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Blow-up for a localized singular parabolic equation with weighted nonlocal nonlinear boundary conditions

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Abstract. This paper deals with the blow-up properties of positive solutions to a localized singular parabolic equation with weighted nonlocal nonlinear boundary conditions. Under certain conditions, criteria of global existence and finite time blow-up are established. Furthermore, when $q = 1$, the global blow-up behavior and the uniform blow-up profile of the blow-up solution are described; we find that the blow-up set is the whole domain $[0, a]$, including the boundary, in contrast to the case of parabolic equations with local sources or with homogeneous Dirichlet boundary conditions.

1. Introduction. In this paper we consider the following localized singular parabolic equation with weighted nonlocal nonlinear boundary conditions:

$$(1.1)\qquad \begin{aligned} &u_t = (x^\alpha u_x)_x + u^p(x_0,t), \qquad x \in (0,a),\ t > 0,\\ &u(0,t) = \int_0^a f(x)u^q(x,t)\,dx, \quad u(a,t) = \int_0^a g(x)u^q(x,t)\,dx, \quad t > 0,\\ &u(x,0) = u_0(x), \qquad x \in [0,a], \end{aligned}$$

where $0 \le \alpha < 1$, a, p and q are positive constants, $x_0 \in (0,a)$ is a fixed point, and $f(x)$ and $g(x)$ are continuous, nonnegative and not identically zero on $[0,a]$.

The equation in (1.1) arises in a large number of physical phenomena. For example, it can be used to describe the heat conduction related to the geometric shape of the body with an internal localized source (see [CC] and the references therein for more details of the physical background). Note that problem (1.1) is singular and degenerate because the coefficients of u_x and u_{xx} tend respectively to ∞ and 0 as $x \to 0$.

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Blow-up singularity, as one of the most remarkable properties that distinguish nonlinear parabolic problems from linear ones, attracted extensive attention of mathematicians in the past few decades. There are many works on global existence and blow-up properties of various degenerate and singular parabolic equations (or systems) with homogeneous Dirichlet boundary conditions (see [BGC, CLX1, CLX2, LCX, Zh] and the references therein).

On the other hand, parabolic equations with nonlocal (or nonlocal nonlinear) boundary conditions come from applied sciences; for instance, in the study of heat conduction with thermoelasticity, Day [Da1, Da2] derived a class of heat equations with nonlocal boundary conditions in one-dimensional space. In this model, the solution $u(x,t)$ describes the entropy of per volume material. Motivated by the work of Day, many mathematicians have recently studied the blow-up behavior of different kinds of parabolic equations with nonlocal boundary conditions. The problem of nonlocal boundary conditions in a multidimensional space for linear parabolic equations of the type

$$(1.2)\qquad \begin{aligned} &u_t - Au = c(x)u, && x\in\Omega,\ t>0,\\ &u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, && x\in\partial\Omega,\ t>0,\\ &u(x,0)=u_0(x), && x\in\Omega, \end{aligned}$$

with a uniformly elliptic operator $A=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}-\sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$ and $c(x)\le 0$ was studied by Friedman [Fr1]. The global existence and monotonic decay of the solution of problem (1.2) were obtained under the condition that $\int_\Omega |k(x,y)|\,dy<1$ for all $x\in\partial\Omega$. Later, problem (1.2) with Au replaced by Δu and the linear term $c(x)u$ replaced by a nonlinear term $g(x,u)$ was discussed by Deng [De]. A comparison principle and local existence were established. On the basis of Deng's work, Seo [Se] investigated the above problem with $g(x,u)=g(u)$; by using the upper and lower solutions technique; he obtained a blow-up criterion for positive solutions, and in the special case $g(u)=u^p$ or $g(u)=e^u$ he also derived blow-up rate estimates.

Parabolic equations with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, the problem

$$(1.3)\qquad \begin{aligned} &u_t - \Delta u = \int_\Omega g(u)\,dx, && x\in\Omega,\ t>0,\\ &u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, && x\in\partial\Omega,\ t>0,\\ &u(x,0)=u_0(x), && x\in\Omega, \end{aligned}$$

was studied by Lin and Liu [LL]. They established local existence, global existence and nonexistence results, and discussed the blow-up properties of solutions. For other works on this topic, we refer the readers to [Pa1, Pa2, WMX, CY, GG1, CL] and the references therein.

However, as far as we know, there are only a few articles concerning the blow-up behavior of solutions of parabolic equations with nonlocal nonlinear boundary conditions. Gladkov and Kim [GK1, GK2] considered

$$\begin{aligned} &u_t = \Delta u + c(x,t)u^p, && x \in \Omega,\ t > 0,\\ &u(x,t) = \int_\Omega K(x,y,t)u^l(y,t)\,dy, && x \in \partial\Omega,\ t > 0,\\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.4}$$

where $p, l > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. First, they obtained uniqueness and nonuniqueness results for local solutions (see [GK2]), then according to the different behavior of the coefficient function $c(x,t)$ and the weight function $K(x,y,t)$ as t tends to infinity, they gave some criteria for solutions of (1.4) to exist globally or to blow up in finite time (see [GK1]). Recently, Gladkov and Guedda studied problem (1.4) with $c(x,t)u^p$ replaced by $-c(x,t)u^p$. They proved existence, uniqueness and nonuniqueness results for local solutions (see [GG3]). What is more, they gave the critical blow-up exponent (see [GG2]).

The main goal of this paper is to investigate the influence of α, p, q and the weight functions $f(x)$ and $g(x)$ on the global existence and blow-up singularity of solutions to problem (1.1). Compared with [GK1] and [LL], we need more techniques to solve the difficulties, which are produced by the degeneracy and singularity of problem (1.1) and the appearance of nonlocal nonlinear boundary conditions. Throughout this paper, we denote

$$B = \max\Big\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\Big\}, \tag{1.5}$$

and let λ_1 be the first eigenvalue and $\xi(x)$ be the corresponding eigenfunction of the eigenvalue problem

$$-(x^\alpha \xi_x)_x = \lambda\xi, \quad 0 < x < a; \qquad \xi(0) = \xi(a) = 0. \tag{1.6}$$

From [CLX1, Mc], we know that the principal eigenvalue λ_1 of (1.6) is the first root of $J_{\frac{1-\alpha}{2-\alpha}}\big(\frac{2\sqrt{\lambda}}{2-\alpha}a^{\frac{2-\alpha}{2}}\big)$, where $J_{\frac{1-\alpha}{2-\alpha}}$ is the Bessel function of the first kind of order $\frac{1-\alpha}{2-\alpha}$. In addition, $\xi(x)$ is a positive smooth function in $(0,a)$, and can be expressed in explicit form as

$$\xi(x) = kx^{(1-\alpha)/2} J_{\frac{1-\alpha}{2-\alpha}}\bigg(\frac{2\sqrt{\lambda}}{2-\alpha}x^{(2-\alpha)/2}\bigg), \tag{1.7}$$

where k is an arbitrary positive parameter. Here, for the sake of convenience, we choose k such that $\int_0^a \xi(x)\,dx = 1$.

