ANNALES
POLONICI MATHEMATICI
114.2 (2015)

A regularity criterion for the 2D MHD and viscoelastic fluid equations

by Zhuan Ye (Beijing)

Abstract. This paper is dedicated to a regularity criterion for the 2D MHD equations and viscoelastic equations. We prove that if the magnetic field B, respectively the local deformation gradient F, satisfies

$$\nabla B,\ \nabla F \in L^q(0,T;L^p(\mathbb{R}^2))$$

for $1/p+1/q=1$ and $2<p\le\infty$, then the corresponding local solution can be extended beyond time T.

1. Introduction. In this paper, we consider the following 2D incompressible MHD equations with zero magnetic diffusivity:

$$\begin{cases} \partial_t u+(u\cdot\nabla)u-\Delta u+\nabla P=(B\cdot\nabla)B, & (x,t)\in\mathbb{R}^2\times(0,\infty),\\ \partial_t B+(u\cdot\nabla)B=(B\cdot\nabla)u, \\ \nabla\cdot u=0, \quad \nabla\cdot B=0, \\ u(x,0)=u_0(x), \quad B(x,0)=B_0(x), \end{cases} \tag{1.1}$$

where $u=u(x,t)\in\mathbb{R}^2$ denotes the velocity, $P=P(x,t)\in\mathbb{R}$ denotes scalar pressure and $B=B(x,t)\in\mathbb{R}^2$ denotes the magnetic field, while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla\cdot u_0(x)=0$ and $\nabla\cdot B_0(x)=0$, respectively.

For system (1.1), Jiu and Niu [JN] established local existence of solutions in 2D for initial data in H^s with $s\ge 3$. Very recently, the local well-posedness result in H^s with only $s>1$ was established by Fefferman et al. [FMRR]. Jiu and Niu [JN] also proved the following regularity condition:

$$B\in L^q(0,T;W^{2,p}(\mathbb{R}^2)), \quad 1/p+2/q\le 2, \quad 2<p\le\infty, \quad 1\le q\le 4/3.$$

Later, Zhou and Fan [ZF] obtained the regularity criterion

$$\nabla B\in L^1(0,T;\mathrm{BMO}(\mathbb{R}^2)).$$

2010 *Mathematics Subject Classification*: Primary 35Q35, 76A10; Secondary 35B44.
Key words and phrases: MHD equations, viscoelastic equations, regularity criterion.

DOI: 10.4064/ap114-2-3

For some other interesting regularity criteria, we refer the readers to [FO, LMZ, Ye].

2. Theorems. Before stating our results, let us say something about local smooth solutions to systems (1.1) and (2.2). A solution pair (u,B) is a local smooth solution of system (1.1) in the interval $[0,T]$ for $(u_0,B_0)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ provided that $(u,B)\in C(0,T;H^2(\mathbb{R}^2))\times C(0,T;H^2(\mathbb{R}^2))$. For system (2.2) the definition is the same.

Now our main theorem can be stated as follows:

Theorem 2.1. *Assume that* $(u_0,B_0)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ *with* $\nabla\cdot u_0=\nabla\cdot B_0=0$. *Let* (u,B) *be a local smooth solution of system* (1.1). *Suppose that*

$$\nabla B\in L^q(0,T;L^p(\mathbb{R}^2)), \tag{2.1}$$

for $1/p+1/q=1$ *and* $2<p\le\infty$. *Then the solution* (u,B) *remains smooth on* $[0,T]$.

Remark 2.2. It should be noted that system (1.1) has the following scaling property: if (u,B) is a solution of (1.1), then for any $\lambda>0$ the functions

$$u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t),\quad B_\lambda(x,t)=\lambda B(\lambda x,\lambda^2 t),$$

also yield a solution of (1.1) with the initial data $u_{0,\lambda}(x)=\lambda u_0(\lambda x)$, $B_{0,\lambda}(x)=\lambda B_0(\lambda x)$. It is important to note that the solution ∇B in the space $L^\alpha(0,T;L^\beta(\mathbb{R}^2))$ with $1/\alpha+1/\beta=1$ is scaling invariant, that is,

$$\|\nabla B(x,t)\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))}=\|\nabla B_\lambda(x,t)\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))}.$$

Thus under the regularity criterion (2.1) the gradient ∇B belongs to the invariant space.

