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Internal characteristics of domains in $\mathbb{C}^n$

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Abstract. This paper is devoted to internal capacity characteristics of a domain $D \subset \mathbb{C}^n$, relative to a point $a \in D$, which have their origin in the notion of the conformal radius of a simply connected plane domain relative to a point. Our main goal is to study the internal Chebyshev constants and transfinite diameters for a domain $D \subset \mathbb{C}^n$ and its boundary ∂D relative to a point $a \in D$ in the spirit of the author's article [Math. USSR-Sb. 25 (1975), 350–364], where similar characteristics have been investigated for compact sets in $\mathbb{C}^n$. The central notion of directional Chebyshev constants is based on the asymptotic behavior of extremal monic "polynomials" and "copolynomials" in directions determined by the arithmetic of the index set $\mathbb{Z}^n$. Some results are closely related to results on the sth Reiffen pseudometrics and internal directional analytic capacities of higher order (Jarnicki–Pflug, Nivoche) describing the asymptotic behavior of extremal "copolynomials" in varied directions when approaching the point a.

1. Introduction. A well-known classical result of geometric function theory (Fekete [F], Szegö [Sz]) is the coincidence of three characteristics of a compact set K in $\mathbb{C}$, which are defined in quite different ways:

$$d(K) = \tau(K) = c(K),$$

where $d(K)$ is the *transfinite diameter* (a geometric characterization), $\tau(K)$ is the *Chebyshev constant* (an approximation theory approach), and $c(K)$ is the *capacity* (a potential theory point of view). Multidimensional analogs of these characteristics were studied intensively in last decades, beginning with Leja's definition of the multivariate transfinite diameter [Lej] and the author's article [Z1], where a multidimensional analog of Fekete's equality $d(K) = \tau(K)$ has been obtained. In [Z5, Section 3], one can find a survey of results on relations among various capacity characteristics of compact sets in $\mathbb{C}^n$.

Our main goal is the study of *internal Chebyshev constants* and *transfinite diameters* for a domain D and its boundary ∂D in $\mathbb{C}^n$ relative to a point $a \in D$ in the spirit of [Z1], applying the general approach, developed

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in Section 4 of [Z5]. Namely, our considerations are based on two systems: the system of monomials

$$e_{i,a}(z) := (z-a)^{k(i)}, \quad i \in \mathbb{N}, \tag{1.1}$$

where $i \mapsto k(i) = (k_1(i), \dots, k_n(i))$ is a standard enumeration of the set $\mathbb{Z}_+^n$ (see Section 2 below) and its biorthogonal system of analytic functionals $\{e'_{i,a}\}_{i\in\mathbb{N}}$ defined by

$$e'_{i,a}(f) = \frac{f^{(k(i))}(a)}{k(i)!}, \quad i \in \mathbb{N},\ f \in A(\{a\}), \tag{1.2}$$

where $A(\{a\})$ is the space of analytic germs at a. We investigate the asymptotic behavior of the *least deviation* (in proper norms related to the domain D) *from zero* δ_i of either (i) "*monic polynomials*" *with respect to the system* (1.2) $e'_{i,a} + \sum_{j<i} c_j e'_{j,a}$ (in order to "measure the size of ∂D viewed from a") or (ii) "*monic copolynomials*", that is, functions whose Taylor expansion at a is of the form $e_{i,a} + \sum_{j>i} c_j e_{j,a}$ (to "measure the size of D relative to a").