Before stating our results, we make some assumptions on the weight functions $f(x), g(x)$ and the initial datum u_0.

- (H_1) $f(x), g(x)$ are continuous and nonnegative on $[0, a]$ with $\int_0^a f(x)\,dx > 0$ and $\int_0^a g(x)\,dx > 0$.
- (H_2) $u_0 \in C^{2+\gamma}(0, a) \cap C[0, a]$ for some $\gamma \in (0, 1)$, $u_0(x) > 0$ in $(0, a)$, $u_0(0) = \int_0^a f(x)u_0(x)\,dx$ and $u_0(a) = \int_0^a g(x)u_0(x)\,dx$.
- (H_3) $(x^\alpha u_{0x})_x \le M$ in $(0, a)$ for some positive constant M.

Our main results read as follows:

Theorem 1.1. *Suppose that $f(x)$, $g(x)$ and $u_0(x)$ satisfy* (H_1) *and* (H_2), *and that* $\max\{p, q\} \le 1$*; if $q = 1$, we further assume that $B < 1$. Then there exists a global solution $u(x, t)$ of problem* (1.1).

Theorem 1.2. *Let hypotheses* (H_1) *and* (H_2) *hold and let $B < 1$. If* $\min\{p, q\} > 1$ *or $p > 1$, $q = 1$, and $u_0(x)$ is sufficiently small, then there exists a global solution $u(x, t)$ of problem* (1.1).

Remark 1.3. In the case $p > 1$ and $q < 1$ (or $q > 1$ and $p \le 1$), we guess that there exists a global solution of problem (1.1) for small initial data, but we cannot prove this with the method of this paper. We hope to address this question in the future.

Theorem 1.4. *Let hypotheses (H_1) and (H_2) hold, and assume that* $\max\{p,\ q\} > 1$.

- (i) *If $q = \max\{p, q\}$ and $g(x) > 0$ on $[0, a]$, then the solution of problem* (1.1) *blows up in finite time provided that $u_0(x)$ is sufficiently large.*
- (ii) *If $p = \max\{p, q\}$ and $q \ge 1$, then the solution of problem* (1.1) *blows up in finite time provided that $u_0(x)$ is sufficiently large.*

Theorem 1.5. *Let hypotheses* (H_1), (H_2) *and* (H_3) *hold, and let $p > 1$, $q = 1$ and $B \le 1$. If the solution $u(x, t)$ of problem* (1.1) *blows up in finite time, then the blow-up set of $u(x, t)$ is the whole domain $[0, a]$. Furthermore, if we denote the blow-up time of the solution $u(x, t)$ of* (1.1) *by T^*, then in the interior of* $(0, a)$,

$$\lim_{t \to T^*} (T^* - t)^{1/(p-1)} u(x, t) = (p - 1)^{-1/(p-1)} \tag{1.8}$$

uniformly on any compact subset of $(0, a)$*; and on the boundary, we have*

$$\begin{aligned} \lim_{t \to T^*} (T^* - t)^{1/(p-1)} u(0, t) &= (p - 1)^{-1/(p-1)} \int_0^a f(x)\,dx, \\ \lim_{t \to T^*} (T^* - t)^{1/(p-1)} u(a, t) &= (p - 1)^{-1/(p-1)} \int_0^a g(x)\,dx. \end{aligned} \tag{1.9}$$

This paper is organized as follows. In Section 2, we show a comparison principle and local existence. In Section 3, some criteria for a positive solution to exist globally or to blow up in finite time are given. In Section 4, a global blow-up result and the asymptotic behavior of the blow-up solution for the special case of $p > 1$, $q = 1$, $B \leq 1$ are obtained.

2. The comparison principle and local existence. In this section we first establish a suitable comparison principle, and then state the existence and uniqueness results for local solutions of problem (1.1). For convenience, we set $Q_T = (0, a) \times (0, T]$ and $\overline{Q}_T = [0, a] \times [0, T]$. We start with the definitions of supersolution and subsolution of problem (1.1).

Definition 2.1. A function $\hat{u}(x, t)$ is called a *subsolution* of problem (1.1) in Q_T if $\hat{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$
\tag{2.1}
\begin{aligned}
&\hat{u}_t \leq (x^\alpha \hat{u}_x)_x + \hat{u}^p(x_0, t), && (x, t) \in Q_T,\\
&\hat{u}(0, t) \leq \int_0^a f(x)\hat{u}(x, t)\, dx, && t \in (0, T],\\
&\hat{u}(a, t) \leq \int_0^a g(x)\hat{u}(x, t)\, dx, && t \in (0, T],\\
&\hat{u}(x, 0) \leq u_0(x), && x \in [0, a].
\end{aligned}
$$

Similarly, $\tilde{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a *supersolution* of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x, t)$ is a *solution* of problem (1.1) if it is both a subsolution and a supersolution of problem (1.1).

Before studying our problem, we give the following maximum principle.

Lemma 2.2. *Assume that $w \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies*

$$
\tag{2.2}
\begin{aligned}
&w_t - (x^\alpha w_x)_x \geq c_1(x, t) w(x_0, t), && (x, t) \in Q_T,\\
&w(0, t) \geq \int_0^a c_2(x, t) w(x, t)\, dx, && t \in (0, T],\\
&w(a, t) \geq \int_0^a c_3(x, t) w(x, t)\, dx, && t \in (0, T],\\
&w(x, 0) > 0, && x \in [0, a],
\end{aligned}
$$

where $c_i(x, t)$ $(i = 1, 2, 3)$ is nonnegative and bounded in Q_T, and $c_i(x, t)$ $(i = 2, 3)$ is not identically zero. Then $w(x, t) > 0$ in $\overline{Q}_T$.