The method may also be adapted with almost no change to the study of the following viscoelastic model:

$$\begin{cases}\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla P=\nabla\cdot(FF^T),\\ \partial_t F+(u\cdot\nabla)F=\nabla uF,\\ \nabla\cdot u=0,\quad \nabla\cdot F^T=0,\\ u(x,0)=u_0(x),\quad F(x,0)=F_0(x),\end{cases} \tag{2.2}$$

where $u(x,t):\mathbb{R}^2\times(0,\infty)\to\mathbb{R}^2$ is the unknown velocity field of the flow, $P(x,t):\mathbb{R}^2\times(0,\infty)\to\mathbb{R}$ is scalar pressure and $F(x,t):\mathbb{R}^2\times(0,\infty)\to\mathbb{M}_{2\times2}$ represents the local deformation gradient of the fluid ($\mathbb{M}_{2\times2}$ denotes the 2×2 matrices). One can consult [Lar, LLZ] for a detailed discussion on the derivation and physical background of the viscoelastic equations. Due to its physical applications and mathematical significance, system (2.2) has been extensively studied and important progress has been made (see e.g. [CZ, LLZ, LLZh] and the references therein). Recently, a large number of papers have been devoted to regularity criteria for the viscoelastic equations (2.2) (see e.g. [HH, QF, Yuan, YL] and the references therein).

It should be noted that

$$(\nabla\cdot F)_j=\partial_i F_{ji}\quad\text{and}\quad(\nabla\cdot F^T)_j=\partial_i F_{ij}=0.$$

Here and below we adopt the Einstein summation convention over repeated indices. The second equation of (2.2) reads

$$\partial_t F_{ij}+(u\cdot\nabla)F_{ij}=\partial_k u_i F_{kj},\quad i,j=1,\dots,n.$$

Letting $F_{.k}=Fe_k$ denote the columns of F we take the divergence of the second equation in (2.2) to arrive at

$$\partial_t(\nabla\cdot F_{.k})+(u\cdot\nabla)(\nabla\cdot F_{.k})=0.$$

Therefore, if $\nabla\cdot F_{.k}(x,0)=0$ initially, it will remain so for later times, namely $\nabla\cdot F_{.k}(x,t)=0$ for any $t>0$. Under this assumption, we can show

$$\nabla\cdot(FF^T)=\sum_{k=1}^n (F_{.k}\cdot\nabla)F_{.k}.$$

Moreover, it is easy to check that (2.2) is equivalent to

$$\begin{cases}\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla P=\sum_{k=1}^n (F_{.k}\cdot\nabla)F_{.k},\\ \partial_t F_{.k}+(u\cdot\nabla)F_{.k}=(F_{.k}\cdot\nabla)u,\quad k=1,\dots,n,\\ \nabla\cdot u=0,\quad \nabla\cdot F_{.k}=0,\\ u(x,0)=u_0(x),\quad F(x,0)=F_0(x).\end{cases}$$

Due to the similar structure to system (1.1), it is not difficult to show that the viscoelastic model (2.2) admits the same conclusion as Theorem 2.1. More precisely, we have

THEOREM 2.3. *Assume that* $(u_0,F_0)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ *with* $\nabla\cdot u_0=\nabla\cdot F_0^T=0$. *Let* (u,F) *be a local smooth solution of system* (2.2). *Suppose that*

$$\nabla F\in L^q(0,T;L^p(\mathbb{R}^2))\tag{2.3}$$

for $1/p+1/q=1$ *and* $2<p\le\infty$. *Then the solution* (u,F) *remains smooth on* $[0,T]$.

3. The proof of Theorem 2.1. Before proving our result, we point out that the local existence can be established without difficulty through the classical theory of symmetric hyperbolic quasi-linear systems, thus the proof is based on the establishment of *a priori* estimates. In this paper, all constants will be denoted by C; they depend only on the quantities specified in the context.

By the basic energy estimate, we easily get

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2\,d\tau \le C < \infty. \tag{3.1}$$

In order to get H^1 estimates, multiplying $(1.1)_1$ and $(1.1)_2$ by Δu and ΔB, respectively, after integration by parts and taking the divergence-free property into account, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 \\ = \int_{\mathbb{R}^2} (u\cdot\nabla B)\cdot\Delta B\,dx - \int_{\mathbb{R}^2}\{(B\cdot\nabla u)\cdot\Delta B + (B\cdot\nabla B)\cdot\Delta u\}\,dx \\ \triangleq N + K,\end{aligned} \tag{3.2}$$

where we have used

$$\int_{\mathbb{R}^2} (u\cdot\nabla u)\cdot\Delta u\,dx = 0.$$

Using the divergence-free condition as well as integration by parts, the terms N and K can be rewritten as follows:

$$N = -\int_{\mathbb{R}^2} \partial_i(u_k\partial_k B_j)\partial_i B_j\,dx = -\int_{\mathbb{R}^2} \partial_i u_k\partial_k B_j\partial_i B_j\,dx \tag{3.3}$$

and

$$\begin{aligned} K &= \int_{\mathbb{R}^2} \{\partial_i(B_k\partial_k u_j)\partial_i B_j + \partial_i(B_k\partial_k B_j)\partial_i u_j\}\,dx \\ &= \int_{\mathbb{R}^2} \{\partial_i B_k\partial_k u_j\partial_i B_j + \partial_i B_k\partial_k B_j\partial_i u_j\}\,dx \\ &\quad + \int_{\mathbb{R}^2} \{B_k\partial_k\partial_i u_j\partial_i B_j + B_k\partial_k\partial_i B_j\partial_i u_j\}\,dx \\ &= \int_{\mathbb{R}^2} \{\partial_i B_k\partial_k u_j\partial_i B_j + \partial_i B_k\partial_k B_j\partial_i u_j\}\,dx \end{aligned} \tag{3.4}$$

Invoking the Hölder inequality and the Gagliardo–Nirenberg inequality, we find

$$\begin{aligned} |N| + |K| &\le C\|\nabla u\|_{L^{2p/(p-2)}}\|\nabla B\|_{L^{4p/(p+2)}}^2 \\ &\le C\|\nabla u\|_{L^2}^{(p-2)/p}\|\Delta u\|_{L^2}^{2/p}\|\nabla B\|_{L^2}\|\nabla B\|_{L^p} \\ &\le \tfrac{1}{2}\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{(p-2)/(p-1)}\|\nabla B\|_{L^2}^{p/(p-1)}\|\nabla B\|_{L^p}^{p/(p-1)} \\ &\le \tfrac{1}{2}\|\Delta u\|_{L^2}^2 + C\|\nabla B\|_{L^p}^{p/(p-1)}(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2). \end{aligned} \tag{3.5}$$

Plugging (3.5) into (3.2) and absorbing the dissipative term, we get

$$(3.6)\qquad \frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 \le C\|\nabla B\|_{L^p}^{p/(p-1)}(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2).$$

Under the assumption of (2.1), by using the Gronwall inequality we obtain

$$(3.7)\qquad \|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 + \int_0^t \|\Delta u(s)\|_{L^2}^2\, ds \le (\|\nabla u_0\|_{L^2}^2 + \|\nabla B_0\|_{L^2}^2)\exp\Big[\int_0^t C\|\nabla B(s)\|_{L^p}^{p/(p-1)}\, ds\Big] \le C < \infty.$$

The estimates (3.1) and (3.7) imply that

$$(3.8)\qquad u, B \in L^\infty(0,T; L^q(\mathbb{R}^2)) \quad \text{for any } 2 \le q < \infty.$$

The Hölder inequality and the Gagliardo–Nirenberg inequality give

$$(3.9)\qquad \|u\cdot\nabla u\|_{L^{(p+2)/2}} \le C\|u\|_{L^{p(p+2)/(p-2)}}\|\nabla u\|_{L^p} \le C\|u\|_{L^{p(p+2)/(p-2)}}\|\nabla u\|_{L^2}^{2/p}\|\Delta u\|_{L^2}^{(p-2)/p}$$

and

$$(3.10)\qquad \|B\cdot\nabla B\|_{L^{(p+2)/2}} \le C\|B\|_{L^{p(p+2)/(p-2)}}\|\nabla B\|_{L^p}.$$

Noticing the bounds (3.1), (3.7) and (3.8), we can immediately obtain

$$(3.11)\qquad \begin{aligned} u\cdot\nabla u &\in L^{2p/(p-2)}(0,T; L^{(p+2)/2}(\mathbb{R}^2)),\\ B\cdot\nabla B &\in L^{p/(p-1)}(0,T; L^{(p+2)/2}(\mathbb{R}^2)). \end{aligned}$$

Recall the first equation of (1.1), namely

$$(3.12)\qquad \partial_t u - \Delta u + \nabla P = f := -(u\cdot\nabla)u + B\cdot\nabla B.$$

From (3.11), we know that

$$(3.13)\qquad f \in L^{p/(p-1)}(0,T; L^{(p+2)/2}(\mathbb{R}^2)).$$

Thanks to the divergence-free condition, we can deduce from (3.12) that

$$\nabla P = -\frac{\nabla\mathrm{div}}{-\Delta} f.$$

Here the notation $\frac{\nabla\mathrm{div}}{-\Delta}$ can be viewed as $\big(\frac{\nabla\mathrm{div}}{-\Delta}\big)_{i,j} = \frac{\partial_{x_i}\partial_{x_j}}{-\Delta} = \mathcal{R}_i\mathcal{R}_j$, where $\mathcal{R}_i$ is the standard Riesz operator.

