.

THEOREM 1. *Assume that* (1) *satisfies the conditions* (L). *Then the function* T *is convex on* $\mathcal{O}^e_{abcd}$.

Proof. It is sufficient to prove that $T'(s)$ is increasing on $\mathcal{O}^e_{abcd}$. By Lemma 7 in [9], the derivative of $T(s)$ is given by (3), where

$$\mu(x, y) = \left(\frac{G(x)}{G'(x)}\right)' + \left(\frac{F(y)}{F'(y)}\right)' - 1 = \alpha(x) + \beta(y).$$

Consider two cycles, γ_{s_1}, γ_{s_2}, with $s_1 < s_2$. The cycle γ_{s_1} is contained in the bounded region having γ_{s_2} as boundary. In order to prove that $T'(s_1) \leq T'(s_2)$, we have to show that

$$\int_0^{T(s_1)} \mu(\gamma_{s_1}(t))\, dt \leq \int_0^{T(s_2)} \mu(\gamma_{s_2}(t))\, dt.$$

The orbits will be decomposed into arcs over which the integration will be performed with respect to x or y.

Let us first compare the terms $\int_0^{T(s_1)} \alpha(\gamma_{s_1}(t))\, dt$ and $\int_0^{T(s_2)} \alpha(\gamma_{s_2}(t))\, dt$.

Since γ_1 encloses the rectangle $[a, b] \times [c, d]$, it meets the line $x = b$ at points (b, c'), (b, d'), with $c' \leq 0 \leq d'$. Also, it meets the line $x = a$ at points (a, c''), (a, d''), with $c'' \leq 0 \leq d''$.

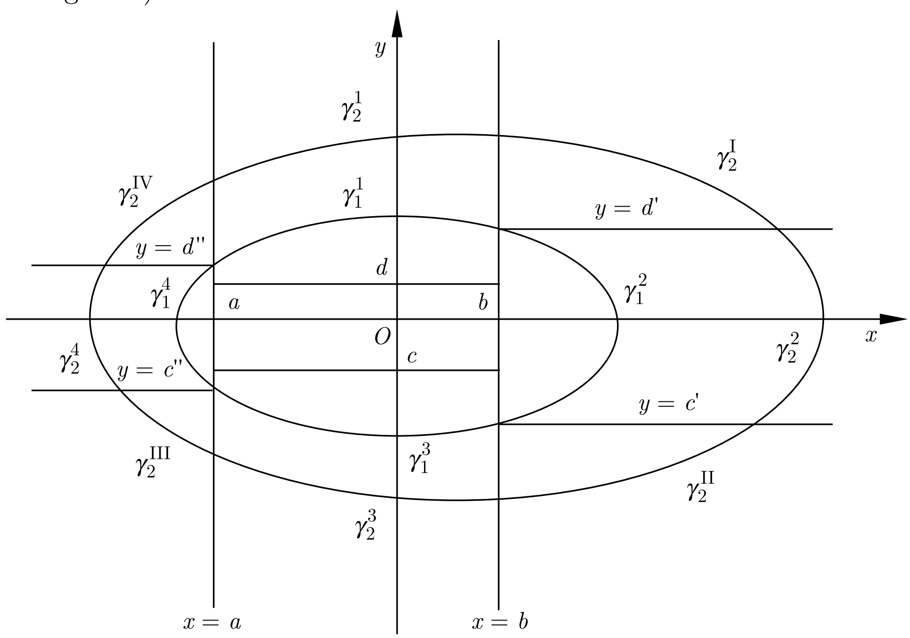
The curve γ_1 is the union of four arcs:

$$\gamma_1^1 \subset \{a \leq x \leq b,\, y > 0\}, \quad \gamma_1^2 \subset \{x \geq b\},$$
$$\gamma_1^3 \subset \{a \leq x \leq b,\, y < 0\}, \quad \gamma_1^4 \subset \{x \leq a\}.$$

The curve γ_2 is the union of eight arcs:

$$\gamma_2^1 \subset \{a \le x \le b,\, y > 0\}, \qquad \gamma_2^2 \subset \{x \ge b,\, c' \le y \le d'\},$$
$$\gamma_2^3 \subset \{a \le x \le b,\, y < 0\}, \qquad \gamma_2^4 \subset \{x \le a,\, c'' \le y \le d''\};$$
$$\gamma_2^{\mathrm{I}} \subset \{x \ge b,\, y \ge d'\}, \qquad \gamma_2^{\mathrm{II}} \subset \{x \ge b,\, y \le c'\},$$
$$\gamma_2^{\mathrm{III}} \subset \{x \le a,\, y \le c''\}, \qquad \gamma_2^{\mathrm{IV}} \subset \{x \le a,\, y \ge d''\}$$

(see Figure 1).