Proof. This can be proved in the same way as [ZK, Lemma 2.1]; we give the proof for completeness. Since $w(x, 0) > 0$ and $w \in C(\overline{Q}_T)$, there exists a constant $\delta > 0$ such that $w(x,t) > 0$ for $(x,t) \in [0,a] \times [0,\delta]$. Set

$$\bar{t} = \sup\{t' \in (0,T] : w(x,t) > 0 \text{ for } (x,t) \in [0,a] \times [0,t']\}.$$

Then $\bar{t} \geq \delta > 0$. We claim that $\bar{t} = T$. In fact, suppose for contradiction that $\bar{t} < T$; then $w(x,t) > 0$ for $(x,t) \in [0,a]\times[0,\bar{t})$ and $w(x,t) \geq 0$ for $(x,t) \in \overline{Q}_{\bar{t}}$. Hence there exists some $\overline{x} \in [0,a]$ such that $w(\overline{x},\bar{t}) = 0 = \inf_{(x,t)\in\overline{Q}_{\bar{t}}} w(x,t)$. If $(\overline{x},\bar{t}) \in Q_{\bar{t}}$, then by the first inequality in (2.2) and the nonnegativity of $c_1(x,t)$ and $w(x,t)$ on $\overline{Q}_{\bar{t}}$, we find that

$$w_t - (x^\alpha w_x)_x \geq c_1(x,t)w(x_0,t) \geq 0, \qquad (x,t) \in Q_{\bar{t}}.$$

Then by the strong maximum principle for parabolic equations (see [Fr2, Chapter 2, Theorems 1 and 5]), we have $w(x,t) \equiv 0$ for $(x,t) \in Q_{\bar{t}}$, a contradiction. If $\overline{x} = 0$ or a, this also leads to a contradiction that

$$0 = w(0,\bar{t}) \geq \int_0^a c_2(x,\bar{t})w(x,\bar{t})\,dx > 0$$

or

$$0 = w(a,\bar{t}) \geq \int_0^a c_3(x,\bar{t})w(x,\bar{t})\,dx > 0,$$

due to $c_2(x,t)$ and $c_3(x,t)$ being nonnegative and not identically zero. This proves $w(x,t) > 0$ in $\overline{Q}_T$. ∎

Remark 2.3. If one of the following conditions holds:

(i) $c_2(x,t) = c_3(x,t) \equiv 0$ for $x \in (0,a)$, $t > 0$,
(ii) $c_i(x,t) \geq 0$, $x \in (0,a)$, $t > 0$, $i = 2,3$, and $\max\{\int_0^a c_2(x,t)\,dx, \int_0^a c_3\ (x,t)\,dx\} \leq 1$, $t > 0$,

and $w(x,t)$ satisfies all the inequalities in (2.2) except the third one, which is replaced by $w(x,0) \geq 0$ on $[0,a]$, then we also have $w(x,t) \geq 0$ in Q_T.

In order to get global existence and finite time blow-up results for problem (1.1), we still need the following comparison principle which is a direct consequence of Lemma 2.2 and Remark 2.3.

Lemma 2.4. *Assume that $\tilde{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a nonnegative supersolution of problem* (1.1) *and $\hat{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a nonnegative subsolution of problem* (1.1). *If $\min\{p,q\} \leq 1$, suppose moreover that there exists a small positive constant η such that $\tilde{u}(x,t) \geq \eta$ on $\overline{Q}_T$ or $\hat{u}(x,t) \geq \eta$ on $\overline{Q}_T$. And assume that $\tilde{u}(x,0) > \hat{u}(x,0)$ on $[0,a]$ or $\tilde{u}(x,0) \geq \hat{u}(x,0)$ on $[0,a]$ if $\max\{\int_0^a f(x)\,dx, \int_0^a f(x)\,dx\} \leq 1$. Then $\tilde{u}(x,t) \geq \hat{u}(x,t)$ on $\overline{Q}_T$.*

Local in time existence of a positive classical solution of problem (1.1) can be obtained by using the regularization method, the representation formula and the fixed point theorem as in [CLX2, Yi]. By the above comparison principle, we can get the uniqueness of the solution to problem (1.1), and then we have

THEOREM 2.5. *Let hypotheses* (H_1) *and* (H_2) *hold. Then there exist* T^* $(0 < T^* \le \infty)$ *and* $u \in C([0,a]\times[0,T^*))\cap C^{2,1}((0,a)\times(0,T^*))$ *such that* $u(x,t)$ *is the unique maximal solution of problem* (1.1). *If* $T^* < \infty$, *then*

$$\limsup_{t\to T^*} \max_{x\in[0,a]} u(x,t) = \infty.$$

The proof is more or less standard, and is therefore omitted here.

3. Global existence and finite time blow-up criteria. In this section, first of all, by constructing some appropriate global supersolutions and using the comparison principle, we obtain the existence of global solutions for problem (1.1), and give the proofs of Theorems 1.1 and 1.2, respectively.

Proof of Theorem 1.1. Consider the following boundary value problem for an ordinary differential equation:

$$\begin{gathered} -(x^\alpha\phi'(x))' = \eta_0, \qquad x\in(0,a), \\ \phi(0) = \int_0^a f(x)\,dx, \qquad \phi(a) = \int_0^a g(x)\,dx, \end{gathered} \tag{3.1}$$

where η_0 is a fixed positive constant, and if $B = \max\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\} < 1$, it is small enough so that the solution of problem (3.1) satisfies $\phi(x) \le 1$ on $[0,a]$. We can easily solve this problem and obtain its solution

$$\begin{aligned} \phi(x) = {} & -\frac{\eta_0}{2-\alpha}x^{2-\alpha} \\ & + \left[\frac{a\eta_0}{2-\alpha} + \frac{1}{a^{1-\alpha}}\int_0^a (g(x)-f(x))\,dx\right]x^{1-\alpha} + \int_0^a f(x)\,dx. \end{aligned} \tag{3.2}$$

According to the elliptic maximum principle (see [Ev, Chapter 2]), we know that the solution $\phi(x)$ of problem (3.1) is positive on $[0,a]$. Let $A_1 = \max_{x\in[0,a]}\phi(x)$ and $A_2 = \min_{x\in[0,a]}\phi(x)$. Then $A_1 > A_2 > 0$, and $A_1 \le 1$ when $B < 1$. Set $\sigma = \phi(x_0)/A_2$ and choose M sufficiently large such that $M \ge (A_1)^{q/(1-q)}$ if $q < 1$ and $M\phi(x) > u_0(x) + 1$. Set $v_1(x,t) = M\phi(x)\exp(\sigma t)$. Then it follows from the choice of M that $v_1(x,t) \ge 1$ for all $x \in [0,a]$, $t \ge 0$ and

$$v_1(x,0) = M\phi(x) > u_0(x), \qquad x\in[0,a]. \tag{3.3}$$

Noticing that $p \le 1$ and that $v_1(x,t) \ge 1$ on $[0,a]\times[0,\infty)$, it follows from (3.1) and the choice of σ that

$$\begin{aligned} v_{1t} - (x^\alpha v_{1x})_x - v_1^p(x_0,t) &= \sigma v_1 + M\eta_0\exp(\sigma t) - M^p\phi^p(x_0)\exp(p\sigma t) \\ &\ge \sigma v_1 - v_1\phi(x_0)A_2^{-1} \ge 0, \qquad x\in(0,a),\ t>0. \end{aligned} \tag{3.4}$$