By analogy with [Z1], for an arbitrary domain $D \subset \mathbb{C}^n$, we introduce in Section 4 the directional Chebyshev constant $\tau(a, D; \theta)$, which describes the asymptotic behavior of extremal monic copolynomials with respect to the system (1.1) in the direction θ, study properties of the characteristics $\tau(a, D; \theta)$ as a function of θ, and define the principal Chebyshev constant $\tau(a, D)$ as the geometric mean of directional ones. In Section 5 we consider, dual in a sense, directional Chebyshev constants $\tau(a, \partial D; \theta)$ and the principal Chebyshev constant $\tau(a, \partial D)$ that describe the asymptotic behavior of extremal monic polynomials (1.2) and "measure the size of ∂D viewed from the point $a \in D$". It is shown that, in the case of a strictly pluriregular domain D, these characteristics are mutually reciprocal and remain the same when the normed spaces used in their definition vary in a wide range of spaces. Applying the theorem on Hilbert scales of analytic functions (see, e.g., [Z3, Z4]), we show in Section 6 that the asymptotics of the leading coefficients of orthonormal bases, obtained by the Gram–Schmidt procedure from the systems (1.1) and (1.2) in appropriate Hilbert spaces, can be expressed through the Chebyshev constants. The transfinite diameter $d(a, \partial D)$ of the boundary ∂D viewed from $a \in D$ is introduced in Section 7 by means of extremal Vandermondians for the sequence (1.2). The equality $d(a, \partial D) = \tau(a, \partial D) = \tau(a, D)^{-1}$ is proved, which can be considered as an internal multivariate analog of the Fekete equality.

In Section 3 we consider the one-dimensional case, which displays a direct connection of the above internal characteristics with the logarithmic capacity of an appropriate compact set and, if D is simply connected, with its conformal radius relative to a point.

Section 8 deals with internal analytic capacities of a domain relative to a point and is closely related to Jarnicki–Pflug's and Nivoche's results [JP1, JP2, JP3, Ni1, Ni3]. Applying the latter, we give an expression of the Robin function in terms of internal orthonormal bases (which can be considered as an internal analog of Zeriahi's result [Ze, Theorem 2]) and consider analogs of Szegö's equality by introducing some natural Chebyshev constants, though different from those considered above. The problem on analogs of Szegö's equality for the Chebyshev constants studied in Sections 4–6 (similar to Rumely's result for compact sets in $\mathbb{C}^n$ [Ru]) remains open; see Section 9, where some other conclusions and generalizations are discussed.

2. Preliminaries and notation. Given an open set $D \subset \mathbb{C}^n$ we denote by $A(D)$ the space of all analytic functions in D with the usual locally convex topology of locally uniform convergence in D. If $K \subset \mathbb{C}^n$ is a compact set then $A(K)$ is the locally convex space of all germs of analytic functions on K, endowed with the standard inductive topology.

Definition 1. A Stein manifold Ω is called *pluriregular* (or *strongly pseudoconvex*, $\mathbb{C}^n$*-pluriregular*, P*-pluriregular*, *hyperconvex*) if there exists a negative function $u \in \mathrm{Psh}(\Omega)$ such that $u(z_j) \to 0$ for every sequence $\{z_j\}$ without limit points in Ω (briefly, if $z \to \partial\Omega$). We say that a domain D in a Stein manifold Ω is *strictly pluriregular* if there is a pseudoconvex domain Δ with $D \Subset \Delta \subset \Omega$ and a continuous function $u \in \mathrm{Psh}(\Delta)$ such that $D = \{z \in \Delta : u(z) < 0\}$. If $\dim \Omega = 1$, we say that D is *strictly regular*.

Notice that "strict pluriregularity" is somewhat weaker than "strict hyperconvexity" considered in [Ni2].

Definition 2. The (generalized) *pluripotential Green function* of a Stein manifold D *with a logarithmic singularity at* $a \in D$ is defined by

$$g_D(a,z) := \limsup_{\zeta\to z} \sup\{u(\zeta) : u \in \mathcal{G}(a,D)\}, \tag{2.1}$$

where $\mathcal{G}(a,D)$ consists of all negative functions $u \in \mathrm{Psh}(D)$ such that $u(z) - \ln|\varphi(z)|$ is bounded from above near a, with $\varphi \in A(D)^n$ representing local coordinates at a such that $\varphi(a) = 0$ (this definition does not depend on the choice of the local coordinates; if $D \subset \mathbb{C}^n$ we take $\varphi(z) = z - a$).