Since $\alpha \ge 0$ off $[a,b]$, one has

$$\int_{\gamma_2^{\mathrm{I}}} \alpha \ge 0, \qquad \int_{\gamma_2^{\mathrm{II}}} \alpha \ge 0, \qquad \int_{\gamma_2^{\mathrm{III}}} \alpha \ge 0, \qquad \int_{\gamma_2^{\mathrm{IV}}} \alpha \ge 0.$$

In order to prove that $\int_0^{T(s_1)} \alpha(\gamma_{s_1}(t))\; dt \le \int_0^{T(s_2)} \alpha(\gamma_{s_2}(t))\; dt$, it is sufficient to prove that

$$\int_{\gamma_1^j} \alpha \le \int_{\gamma_2^j} \alpha, \qquad j = 1, \ldots, 4.$$

We give the details only for the arcs γ_1^1, γ_1^2, γ_2^1, γ_2^2, since the other four arcs can be treated in a similar way. Since for $a \le x \le b$ one has $dx/dt = F'(y) > 0$, along $\gamma_1^1(t)$ one can express t as a function of x and integrate with respect to x. Writing $F(y)$ for $F(y(t(x)))$, one has

$$\int_{\gamma_1^1} \alpha(\gamma_{s_1}(t))\, dt = \left[\int_a^b \frac{\alpha(x)\, dx}{F'(y)} \right]_{\gamma_1^1}.$$

Since $\alpha(x)F''(y) \le 0$ for $x \in [a,b]$, $y \notin [c,d]$, one has

$$\frac{\partial}{\partial y}\frac{\alpha(x)}{F'(y)} = -\frac{\alpha(x)F''(y)}{F'(y)^2} \ge 0,$$

so that $\alpha(x)/F'(y)$ is an increasing function of y. As γ_2 is external with respect to γ_1, it follows that

$$\int_{\gamma_1^1} \alpha(\gamma_{s_1}(t))\,dt = \left[\int_a^b \frac{\alpha(x)\,dx}{F'(y)}\right]_{\gamma_1^1} \le \left[\int_a^b \frac{\alpha(x)\,dx}{F'(y)}\right]_{\gamma_2^1} = \int_{\gamma_2^1} \alpha(\gamma_{s_2}(t))\,dt.$$

Now consider the arcs γ_1^2, γ_2^2, along which one has $dy/dt = -G'(x) < 0$, so that one can express t as a function of y, and integrate with respect to y,

$$\int_{\gamma_1^2} \alpha(\gamma_{s_1}(t))\,dt = \left[\int_{d'}^{c'} \frac{\alpha(x)\,dy}{-G'(x)}\right]_{\gamma_1^2} = \left[\int_{c'}^{d'} \frac{\alpha(x)\,dy}{G'(x)}\right]_{\gamma_1^2}.$$

By (L_4), one has

$$\frac{\partial}{\partial x}\left(\frac{\alpha(x)}{G'(x)}\right) \ge 0,$$

hence $\alpha(x)/G'(x)$ is an increasing function, and as above

$$\int_{\gamma_1^2} \alpha(\gamma_{s_1}(t))\,dt = \left[\int_{d'}^{c'} \frac{\alpha(x)\,dy}{-G'(x)}\right]_{\gamma_1^2} \le \left[\left[\int_{d'}^{c'} \frac{\alpha(x)\,dy}{-G'(x)}\right]_{\gamma_1^2}\right]_{\gamma_2^2} = \int_{\gamma_2^2} \alpha(\gamma_{s_2}(t))\,dt.$$

The same argument works as well for the arcs γ_1^3, γ_1^4, γ_2^3, γ_2^4. Summing up, one has

$$\int_0^{T(s_1)} \alpha(\gamma_{s_1}(t))\,dt \le \int_0^{T(s_2)} \alpha(\gamma_{s_2}(t))\,dt.$$

Now consider the integrals involving β. We can work as we did for α, with the lines $y = c$, $y = d$ playing the role of $x = a$, $x = b$. Computations are similar, and lead to a similar conclusion,

$$\int_0^{T(s_1)} \beta(\gamma_{s_1}(t))\,dt \le \int_0^{T(s_2)} \beta(\gamma_{s_2}(t))\,dt. \ \blacksquare$$

The term -1 appearing in μ can be absorbed in different ways by α and β. In general, for a given $\kappa \in \mathbb{R}$, we may write

$$\mu(x,y) = \left[\left(\frac{G(x)}{G'(x)}\right)' + \kappa\right] + \left[\left(\frac{F(y)}{F'(y)}\right)' - 1 - \kappa\right] = \alpha(x) + \beta(y).$$

Denote by $(-L_j)$, $j = 2,\ldots,5$, the conditions obtained from (L_j), $j = 2,\ldots,5$, by reversing the inequalities. We have the following analogue of Theorem 1 for the concavity of the period function.