On the other hand, again utilizing (3.1) and the fact that $v_1(x,t) \geq 1$ on $[0,a] \times [0,\infty)$, noting the choice of M for $q<1$, and noting that $B<1$ and $\phi(x) \leq 1$ on $[0,a]$ for $q=1$, we obtain

$$
\begin{aligned}
v_1(0,t) &= M\phi(0)\exp(\sigma t) = M\int_0^a f(x)\,dx\exp(\sigma t) \\
&\geq \begin{cases} M^q A_1^q \int_0^a f(x)\,dx\exp(q\sigma t) & \text{if } q<1, \\ M\int_0^a f(x)\phi(x)\,dx\exp(\sigma t) & \text{if } q=1 \end{cases} \qquad (3.5) \\
&\geq \int_0^a f(x)v_1^q(x,t)\,dx, \qquad t>0.
\end{aligned}
$$

Similarly, we can also obtain

$$
v_1(a,t) \geq \int_0^a g(x)v_1^q(x,t)\,dx, \qquad t>0. \tag{3.6}
$$

From (3.3)–(3.6), we know that $v_1(x,t)$ is a global supersolution of problem (1.1). Noticing $v_1(x,t) \geq 1$ on $[0,a]\times[0,\infty)$ and (3.3), by the comparison principle (Lemma 2.4), we get the global existence result for problem (1.1). ∎

In order to prove Theorem 1.2, we first give the following lemma.

Lemma 3.1. *Let hypothesis* (H$_1$) *hold, and assume that*

$$
\begin{aligned}
&\int_0^a f(x)x^{1-\alpha}\,dx\Big[\int_0^a g(x)\,dx - 1\Big] \\
&\qquad \neq \Big[\int_0^a g(x)x^{1-\alpha}\,dx - a^{1-\alpha}\Big]\Big[\int_0^a f(x)\,dx - 1\Big].
\end{aligned} \tag{3.7}
$$

Then there exists a unique solution to the elliptic problem

$$
\begin{aligned}
&-(x^\alpha\psi'(x))' = 1, \qquad x\in(0,a), \\
&\psi(0) = \int_0^a f(x)\psi(x)\,dx, \qquad \psi(a) = \int_0^a g(x)\psi(x)\,dx.
\end{aligned} \tag{3.8}
$$

Moreover, if $\max\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\} < 1$, *then* $\psi(x)$ *is positive.*

Proof. It is easy to verify that

$$
\psi(x) = -\frac{1}{2-\alpha}x^{2-\alpha} + c_1x^{1-\alpha} + c_2
$$

is the general solution of the equation in (3.8). Substituting this expression into the boundary conditions in (3.8), and noting condition (3.7), we can determine the constants c_1 and c_2 uniquely, and therefore under condition (3.7), there exists a unique solution $\psi(x)$ to problem (3.8). Furthermore, if $\max\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\} < 1$, then condition (3.7) holds, so according to the elliptic maximum principle (see [Ev, Chapter 2]), the solution $\psi(x)$ of problem (3.8) is unique and positive on $[0,a]$. ∎

Proof of Theorem 1.2. Let $\psi(x)$ be a solution of problem (3.8). From Lemma 3.1 we know that under the hypothesis

$$B = \max\Big\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\Big\} < 1$$

in the theorem, $\psi(x)$ is unique and positive on $[0,a]$. Let $K_1 = \max_{x\in[0,a]} \psi(x)$ and $K_2 = \min_{x\in[0,a]} \psi(x)$. Then $K_1 > K_2 > 0$. Let

$$M = \min\{\psi(x_0)^{-p/(p-1)}, K_1^{-1}\},$$

and set

$$v_2(x,t) = M\psi(x). \tag{3.9}$$

Noting that $M \le \psi(x_0)^{-p/(p-1)}$, for $x \in \Omega$ and $t > 0$ we have

$$v_{2t} - (x^\alpha v_{2x})_x - v_2^p(x_0,t) = M - M^p\psi^p(x_0) \ge 0. \tag{3.10}$$

On the other hand, since $\psi(x)$ is the solution of problem (3.8) and $M\psi(x) \le 1$, we have

$$\begin{aligned} v_2(0,t) &= M\psi(0) = M\int_0^a f(x)\psi(x)\,dx \ge \int_0^a f(x)v_2^q(x,t)\,dx,\\ v_2(a,t) &= M\psi(a) = M\int_0^a g(x)\psi(x)\,dx \ge \int_0^a g(x)v_2^q(x,t)\,dx. \end{aligned} \tag{3.11}$$

Then from (3.10) and (3.11) we deduce that $v_2(x,t)$ is a supersolution of problem (1.1) provided that $u_0(x) \le M\psi(x)$. Since $v_2(x,t) \ge MK_2 > 0$, $v_2(x,0) \ge u_0(x)$, $B < 1$, and $v_2(x,t)$ exists globally, from Lemma 2.4 we find that $u(x,t) \le v_2(x,t)$. Thus $u(x,t)$ exists globally. ∎

Next, by using a slight variant of the eigenfunction method (Kaplan's method), introduced by Kaplan [Ka], we will discuss the blow-up in finite time for problem (1.1) with $\max\{p,q\} > 1$ and sufficiently large initial data.

Proof of Theorem 1.4. Let $\xi(x)$ be the first eigenfunction of the eigenvalue problem (1.6) whose expression is given by (1.7).