Hence, we rewrite (3.12) as

$$(3.14)\qquad \partial_t u - \Delta u = \left(I + \frac{\nabla\mathrm{div}}{-\Delta}\right) f.$$

Now we recall the following maximal $L_t^q L_x^p$ regularity for the heat kernel (see, e.g., [Lem]), Proposition 3.1. *The operator A defined by*

$$Af(x,t) \triangleq \int_0^t e^{(t-s)\Delta} \Delta f(s,x)\, ds$$

is bounded from $L^p(0,T;L^q(\mathbb{R}^n))$ to $L^p(0,T;L^q(\mathbb{R}^n))$ for every $(p,q) \in (1,\infty)\times(1,\infty)$ and $T \in (0,\infty]$.

Applying the operator Δ to equality (3.14), one arrives at

$$\partial_t(\Delta u) - \Delta(\Delta u) = \Delta\Big(I + \frac{\nabla \mathrm{div}}{-\Delta}\Big) f.$$

By Duhamel's Principle, the above equation can be solved by

$$\Delta u(x,t) = e^{t\Delta}\Delta u_0(x) + \int_0^t e^{(t-s)\Delta}\Delta\Big(I + \frac{\nabla \mathrm{div}}{-\Delta}\Big) f(x,s)\, ds. \tag{3.15}$$

By Proposition 3.1, one can conclude from (3.15) that (with $H(t,x) = \frac{1}{2\pi t}\exp\big[-\frac{|x|^2}{4t}\big]$)

$$\begin{aligned}
\|\Delta u\|_{L_T^{p/(p-1)} L_x^{(p+2)/2}} &\le \|e^{t\Delta}\Delta u_0\|_{L_T^{p/(p-1)} L_x^{(p+2)/2}} \\
&\quad + \Big\| \int_0^t e^{(t-s)\Delta}\Delta\Big(I + \frac{\nabla \mathrm{div}}{-\Delta}\Big) f(x,s)\, ds \Big\|_{L_T^{p/(p-1)} L_x^{(p+2)/2}} \\
&\le C\|H(t,x)\|_{L_T^{p/(p-1)} L_x^{(2p+4)/(p+6)}} \|\Delta u_0\|_{L_x^2} \\
&\quad + C\Big\| \Big(I + \frac{\nabla \mathrm{div}}{-\Delta}\Big) f(x,s)\Big\|_{L_T^{p/(p-1)} L_x^{(p+2)/2}} \\
&\le C(T)\|u_0\|_{H^2} + C\|f\|_{L_T^{p/(p-1)} L_x^{(p+2)/2}} \le C < \infty,
\end{aligned} \tag{3.16}$$

where in the third inequality we have used the boundedness of the Calderón–Zygmund operator between the $L_x^{(p+2)/2}$ $(1 < p < \infty)$ spaces. The estimates (3.7) and (3.16) imply that

$$u \in L^{p/(p-1)}(0,T; W^{2,(p+2)/2}(\mathbb{R}^2)). \tag{3.17}$$

As $p > 2$, we have

$$\nabla u \in L^1(0,T; L^\infty(\mathbb{R}^2)). \tag{3.18}$$

The global H^1-bound (3.7) and the estimate (3.18) allow us to derive the H^2-bound for u and B. Now we apply Λ^2 $(\Lambda := (-\Delta)^{1/2})$ to system (1.1) and multiply the resulting equations by $\Lambda^2 u$ and $\Lambda^2 B$ respectively and add them up to obtain

$$
\begin{aligned}
(3.19)\qquad \frac{1}{2}\frac{d}{dt}(\|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^2 B(t)\|_{L^2}^2) + \|\Lambda^3 u\|_{L^2}^2 \\
= -\int_{\mathbb{R}^2} [\Lambda^2, u\cdot\nabla]u\cdot\Lambda^2 u\,dx + \int_{\mathbb{R}^2} [\Lambda^2, B\cdot\nabla]B\cdot\Lambda^2 u\,dx \\
- \int_{\mathbb{R}^2} [\Lambda^2, u\cdot\nabla]B\cdot\Lambda^2 B\,dx + \int_{\mathbb{R}^2} [\Lambda^2, B\cdot\nabla]u\cdot\Lambda^2 B\,dx \\
:= J_1 + J_2 + J_3 + J_4;
\end{aligned}
$$