THEOREM 2. *Assume that* (1) *satisfies the conditions* (L_1), $(-L_j)$, $j = 2, \dots, 5$. *Then the function* T *is concave on* $\mathcal{O}^e_{abcd}$.

Proof. As in Theorem 1, reversing the integral inequalities. ∎

The next four corollaries are concerned with the strict convexity on $\mathcal{O}^e_{abcd}$. This property implies the uniqueness of critical orbits on $\mathcal{O}^e_{abcd}$, if they exist.

COROLLARY 1. *Assume that the hypotheses of Theorem* 1 *hold. Let the cycle* $\overline{\gamma}$ *pass through a point* $(\overline{x}, \overline{y})$ *such that at least one of the inequalities in* (L_j), $j = 2, \dots, 5$, *is strict. Then* T *is strictly convex in a neighbourhood of* $\overline{\gamma}$.

Proof. At least one of the integral inequalities of the proof of Theorem 1 is strict at $(\overline{x}, \overline{y})$. By continuity, this holds in a neighbourhood of $(\overline{x}, \overline{y})$, hence $T'(s)$ is strictly increasing in a neighbourhood of $\overline{\gamma}$. ∎

For instance, if there exists $\overline{x} > b$ such that $\alpha(\overline{x}) > 0$, then T is strictly convex at every orbit intersecting the line $x = \overline{x}$. As a consequence, one has at most one critical orbit intersecting the line $x = \overline{x}$. A similar statement can be proved about strict concavity.

COROLLARY 2. *Under the hypotheses of Theorem* 1, *assume that one of the following holds*:

(i) *there exist* $x_n > b$ *with* $\lim_{n\to\infty} x_n = b$ *such that* $\alpha(x_n) > 0$ ($x_n < a$ *with* $\lim_{n\to\infty} x_n = a$ *such that* $\alpha(x_n) > 0$);
(ii) *there exist* $y_n > d$ *with* $\lim_{n\to\infty} y_n = d$ *such that* $\beta(y_n) > 0$ ($y_n < c$ *with* $\lim_{n\to\infty} y_n = c$ *such that* $\beta(y_n) > 0$).

Then the function T *is strictly convex on* $\mathcal{O}^e_{abcd}$.

Proof. This is an immediate consequence of Corollary 1, since every cycle in $\mathcal{O}^e_{abcd}$ has to meet at least one of the lines $x = x_n$ ($y = y_n$). ∎

COROLLARY 3. *Under the hypotheses of Theorem* 1, *assume that one of the following holds*:

(i) *there exists* $\overline{x} \in [a, b]$ *such that* $\alpha(\overline{x}) < 0$ *and* $F''(y) > 0$ *for* $y > d$ ($F''(y) > 0$ *for* $y < c$);
(ii) *there exists* $\overline{y} \in [c, d]$ *such that* $\beta(\overline{y}) < 0$ *and* $G''(x) > 0$ *for* $x > b$ ($G''(x) > 0$ *for* $x < a$).

Then the function T *is strictly convex on* $\mathcal{O}^e_{abcd}$.

Proof. (i) is an immediate consequence of Corollary 1, since every cycle in $\mathcal{O}^e_{abcd}$ has to meet the half-line $x = \overline{x}$, $y > d$ ($x = \overline{x}$, $y < c$). Item (ii) can be proved similarly. ∎

Strict convexity (concavity) can also be proved for analytic systems. We recall that monotonicity is not assumed to be strict, so that a constant period function is monotone.