Case (i): $q = \max\{p,q\} > 1$ and $g(x) > 0$ on $[0,a]$. Set $U(t) = \int_0^a u(x,t)\xi(x)\,dx$. Multiplying both sides of the equation in problem (1.1) by $\xi(x)$, and integrating the resulting equation over $[0,a]$ with respect to x, we get

$$U'(t) = \int_0^a [(x^\alpha u_x)_x + u^p(x_0, t)]\xi(x)\,dx$$
$$\geq -\lambda_1 U(t) + u^p(x_0, t) - a^\alpha \xi_x(a) \int_0^a g(x) u^q(x, t)\,dx, \qquad t > 0,$$

where λ_1 is the first eigenvalue of (1.6). As $\xi_x(a) < 0$, using Jensen's inequality, from the above inequality we get

$$U'(t) \geq -\lambda_1 U(t) - \frac{a^\alpha \xi_x(a) \min_{x\in[0,a]} g(x)}{\max_{x\in[0,a]} \xi(x)} \int_0^a u^q(x,t)\xi(x)\,dx$$
$$\geq -\lambda_1 U(t) - \frac{a^\alpha \xi_x(a) \min_{x\in[0,a]} g(x)}{\max_{x\in[0,a]} \xi(x)} U^q(t). \tag{3.12}$$

Denote $C_0 = -a^\alpha \xi_x(a) \min_{x\in[0,a]} g(x)/\max_{x\in[0,a]} \xi(x)$. Then $C_0 > 0$. Solving inequality (3.12), we obtain

$$U(t) \geq \left\{ \frac{\lambda_1}{C_0 - [C_0 - \lambda_1 U^{1-q}(0)] e^{\lambda_1 (q-1)t}} \right\}^{1/(q-1)}. \tag{3.13}$$

From (3.13), we see that if

$$U(0) = \int_0^a u_0(x)\xi(x)\,dx > \left(\frac{\lambda_1}{C_0}\right)^{1/(q-1)}, \tag{3.14}$$

then $\lim_{t\to T_U^*} U(t) = \infty$, where

$$T_U^* \leq \frac{1}{\lambda_1(q-1)} \ln \frac{C_0 U^{q-1}(0)}{C_0 U^{q-1}(0) - \lambda_1}. \tag{3.15}$$

Therefore the solution $u(x,t)$ of problem (1.1) blows up in finite time in the case $q = \max\{p, q\} > 1$ provided the initial data $u_0(x)$ is sufficiently large so that (3.14) holds.

Case (ii): $p = \max\{p, q\} > 1$ and $q \geq 1$. Let $C_1 = \xi^p(x_0)/\max_{x\in[0,a]} \xi(x)$. Then $C_1 > 0$. Let $h(t)$ be the solution of the ordinary differential equation

$$\begin{aligned} &h'(t) + \lambda_1 h(t) - C_1 h^p(t) = 0, \qquad t > 0, \\ &h(0) = (2\lambda_1/C_1)^{1/(p-1)}. \end{aligned} \tag{3.16}$$

By solving this problem, we can easily get the expression of $h(t)$:

$$h(t) = \left[\frac{C_1}{\lambda_1} - \left(\frac{C_1}{\lambda_1} - h^{1-p}(0)\right) e^{\lambda_1 (p-1)t}\right]^{-1/(p-1)}. \tag{3.17}$$

Since $h(0) = (2\lambda_1/C_1)^{1/(p-1)}$, $C_1/\lambda_1 - h^{1-p}(0) = C_1/\lambda_1 - C_1/(2\lambda_1) = C_1/(2\lambda_1) > 0$, we know that $h(t)$ is increasing in t and blows up in finite time $T_h^* = \frac{\ln 2}{\lambda_1(p-1)}$.

Let $v_3(x,t) = \xi(x)h(t)$. Then

$$
(3.18)\qquad
\begin{aligned}
&v_{3t}(x,t) - (x^\alpha v_{3x}(x,t))_x - v_3^p(x_0,t)\\
&\qquad\qquad = \xi(x)h'(t) + \lambda_1\xi(x)h(t) - \xi^p(x_0)h^p(t)\\
&\qquad\qquad \le \max_{x\in[0,a]} \xi(x)(h'(t) + \lambda_1 h(t)) - \xi^p(x_0)h^p(t)\\
&\qquad\qquad = 0, \quad x \in (0,a),\ t > 0,\\
&v_3(0,t) = 0 \le \int_0^a f(x)v_3^q(x,t)\,dx, \qquad t > 0,\\
&v_3(a,t) = 0 \le \int_0^a g(x)v_3^q(x,t)\,dx, \qquad t > 0.
\end{aligned}
$$

We see from the above inequalities that if $u_0(x) > h(0)\xi(x)$ on $[0,a]$, then $v_3(x,t)$ is a subsolution of problem (1.1). Utilizing the comparison principle (Lemma 2.4), we know that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \ge v_3(x,t)$ for $x \in [0,a]$, $t > 0$. Hence $u(x,t)$ blows up in finite time. This completes the proof of Theorem 1.4. ∎

4. The blow-up set and the uniform blow-up profile. In this section we discuss the blow-up set and the uniform blow-up profile of the blow-up solution of problem (1.1), and give the proof of Theorem 1.5. Throughout this section we assume that $p > 1$, $q = 1$ and

$$B = \max\Big\{\int_0^a f(x)\,dx, \int_0^a g(x)\,dx\Big\} \le 1.$$

From Theorem 1.4, we see that the solution $u(x,t)$ of problem (1.1) blows up in finite time for large initial data. We denote by T^* the blow-up time of the blow-up solution $u(x,t)$ of problem (1.1) and write $v \sim w$ for

$$\lim_{t\to T^*} \frac{v(t)}{w(t)} = 1.$$

First, we give the following two preliminary lemmas.

LEMMA 4.1. *Let hypotheses* (H$_1$), (H$_2$) *and* (H$_3$) *hold, assume that* $p > 1$, $q = 1$, $B \le 1$ *and that* $u(x,t)$ *is the blow-up solution of problem* (1.1). *Then* $(x^\alpha u_x)_x \le M$ *in* $(0,a) \times (0,T^*)$, *where* M *is the positive constant given in hypothesis* (H$_3$).