The following assertion will be needed (see, e.g., [Z5, Lemma 2.1]).

Lemma 3. *Suppose X, Y is a pair of locally convex spaces and $J : X \to Y$ is an injective continuous linear operator with dense image. Then the adjoint operator $J^* : Y^* \to X^*$ is also injective and, if X is reflexive, the image $J^*(Y^*)$ is dense in X^*.*

Remark 4. In what follows, we always treat the operator J as an identical embedding, identifying x with Jx and using the notation $X \hookrightarrow Y$ for a continuous linear embedding. In particular, we also write $Y^* \hookrightarrow X^*$ in the situation of Lemma 3.

We write $|f|_E := \sup\{|f(z)| : z \in E\}$ for a function $f : E \to \mathbb{C}$. Denote by $\mathbb{Z}_+^n$ the set of all integer-valued vectors $k = (k_1, \dots, k_n)$ with non-negative coordinates. Let $|k| := k_1 + \dots + k_n$ be the degree of the multiindex k. Introduce an enumeration $\{k(i)\}_{i\in\mathbb{N}}$ of the set $\mathbb{Z}_+^n$ via the conditions: the sequence $s(i) := |k(i)|$ is non-decreasing and on each set $\mathcal{K}_s := \{|k(i)| = s\}$ the enumeration coincides with the lexicographic order relative to $k_1, \dots, k_n$. Denote by $i(k)$ the number assigned to k under this ordering. Notice that the number of multiindices of degree not larger than s is $m_s := \binom{s+n}{s}$, and the number of those of degree s is $N_s := m_s - m_{s-1} = \binom{s+n-1}{s}$ for $s \ge 1$, and $N_0 = 1$. Set

$$l_s := \sum_{q=0}^{s} qN_q \tag{2.2}$$

for $s = 0, 1, \dots$.

We consider the standard $(n-1)$-simplex

$$\varSigma := \Big\{\theta = (\theta_\nu) \in \mathbb{R}^n : \theta_\nu \ge 0,\ \nu = 1, \dots, n;\ \sum_{\nu=1}^{n} \theta_\nu = 1\Big\} \tag{2.3}$$

and its interior $\varSigma^\circ$ (in the relative topology on the hyperplane containing $\varSigma$). For $\theta \in \varSigma$ we denote by $\mathcal{L}_\theta$ the set of all infinite sequences $L \subset \mathbb{N}$ such that $k(i)/s(i) \xrightarrow{L} \theta$. We set $k! := k_1! \cdots k_n!$ for $k = (k_\nu) \in \mathbb{Z}_+^n$. We also use the notation $|z| := (\sum_{\nu=1}^n |z_\nu|^2)^{1/2}$.

Given a pair of Hilbert spaces $H_1 \hookrightarrow H_0$ with dense embedding, we denote by

$$H^\alpha = (H_0)^{1-\alpha}(H_1)^\alpha, \qquad \alpha \in \mathbb{R},$$

the Hilbert scale generated by the pair H_0, H_1 (see, e.g., [KPS]).

By $H^\infty(D)$ we denote the space of all bounded functions $f \in A(D)$ with the uniform norm $\|f\|_{H^\infty(D)} := |f|_D$. If D is bounded we consider its subspace $AC(\overline{D})$ that consists of functions continuously extendible onto $\overline{D}$, and the Bergman space $AL^2(D)$ of all analytic functions square integrable with respect to the Lebesgue measure on D. By $\mathbb{U}_r(a)$ we denote the equilateral polydisk of radius $r > 0$ centered at $a \in \mathbb{C}^n$.