Corollary 4. *Assume that the hypotheses of Theorem* 1 *hold. If F and G are analytic functions*, *and T is not monotone on $\mathcal{O}^e_{abcd}$, then T is strictly convex on $\mathcal{O}^e_{abcd}$.*

Proof. $T(s) = T(\delta(s))$ is an analytic function. By Theorem 1, T is convex on $\mathcal{O}^e_{abcd}$, hence $T''(s) \geq 0$. Moreover, $T''(s)$ is not identically zero: otherwise there would exist $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $T(s) = \kappa_1 s + \kappa_2$, which would imply monotonicity. By analyticity, the zeroes of $T''(s)$ are isolated, so $T'(s)$ is strictly increasing, which gives the strict convexity of T. ∎

Example 1. Setting

$$G(x) = \frac{|x|^{11/2}}{x^4+1},$$

consider the Hamiltonian

$$H(x,y) = G(x) + G(y) = \frac{|x|^{11/2}}{x^4+1} + \frac{|y|^{11/2}}{y^4+1}.$$

Since $G(x)$ is an even function, one may consider only its derivatives for $x > 0$, which simplifies the computations. One has, for $x > 0$,

$$G'(x) = x^{9/2}\,\frac{3x^4+11}{2(x^4+1)^2}, \qquad G''(x) = x^{7/2}\,\frac{3x^8-26x^4+99}{4(x^4+1)^3},$$

hence the origin is a global center, with convex Hamiltonian. Moreover, setting

$$\alpha(x) = \frac{3x^8+118x^4-77}{2(3x^4+11)^2},$$

one has

$$\mu(x,y) = \alpha(x) + \alpha(y) = \frac{3x^8+118x^4-77}{2(3x^4+11)^2} + \frac{3y^8+118y^4-77}{2(3y^4+11)^2}.$$

The function $\alpha(x)$ vanishes for $x_0 = \left(-\frac{59}{3} + \frac{8}{3}\sqrt{58}\right)^{1/4}$, which is approximately 0.9. One has $\alpha(x) < 0$ for $x \in (-x_0, x_0)$, and $\alpha(x) > 0$ for $x \notin [-x_0, x_0]$. The period function of the Hamiltonian system is decreasing on the orbits contained in the square $Q = [-x_0, x_0] \times [-x_0, x_0]$, and is strictly convex on the orbits enclosing Q.

The next corollary is concerned with conservative second order differential equations,

$$x'' + G'(x) = 0. \tag{4}$$

As in [1], we set

$$N(x) = 6G(x)G''^2(x) - 3G'(x)^2G''(x) - 2G(x)G'(x)G'''(x).$$

In what follows, we choose $c = 0 = d$.

COROLLARY 5. *Let $G \in C^3(I,\mathbb{R})$ with $xG'(x) > 0$ for $x \neq 0$. If there exist $a, b \in I$, $a \leq 0 \leq b$, such that*

(i) *$G'(x)^2 - 2G(x)G''(x) \leq 0$ for $x \in [a,b]$, and $G'(x)^2 - 2G(x)G''(x) \geq 0$ for $x \notin [a,b]$,*
(ii) *$N(x) \geq 0$ for $x \notin [a,b]$,*

then the period function $T(s)$ is convex on $\mathcal{O}^e_{ab00}$.

Reversing the above inequalities implies the concavity of $T(s)$ on $\mathcal{O}^e_{ab00}$.

Proof. The equation (4) is a special case of (1), with $F(y) = y^2/2$, $c = 0 = d$, $\beta(y) = 0$. Then $\alpha = (G'^2 - 2GG'')/2G'^2$, and

$$\left(\frac{\alpha}{G'}\right)' = \frac{6GG''^2 - 3G'^2G'' - 2GG'G'''}{2G'^4} = \frac{N}{2G'^4}.$$

Conditions (i)–(ii) ensure that the hypotheses of Theorem 1 hold. ∎

A simple additional condition allows us to prove the uniqueness of critical orbits of (4) on all of N_O. In the situation considered in the next corollary, one has $N_O = \mathcal{O}^i_{ab00} \cup \mathcal{O}^e_{ab00}$.

COROLLARY 6. *Suppose that* (4) *is a non-linear equation. Under the hypotheses of Corollary* 5, *assume additionally that $G(a) = G(b)$. If the hypotheses of one of the corollaries* 2 *or* 4 *hold, then* (4) *has at most one critical orbit in N_O, contained in the set $G(x) + y^2/2 > G(a)$.*