Proof. Set $v(x,t) = (x^\alpha u_x(x,t))_x - M$. Then the equation in (1.1) yields

$$(4.1)\qquad v_t - (x^\alpha v_x)_x = 0, \qquad (x,t) \in (0,a) \times (0,T^*).$$

On the other hand, by using the assumption $B \le 1$, we have

$$\begin{aligned}\lim_{x\to 0} v(x,t) &= u_t(0,t) - u^p(x_0,t) - M = \int_0^a f(x)u_t(x,t)\,dx - u^p(x_0,t) - M\\ &= \int_0^a f(x)v(x,t)\,dx + \Big(\int_0^a f(x)\,dx - 1\Big)[u^p(x_0,t)+M]\\ &\le \int_0^a f(x)v(x,t)\,dx\end{aligned}$$

and

$$\begin{aligned}v(a,t) &= \int_0^a g(x)u_t(x,t)\,dx - u^p(x_0,t) - M\\ &= \int_0^a g(x)v(x,t)\,dx + \Big(\int_0^a g(x)\,dx - 1\Big)[u^p(x_0,t)+M]\\ &\le \int_0^a g(x)v(x,t)\,dx.\end{aligned}$$

Since $v(x,0) = (x^\alpha u_{0x}(x))_x - M \le 0$ in $(0,a)$, by the maximum principle, $v(x,t) \le 0$ in $(0,a)\times(0,T^*)$. (Notice that equations (4.1) and (1.1) are degenerate and singular at $x=0$. However, we can always approximate them with regular parabolic equations, and then apply the maximum principle; see [GH, Lemma 2.1 and its proof].) That is, $(x^\alpha u_x(x,t))_x \le M$ in $(0,a)\times(0,T^*)$. ∎

Set

$$h(t) = u^p(x_0,t), \qquad H(t) = \int_0^t h(s)\,ds. \tag{4.2}$$

Then we have

Lemma 4.2. *Under the assumptions of Lemma* 4.1, *we have*

$$\lim_{t\to T^*} h(t) = \lim_{t\to T^*} H(t) = \infty,$$

and there exists a positive constant C_2 *such that*

$$u(x_0,t) \ge C_2(T^*-t)^{-1/(p-1)}, \qquad t\in(0,T^*). \tag{4.3}$$

That is, x_0 *is a blow-up point of the blow-up solution* $u(x,t)$ *of problem* (1.1).

Proof. Noting (4.2) and integrating the equation in (1.1) over $(0,t)$, we get

$$u(x,t) = u_0(x) + \int_0^t (x^\alpha u_x(x,t))_x\,dt + H(t), \qquad (x,t)\in(0,a)\times(0,T^*).$$

In view of Lemma 4.1, we have

$$(4.4)\qquad u(x,t) \le C_3 + H(t), \qquad (x,t) \in (0,a) \times (0,T^*),$$

where $C_3 = \max_{x\in[0,a]} u_0(x) + MT^*$. As $\limsup_{t\to T^*} \max_{x\in[0,a]} u(x,t) = \infty$, the above inequality ensures that $\lim_{t\to T^*} H(t) = \infty$. Since $T^* < \infty$, from the definition of $H(t)$ in (4.2) we find that $\lim_{t\to T^*} h(t) = \infty$.

To show the second conclusion, by applying Lemma 4.1 to equation (1.1) we get

$$(4.5)\qquad u_t(x_0,t) \le M + u^p(x_0,t), \qquad (x,t) \in (0,a) \times (0,T^*).$$

From the definition of $h(t)$ and $\lim_{t\to T^*} h(t) = \infty$ we infer that $\lim_{t\to T^*} u(x_0,t) = \infty$. Utilizing hypotheses ($\mathrm{H}_1$) and ($\mathrm{H}_2$), we find that $u_0(x) > 0$ on $[0,a]$, and therefore there exists a positive constant K such that $M \le Ku^p(x_0,t)$. Integrating (4.5) over (t,T^*) and noting $\lim_{t\to T^*} u(x_0,t) = \infty$, we get

$$u(x_0,t) \ge C_2(T^* - t)^{-1/(p-1)}, \qquad t \in (0,T^*),$$

where $C_2 = [(K+1)(p-1)]^{-1/(p-1)}$. ∎

In order to get the global blow-up result, we transfer problem (1.1) via

$$(4.6)\qquad u(x,t) = w(z,t), \qquad x = [(1-\alpha)z]^{1/(1-\alpha)}$$

to a new problem

$$(4.7)\qquad \begin{aligned} &w_t - d_0 z^{-\gamma} w_{zz} = h(t), \qquad (z,t) \in (0,l) \times (0,T^*),\\ &w(0,t) = \int_0^l f_1(z)w(z,t)\,dz, \quad w(l,t) = \int_0^l g_1(z)w(z,t)\,dz, \quad t \in (0,T^*),\\ &w(z,0) = w_0(z), \qquad z \in [0,l], \end{aligned}$$

where $d_0 = (1-\alpha)^{-\gamma}$, $\gamma = \frac{\alpha}{1-\alpha}$, $l = \frac{1}{1-\alpha}a^{1-\alpha}$, $w_0(z) = u_0(((1-\alpha)z)^{1/(1-\alpha)})$,

$$f_1(z) = f(((1-\alpha)z)^{1/(1-\alpha)})[(1-\alpha)z]^\gamma, \quad g_1(z) = g(((1-\alpha)z)^{1/(1-\alpha)})[(1-\alpha)z]^\gamma$$

and $h(t)$ is given by (4.2). It is obvious that $\int_0^l f_1(z)\,dz = \int_0^a f(x)\,dx$ and $\int_0^l g_1(z)\,dz = \int_0^a g(x)\,dx$. Consider the following eigenvalue problem:

$$(4.8)\qquad -(z^{-\gamma}\varphi(z))'' = \mu\varphi(z), \quad z \in (0,l); \quad \varphi(0) = \varphi(l) = 0.$$

On setting $\varphi(z) = z^{\gamma+1/2}\eta(y)$, $z = y^{2/(2+\gamma)}$, the above problem becomes

$$(4.9)\qquad \begin{aligned} &y^2\eta''(y) + y\eta'(y) + \left[\frac{4\mu y^2}{(2+\gamma)^2} - \frac{1}{(2+\gamma)^2}\right]\eta(y) = 0, \qquad y \in (0,b),\\ &\eta(0) = \eta(b) = 0, \end{aligned}$$

where $b = l^{(2+\gamma)/2}$. Equation (4.9) is a Bessel equation, its general solution is given by

$$\eta(y) = AJ_{1/(2+\gamma)}\left(\frac{2\sqrt{\mu}}{2+\gamma}y\right) + BJ_{-1/(2+\gamma)}\left(\frac{2\sqrt{\mu}}{2+\gamma}y\right).$$

Let μ_1 be the first root of $J_{1/(2+\gamma)}\big(\frac{2\sqrt{\mu}}{2+\gamma}b\big) = 0$; by McLachlan [Mc, pp. 29 and 75], it is positive. It is obvious that μ_1 is the first eigenvalue of problem (4.8); also we can easily obtain the corresponding eigenfunction

$$\varphi_1(z) = kz^{\gamma+1/2}J_{1/(2+\gamma)}\left(\frac{2\sqrt{\mu_1}}{2+\gamma}z^{(2+\gamma)/2}\right) \tag{4.10}$$

which is positive for $z \in (0,l)$, where $k > 0$ is chosen so that $\int_0^l \varphi_1(z)\,dz = 1$.