3. One-dimensional case: internal capacity characteristics. The *conformal radius of a simply connected domain* $D \subset \overline{\mathbb{C}}$ *with respect to a point* $a \in D$ is the number $r(a, D) := 1/|\omega'(a)|$, where $\omega : D \to \mathbb{U}$ is a biholomorphic mapping such that $\omega(a) = 0$; it is supposed here that $\omega'(\infty) := \frac{d\omega(1/\zeta)}{d\zeta}\big|_{\zeta=0}$ if $a = \infty$; the number $r(\infty, D)^{-1}$ is also called the *conformal radius of the compact set* $K := \overline{\mathbb{C}} \setminus D$ (see, e.g., [P]).

The *capacity of D relative to $a \in D$* is defined by $c(a, D) := \exp(-\rho(a, D))$, where $\rho(a, D) := \lim_{z\to a}(g_D(a, z) - \ln|z - a|)$ is the *Robin constant of D relative to $a \in D$* and $g_D(a, z)$ is the generalized (subharmonic!) Green function of D with the normalized (negative) logarithmic singularity at a. If D is a simply connected domain in $\mathbb{C}$ and $a \in D$, then the conformal radius $r(a, D)$ coincides with the capacity $c(a, D)$.

The characteristic $c(a, D)$ was considered by many authors also under the name "interior (or inner) radius of D relative to a" (see, e.g., [M]). A related capacity characteristic, named "radius of ∂D viewed from $a \in D$", was also under consideration: $c(a, \partial D) := \exp \rho(a, D) = 1/c(a, D)$.

Using the mapping $\varphi_a(z) = 1/(z - a)$, $a \neq \infty$, these capacities can be reduced to the logarithmic capacity of the compact set $K_a := \varphi_a(\overline{\mathbb{C}} \setminus D)$ (for $a = \infty$ we have to take $K_\infty = \overline{\mathbb{C}} \setminus D$):

$$c(a, D) = 1/c(K_a), \qquad c(a, \partial D) = c(K_a), \tag{3.1}$$

where $c(K)$ is the logarithmic capacity of a compact set K in $\mathbb{C}$, which coincides, by the Fekete–Szegö result, with its transfinite diameter $d(K)$ and Chebyshev constant $\tau(K)$.

For a fixed $a \in \mathbb{C}$ we consider the system of functions $e_{s,a}(z) := 1/(z-a)^s$, $s \in \mathbb{N}$ if $a \neq \infty$, and $e_{s,\infty}(z) = z^s$, $s \in \mathbb{N}$, otherwise. Given a domain $D \neq \overline{\mathbb{C}}$ and $a \in D$ we define the *Chebyshev constant of ∂D viewed from a* by

$$\tau(a, \partial D) := \lim_{s\to\infty} \inf\Big\{\Big(\Big|e_{s,a} + \sum_{0\le j<s} c_j e_{j,a}\Big|_{\partial D}\Big)^{1/s} : c_j \in \mathbb{C}\Big\}. \tag{3.2}$$

and the *transfinite diameter of ∂D viewed from a* by

$$d(a, \partial D) := \lim_{s\to\infty} \big(\sup\big\{|\det{(e_{\mu,a}(\zeta_\nu))}_{\mu,\nu=0}^{s}| : (\zeta_\nu) \in (\mathbb{C} \setminus D)^s\big\}\big)^{2/s(s-1)}. \tag{3.3}$$

Changing variables $z = a + 1/w$ we obtain

$$\tau(a, \partial D) = d(a, \partial D) = \tau(K_a) = c(K_a) = c(a, \partial D). \tag{3.4}$$

The representations (3.2) and (3.3) give a motivation for the notions of multivariate internal Chebyshev constants and transfinite diameter of ∂D viewed from $a \in D$, which we consider in the next sections. However, for $n \geq 2$, one has to deal (see Section 7) with appropriate analytic functionals instead of the functions $1/(z-a)^k$, $k \in \mathbb{Z}^n$, which are not defined on $D \setminus \{a\}$ as analytic functions. Since evaluation at a point makes no sense for analytic functionals, we need to apply, in the definition of the transfinite diameter, the general approach suggested in Section 4 of [Z5]. As an application, we obtain an expression of the capacity $c(a, D)$ via extremal Wronskians at a (Section 7, Corollary 20).