Proof. The cycles are contained in level sets of the first integral $G(x) + y^2/2$. If $G(a) = G(b)$, then there exists a cycle γ_{ab} passing through $(a,0)$ and $(b,0)$. All the other cycles either meet both the lines $x = a$ and $x = b$, or are contained in the strip $a < x < b$, hence $N_O = \mathcal{O}^i_{ab00} \cup \mathcal{O}^e_{ab00}$. One has $T'(s) \leq 0$ for every cycle $\gamma_s \in \mathcal{O}^i_{ab00}$, because $\alpha(x) \leq 0$ on $[a,b]$. We claim that actually $T'(s) < 0$ on $\mathcal{O}^i_{ab00}$. In fact, assume that $\alpha \equiv 0$ on $[a,b]$. Then $G'^2 - 2GG'' \equiv 0$ on $[a,b]$, so that, on the interval $(0,b)$, where both G and G' are positive, one has

$$\frac{G'}{G} = 2\,\frac{G''}{G'}.$$

Integrating gives $\ln G = 2 \ln G' + \kappa_0$, $\kappa_0 \in \mathbb{R}$, hence $G = \kappa_1 {G'}^2$, $\kappa_1 > 0$. Integrating the equation $G = \kappa_1 {G'}^2$ gives $G(x) = (\kappa_2 x + \kappa_3)^2$. Since $G(x)$ vanishes at 0, one has $\kappa_3 = 0$, so that $G(x) = (\kappa_2 x)^2$, contradicting the non-linearity of (4). This proves that $\alpha(x)$ vanishes identically on no interval $[0, b_1) \subset [0, b)$. As a consequence, T' is strictly negative on $\mathcal{O}^i_{ab00}$. In particular, T' is strictly negative on the orbit γ_{ab}, and, by continuity, on a neighbourhood of γ_{ab}. Hence a critical orbit cannot be contained in the sublevel set $G(x) + y^2/2 \leq G(a)$, but, if it exists, it has to belong to $\mathcal{O}^e_{ab00}$, where T is strictly convex, by Corollary 2 or 4. This gives the uniqueness. ∎

Example 2. The potential $G(x) = x^2 + x^4 - x^6$ generates the system

$$x' = y, \qquad y' = -2x - 4x^3 + 6x^5. \tag{5}$$

We take $I = [-1, 1]$ and $J = \mathbb{R}$. The system (5) has a center at the origin, with central region contained in the rectangle $[-1, 1] \times [-\sqrt{2}, \sqrt{2}]$.

One has

$$G'^2 - 2GG'' = -4x^4(3 - 8x^2 - 9x^4 + 6x^6),$$
$$N = -24x^4(1 - 18x^2 + 34x^4 - 52x^6 - 59x^8 + 30x^{10}).$$

Applying the Sturm procedure, one can show that in the interval $[-1, 1]$, $G'^2 - 2GG''$ has exactly two zeroes $-x_1 < 0 < x_1$, as does N, which vanishes at $-x_2 < 0 < x_2$. One has $-x_1 < -x_2 < 0 < x_2 < x_1$, so that if we take $a = -x_1$, $b = x_1$, the system (5) satisfies all the hypotheses of Corollary 6. Its period function is strictly decreasing in a neighbourhood of the origin, it is strictly convex on $\mathcal{O}_{-x_1x_100}$, it tends to $+\infty$ approaching the boundary ∂N_O, and there exists exactly one critical orbit. A numerical approximation shows that x_1 is approximately 0.544, while x_2 is approximately 0.249.

Example 3. The potential $G(x) = \frac{x^4}{x^4+1}$ generates the system

$$x' = y, \qquad y' = -\frac{4x^3}{(x^4+1)^2}. \tag{6}$$

We take $I = \mathbb{R}$, $J = (-\sqrt{2}, \sqrt{2})$. The system (6) has a center at the origin, with central region contained in the strip $I \times J$. One has

$$G'^2 - 2GG'' = \frac{8x^6(5x^4 - 1)}{(x^4 + 1)^4}.$$

The right hand side is negative for $x \in (-1/5^{1/4}, 1/5^{1/4})$, and positive for $x \notin [-1/5^{1/4}, 1/5^{1/4}]$. Moreover, one has

$$N = 96x^8(15x^8 + 1)/(x^4 + 1)^7,$$

which is positive for $x \neq 0$. Also in this example $T'(s) < 0$ on the cycles contained in the strip $x \in [-1/5^{1/4}, 1/5^{1/4}]$, and T is strictly convex on the cycles meeting both the lines $x = \pm 1/5^{1/4}$. As a consequence, the system (6) has exactly one critical cycle, meeting both the lines $x = \pm 1/5^{1/4}$.

Remark 1. The above example shows that Theorem A in [1] cannot be extended to non-degenerate centers. In fact, the function $N(x)$ is positive everywhere but at 0 while T is strictly decreasing in a neighbourhood of the origin. The proof of Theorem A in [1] does not apply because the center of (6) is degenerate, and the change of variables on which the proof is based cannot be defined.

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Reçu par la Rédaction le 1.1.2005
Révisé le 10.2.2005 (1550)