Now we can give the proof of Theorem 1.5.

Proof of Theorem 1.5. Define $I(t) = \int_0^t H(s)\,ds$, $v(z,t) = H(t) - w(z,t)$ and

$$\beta(t) = \int_0^l v(z,t)\varphi_1(z)\,dz, \qquad t \in (0,T^*), \tag{4.11}$$

where $w(z,t)$ is given by (4.6) and it satisfies (4.7), μ_1 and $\varphi_1(z)$ are the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem (4.8), and $\varphi_1(x)$ is given by (4.10) and satisfies $\int_0^l \varphi_1(z)\,dz = 1$. Then by (4.7) and (4.8), and using the known facts that $w(z,t) \ge 0$ on $[0,a]\times[0,T^*)$, $(z^{-\gamma}\varphi_1(z))'|_{z=0} \ge 0$ and $(z^{-\gamma}\varphi_1(z))'|_{z=l} \le 0$, we obtain

$$\begin{aligned}\beta'(t) &= \int_0^l (h(t) - w_t(z,t))\varphi_1(z)\,dz = \int_0^l -d_0 z^{-\gamma}w_{zz}(z,t)\varphi_1(z)\,dz \\ &= -d_0\int_0^l w(z,t)(z^{-\gamma}\varphi_1(z))''\,dz + d_0 w(z,t)(z^{-\gamma}\varphi_1(z))'|_0^l \\ &\le d_0\mu_1\int_0^l w(z,t)\varphi_1(z)\,dz = -d_0\mu_1\beta(t) + d_0\mu_1 H(t), \qquad t \in (0,T^*).\end{aligned}$$

Integrating the above inequality over $(0,t)$ and noting that

$$\beta(0) = \int_0^l v(z,0)\varphi_1(z)\,dz = -\int_0^l w_0(z)\varphi_1(z)\,dz \le 0,$$

we get

$$\begin{aligned}\beta(t) &\le \beta(0)e^{-d_0\mu_1 t} + d_0\mu_1\int_0^t e^{d_0\mu_1(s-t)}H(s)\,ds \\ &\le d_0\mu_1 I(t), \qquad t \in (0,T^*).\end{aligned} \tag{4.12}$$

On the other hand, (4.4) implies that

$$\inf_{z\in(0,l)} v(z,t) = \inf_{z\in(0,l)} (H(t) - w(z,t)) = \inf_{x\in(0,a)} (H(t) - u(x,t)) \geq -C_3, \quad t \in (0,T^*). \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\int_0^l |v(z,t)|\varphi(z)\,dz \leq C_4(1 + I(t)), \quad t \in (0,T^*), \tag{4.14}$$

where $C_4 = \max\{2C_3, d_0\mu_1\}$ is a positive constant.

In view of Lemma 4.1, we have

$$(x^\alpha u_x(x,t))_x \leq M \quad \text{in } (0,a) \times (0,T^*).$$

Then it follows from the transformation (4.6) that

$$d_0 z^{-\gamma} w_{zz}(z,t) \leq M, \quad (z,t) \in (0,l) \times (0,T^*).$$

Therefore

$$v_{zz}(z,t) = -w_{zz}(z,t) \geq -C_5, \quad (z,t) \in (0,l) \times (0,T^*), \tag{4.15}$$

where $C_5 = (M/d_0)l^\gamma$. For any subset $[c,d] \subset (0,a)$, let $l_1 = \frac{1}{1-\alpha}c^{1-\alpha}$ and $l_2 = \frac{1}{1-\alpha}d^{1-\alpha}$; then $[l_1,l_2] \subset (0,l)$. Let $r = \frac{1}{2}\min\{l_1, l - l_2\}$; then $r > 0$. For any fixed $x \in [c,d]$, let $z = \frac{1}{1-\alpha}x^{1-\alpha}$; then $z \in [l_1,l_2]$. We define

$$\zeta(y,t) = v(y,t) + \frac{C_5}{2}(y-z)^2, \quad (y,t) \in (z-r, z+r) \times (0,T^*). \tag{4.16}$$

Then by (4.15), we get

$$\zeta_{yy}(y,t) \geq 0, \quad (y,t) \in (z-r, z+r) \times (0,T^*).$$

The mean-value inequality for subharmonic functions yields

$$v(z,t) = \zeta(z,t) \leq \frac{1}{2r}\int_{z-r}^{z+r} \zeta(y,t)\,dy.$$

Then

$$v(z,t) \leq \frac{1}{2r}\left(\int_{z-r}^{z+r} v(y,t)\,dy + \frac{C_5}{3}r^3\right), \quad t \in (0,T^*). \tag{4.17}$$

From (4.10), there exists a positive constant C_6 such that

$$\inf_{y\in[l_1-r,l_2+r]} \varphi_1(y) \geq C_6 r^{\gamma+1}.$$

This together with (4.17) and (4.14) implies that

$$
\begin{aligned}
(4.18)\qquad v(z,t) &\le \frac{1}{2r}\Big(\int_{z-r}^{z+r} |v(y,t)|\,dy + \frac{C_5}{3}r^3\Big)\\
&\le \frac{1}{2r}\Big(\frac{1}{C_6}r^{-\gamma-1}\int_{z-r}^{z+r} |v(y,t)|\varphi_1(y)\,dy + \frac{C_5}{3}l^3\Big)\\
&\le C_7 r^{-\gamma-2}\Big(\int_0^l |v(y,t)|\varphi_1(y)\,dy + 1\Big)\\
&\le C_8 r^{-\gamma-2}(1+I(t)), \qquad t\in(0,T^*),
\end{aligned}
$$

where $C_7 = \max\{\frac{1}{2C_6}, \frac{C_5}{6}l^{\gamma+4}\}$ and $C_8 = C_7(C_4+1)$.