4. Internal Chebyshev constants. Given a domain D in $\mathbb{C}^n$ and a point $a \in D$ we define

$$\delta_i = \delta_i(a, D) := \inf\{|f|_D : f \in \mathcal{N}_i\}, \quad i \in \mathbb{N}, \tag{4.1}$$

where $\mathcal{N}_i = \mathcal{N}_i(a, D) := \{f \in H^\infty(D) : e'_{j,a}(f) = 0,\ j < i;\ e'_{i,a}(f) = 1\}$ and the functionals $e'_{i,a}$ are defined in (1.2). Hereafter it is assumed that $\inf \emptyset = +\infty$ (this may happen, for instance, if $H^\infty(D)$ consists only of constants).

Definition 5. The *directional Chebyshev constant of D relative to a point $a \in D$ in a direction $\theta \in \Sigma$* is the constant

$$\tau(a, D; \theta) := \limsup_{k(i)/|k(i)| \to \theta} \delta_i^{1/s(i)} := \sup_{L \in \mathcal{L}_\theta} \limsup_{i \in L} \delta_i^{1/s(i)} \tag{4.2}$$

with δ_i defined in (4.1).

Lemma 6. *The set $\Sigma(a, D) := \{\theta \in \Sigma : \tau(a, D; \theta) < \infty\}$ is convex and the function $\ln \tau(a, D; \theta)$ is convex on $\Sigma(a, D)$.*

Proof. Given $\theta, \theta' \in \Sigma(a, D)$ and $0 < \alpha < 1$, take natural-valued sequences i_q, j_q, and natural numbers $r_q < R_q$ so that $s(i_q) = s(j_q)$ and

$$\frac{k(i_q)}{s(i_q)} \to \theta, \quad \frac{k(j_q)}{s(j_q)} \to \theta', \quad \frac{r_q}{R_q} \to \alpha \quad \text{as } q \to \infty.$$

For arbitrary $\varepsilon > 0$ find functions $f_{\varepsilon,q} \in \mathcal{N}_{i_q}$ and $g_{\varepsilon,q} \in \mathcal{N}_{j_q}$ such that

$$|f_{\varepsilon,q}|_D < \delta_{i_q}(1 + \varepsilon), \quad |g_{\varepsilon,q}|_D < \delta_{j_q}(1 + \varepsilon).$$

Then the function $F(z) = (f_{\varepsilon,q})^{r_q}(g_{\varepsilon,q})^{R_q - r_q}$ belongs to the subspace $\mathcal{N}_{l_q}$, where $l_q = i(k^{(q)})$ is the number corresponding to the multiindex $k^{(q)} = r_q k(i_q) + (R_q - r_q)k(j_q)$ in the enumeration of the Preliminaries. Therefore

$$\delta_{l_q} \le |F|_D \le (\delta_{i_q}(1 + \varepsilon))^{r_q}(\delta_{j_q}(1 + \varepsilon))^{R_q - r_q}.$$

Then we take the logarithm and divide by $s(l_q) = R_q s(i_q)$. By construction, $k^{(q)}/s(l_q) \to \alpha\theta + (1 - \alpha)\theta'$, therefore, after passage to the upper limit as $q \to \infty$, taking into account that $\varepsilon > 0$ is arbitrary, we obtain

$$\ln \tau(a, D; \alpha\theta + (1 - \alpha)\theta') \le \alpha \ln \tau(a, D; \theta) + (1 - \alpha) \ln \tau(a, D; \theta') < \infty.$$

Therefore $\alpha\theta + (1 - \alpha)\theta \in \Sigma(a, D)$ for every $\alpha \in (0, 1)$, hence $\Sigma(a, D)$ is convex and the function $\tau(a, D; \theta)$ is convex on this set. ∎

Corollary 7. *The function $\ln \tau(a, D; \theta)$ is continuous on the interior of $\Sigma(a, D)$.*

Lemma 8. *Let r be the radius of an inscribed equilateral polydisc for D, centered at a. Then $\tau(a, D; \theta) \ge r$ for all $\theta \in \Sigma$. If the domain D is bounded and R is the radius of a circumscribed equilateral polydisc for D, centered at a, then $\tau(a, D; \theta)$ is uniformly bounded from above by R.*

By Lemmas 6 and 8 the function $\tau(a,D;\theta)$ is measurable and bounded from below. Therefore the following definition makes sense.