here and in what follows, $[\Lambda^2, f]g$ stands for the standard commutator notation, $[\Lambda^2, f]g = \Lambda^2(fg) - f\Lambda^2 g$. Moreover, we have used the identities

$$
\int_{\mathbb{R}^2} u\cdot\nabla\Lambda^2 u\cdot\Lambda^2 u\,dx = \int_{\mathbb{R}^2} u\cdot\nabla\Lambda^2 B\cdot\Lambda^2 B\,dx = 0,
$$

$$
\int_{\mathbb{R}^2} B\cdot\nabla\Lambda^2 B\cdot\Lambda^2 u\,dx + \int_{\mathbb{R}^2} B\cdot\nabla\Lambda^2 u\cdot\Lambda^2 B\,dx = 0.
$$

In order to estimate the terms J_1–J_4, we need the following bilinear commutator estimate (see [KP]):

$$
(3.20)\qquad \|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),
$$

with $s > 0$ and $p_2, p_3 \in (1, \infty)$ such that

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
$$

According to inequality (3.20), we have

$$
J_1 \le \|[\Lambda^2, u\cdot\nabla]u\|_{L^2}\|\Lambda^2 u\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|\Lambda^2 u\|_{L^2}^2.
$$

The Gagliardo–Nirenberg inequality as well as inequality (3.20) allow us to show that

$$
\begin{aligned}
J_2 &\le \|[\Lambda^2, B\cdot\nabla]B\|_{L^{4/3}}\|\Lambda^2 u\|_{L^4} \\
&\le C\|\nabla B\|_{L^4}\|\Lambda^2 B\|_{L^2}\|\Lambda^2 u\|_{L^4} \\
&\le C\|\nabla B\|_{L^2}^{1/2}\|\Lambda^2 B\|_{L^2}^{1/2}\|\Lambda^2 B\|_{L^2}\|\Lambda^2 u\|_{L^2}^{1/2}\|\Lambda^3 u\|_{L^2}^{1/2} \\
&\le \tfrac{1}{8}\|\Lambda^3 u\|_{L^2}^2 + C\|\nabla B\|_{L^2}^{2/3}\|\Lambda^2 u\|_{L^2}^{2/3}\|\Lambda^2 B\|_{L^2}^2, \\
J_3, J_4 &\le \|[\Lambda^2, u\cdot\nabla]B\|_{L^2}\|\Lambda^2 B\|_{L^2} + \|[\Lambda^2, B\cdot\nabla]u\|_{L^2}\|\Lambda^2 B\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^2 B\|_{L^2}^2 + C\|\nabla B\|_{L^4}\|\Lambda^2 u\|_{L^4}\|\Lambda^2 B\|_{L^2} \\
&\le \tfrac{1}{8}\|\Lambda^3 u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\Lambda^2 B\|_{L^2}^2 + C\|\nabla B\|_{L^2}^{2/3}\|\Lambda^2 u\|_{L^2}^{2/3}\|\Lambda^2 B\|_{L^2}^2.
\end{aligned}
$$

Combining all the above estimates, we immediately arrive at

$$\begin{aligned}(3.21)\qquad \frac{d}{dt}(\|\Lambda^2u(t)\|_{L^2}^2+\|\Lambda^2B(t)\|_{L^2}^2)+\|\Lambda^3u\|_{L^2}^2 &\\ \le C\|\nabla u\|_{L^\infty}(\|\Lambda^2u\|_{L^2}^2+\|\Lambda^2B\|_{L^2}^2)&\\ +C\|\nabla B\|_{L^2}^{2/3}\|\Lambda^2u\|_{L^2}^{2/3}\|\Lambda^2B\|_{L^2}^2.&\end{aligned}$$

The following H^2-bound is an easy consequence of the Gronwall inequality and the key bound (3.18):

$$(3.22)\qquad \max_{0\le t\le T}(\|u(t)\|_{H^2}+\|B(t)\|_{H^2})<\infty.$$

Therefore, the solution (u,θ) remains smooth on $[0,T]$. Thus, the proof of Theorem 2.1 is complete. ■

Acknowledgements. The author is grateful to the anonymous referees and the editor for their constructive comments and helpful suggestions that have contributed to the final preparation of the paper. The author was partially supported by NSFC (No. 11371059), BNSF (No. 2112023) and the Fundamental Research Funds for the Central Universities of China.

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Zhuan Ye
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875, People's Republic of China
E-mail: yezhuan815@126.com

Received 2.10.2014
and in final form 24.5.2015 (3509)