Since $I(t) > 0$ for $t \in (0,T^*)$, combining (4.4) with (4.18) we obtain

$$
-\frac{C_3}{H(t)} \le 1 - \frac{u(x,t)}{H(t)} = 1 - \frac{w(z,t)}{H(t)} \le C_8 r^{-\gamma-2}\frac{1+I(t)}{H(t)}
$$

for $t \in (0,T^*)$. Using the conclusion $H(t)\to\infty$ as $t\to T^*$ of Lemma 4.2 and the fact that $T^* < \infty$, we get $\lim_{t\to T^*} I(t)/H(t) = 0$. Then by the arbitrariness of $x\in[c,d]$, we deduce that

$$
\lim_{t\to T^*}\frac{u(x,t)}{H(t)} = 1
$$

uniformly on any compact subset $[c,d]$ of $(0,a)$. Then we have

$$
H'(t) = h(t) = u^p(x_0,t) \sim H^p(t) \qquad \text{as } t\to T^*.
$$

Therefore

$$
\lim_{t\to T^*} u(x,t)(T^*-t)^{1/(p-1)} = \lim_{t\to T^*}\|u(\cdot,t)\|_\infty (T^*-t)^{-1/(p-1)} = (p-1)^{-1/(p-1)}
$$

uniformly on any compact subset $[c,d]$ of $(0,a)$. Utilizing hypothesis (H_1), the above equality and Lebesgue's convergence theorem, we also obtain

$$
\begin{aligned}
\lim_{t\to T^*}(T^*-t)^{1/(p-1)}u(0,t) &= \int_0^a f(x)\lim_{t\to T^*}(T^*-t)^{1/(p-1)}u(x,t)\,dx\\
&= (p-1)^{-1/(p-1)}\int_0^a f(x)\,dx,\\
\lim_{t\to T^*}(T^*-t)^{1/(p-1)}u(a,t) &= \int_0^a g(x)\lim_{t\to T^*}(T^*-t)^{1/(p-1)}u(x,t)\,dx\\
&= (p-1)^{-1/(p-1)}\int_0^a g(x)\,dx.
\end{aligned}
$$

All these show that the blow-up set of the blow-up solution $u(x,t)$ is the whole domain $[0,a]$, including the boundaries. This differs from the case of parabolic equations with local sources or with homogeneous Dirichlet boundary conditions. The proof of Theorem 1.5 is complete. ■

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References

[BGC] C. Budd, V. A. Galaktionov and J. Chen, *Focusing blow-up for quasilinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 965–992.

[CC] C. Y. Chan and C. S. Chen, *A numerical method for semilinear singular parabolic quenching problem*, Quart. Appl. Math. 47 (1989), 45–57.

[CL] Y. P. Chen and L. H. Liu, *The blow-up profile for a nonlocal nonlinear parabolic equation with a nonlocal boundary condition*, Quart. Appl. Math. 70 (2012), 759–772.

[CLX1] Y. P. Chen, Q. L. Liu and C. H. Xie, *The blow-up properties for a degenerate semilinear parabolic equation with nonlocal source*, Appl. Math. J. Chinese Univ. Ser. B 17 (2002), 413–424.

[CLX2] Y. P. Chen, Q. L. Liu and C. H. Xie, *Blow-up for degenerate parabolic equations with nonlocal source*, Proc. Amer. Math. Soc. 132 (2004), 135–145.

[CY] Z. Cui and Z. Yang, *Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition*, J. Math. Anal. Appl. 342 (2008), 559–570.

[Da1] W. A. Day, *Extensions of a property of the heat equation to linear thermoelasticity and other theories*, Quart. Appl. Math. 40 (1982), 319–330.

[Da2] W. A. Day, *A decreasing property of solutions of parabolic equations with applications to thermoelasticity*, Quart. Appl. Math. 40 (1983), 468–475.

[De] K. Deng, *Comparison principle for some nonlocal problems*, Quart. Appl. Math. 50 (1992), 517–522.

[Ev] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1998.

[Fr1] A. Friedman, *Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions*, Quart. Appl. Math. 44 (1986), 401–407.

[Fr2] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[GG1] Y. Z. Gao and W. J. Gao, *Existence and blow-up of solutions for a porous medium equation with nonlocal boundary condition*, Appl. Anal. 90 (2011), 799–809.

[GG2] A. Gladkov and M. Guedda, *Blow-up problem for semilinear heat equation with absorption and a nonlocal boundary condition*, Nonlinear Anal. 74 (2011), 4573–4580.

[GG3] A. Gladkov and M. Guedda, *Semilinear heat equation with absorption and a nonlocal boundary condition*, Appl. Anal. 91 (2012), 2267–2276.

[GK1] A. Gladkov and K. I. Kim, *Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition*, J. Math. Anal. Appl. 338 (2008), 264–273.

[GK2] A. Gladkov and K. I. Kim, *Uniqueness and nonuniqueness for reaction-diffusion equation with nonlocal boundary condition*, Adv. Math. Sci. Appl. 19 (2009), 39–49.

[GH] J. S. Guo and B. Hu, *The profile near quenching time for the solution of a singular semilinear heat equation*, Proc. Edinburgh Math. Soc. 40 (1997), 437–456.

[Ka] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. 16 (1963), 305–330.

[LL] Z. G. Lin and Y. R. Liu, *Uniform blowup profiles for diffusion equations with nonlocal source and nonlocal boundary*, Acta Math. Sci. Ser. B English Ed. 24 (2004), 443–450.

[LCX] Q. L. Liu, Y. P. Chen and C. H. Xie, *Blow-up for a degenerate parabolic equation with a nonlocal source*, J. Math. Anal. Appl. 285 (2003), 487–505.

[Mc] N. W. McLachlan, *Bessel Functions for Engineers*, 2nd ed., Clarendon Press, Oxford, 1955.

[Pa1] C. V. Pao, *Dynamics of reaction-diffusion equations with nonlocal boundary conditions*, Quart. Appl. Math. 50 (1995), 173–186.

[Pa2] C. V. Pao, *Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions*, J. Comput. Appl. Math. 88 (1998), 225–238.

[Se] S. Seo, *Blowup of solutions to heat equations with nonlocal boundary conditions*, Kobe J. Math. 13 (1996), 123–132.

[WMX] Y. Wang, C. Mu and Z. Xiang, *Blowup of solutions to a porous medium equation with nonlocal boundary condition*, Appl. Math. Comput. 192 (2007), 579–585.

[Yi] Y. F. Yin, *On nonlinear parabolic equations with nonlocal boundary condition*, J. Math. Anal. Appl. 185 (1994), 54–60.

[ZK] S. N. Zheng and L. H. Kong, *Roles of weight functions in a nonlinear nonlocal parabolic system*, Nonlinear Anal. 68 (2008), 2406–2416.

[Zh] J. Zhou, *Blowup for a degenerate and singular parabolic equation with nonlocal source and absorption*, Glasgow Math. J. 52 (2010), 209–225.

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