DEFINITION 9. The *principal Chebyshev constant of D relative to $a \in D$* is

$$\tau(a,D) := \exp\Big(\int_{\Sigma} \ln \tau(a,D;\theta)\, d\sigma(\theta)\Big), \tag{4.3}$$

where σ is the normalized Lebesgue measure on Σ.

In general, $\tau(a,D)$ may be equal to $+\infty$, but if D is bounded then $\tau(a,D) \le R$, where R is defined in Lemma 8.

LEMMA 10. *Let D be a bounded domain in $\mathbb{C}^n$. Then the usual limit exists in* (4.2) *for every $\theta \in \Sigma^\circ$.*

Proof. Suppose that there exist two subsequences $\{i_q\}$ and $\{j_q\}$ such that

$$\lim_{q\to\infty} \frac{k(i_q)}{s(i_q)} = \lim_{q\to\infty} \frac{k(j_q)}{s(j_q)} = \theta \in \Sigma^\circ, \tag{4.4}$$

but

$$\lim(\delta_{i_q})^{1/s(i_q)} =: \alpha < \beta := \lim(\delta_{j_q})^{1/s(j_q)}.$$

Going to subsequences if necessary, we assume that

$$k_\nu(j_q) \ge k_\nu(i_q) > 0, \quad \nu = 1,\dots,n; \quad s(j_q)/s(i_q) \nearrow \infty.$$

Setting

$$r(q) := \inf\bigg\{\bigg[\frac{k_\nu(j_q)}{k_\nu(i_q)}\bigg] : \nu = 1,\dots,n\bigg\}, \qquad l(q) := k(j_q) - r(q)k(i_q), \tag{4.5}$$

we see from (4.4), (4.5) that $l_q \in \mathbb{Z}_+^n$ and

$$r(q) \sim s(j_q)/s(i_q), \qquad |l(q)| = o(s(j_q)) \quad \text{as } q \to \infty. \tag{4.6}$$

Given $\varepsilon > 0$ choose $f_{\varepsilon,q} \in \mathcal{N}_{i_q}$ so that $|f_{\varepsilon,q}|_D < \delta_{i_q} + \varepsilon$. Then the function $F(z) := (z-a)^{l(q)}(f_{\varepsilon,q}(z))^{r(q)}$ satisfies

$$|F|_D \le C^{|l(q)|}(\delta_{i_q} + \varepsilon)^{r(q)}, \tag{4.7}$$

where $C = \max\{|z-a| : z \in \overline{D}\}$. On the other hand, by the feature of the enumeration $k(i)$, we have $F \in \mathcal{N}_{j_q}$, hence

$$|F|_D \ge \delta_{j_q}. \tag{4.8}$$

Since $\varepsilon > 0$ is arbitrary, combining (4.6)–(4.8) we obtain $\beta \le \alpha$, contrary to assumption. ∎

5. Strictly pluriregular domains. In this section and the next we suppose that D is a strictly pluriregular domain. We show that, in Definition 5, one can change $H^\infty(D)$ to a wide range of Banach spaces so that the defined characteristic remains the same. This allows us to introduce the notions of Chebyshev constants of ∂D viewed from $a \in D$. On the other hand, this permits applying Hilbert space methods.

Let Y be a Banach space with a dense embedding $Y \hookrightarrow A(\{a\})$, $a \in \mathbb{C}^n$. Then, by Lemma 3, $A(\{a\})^* \hookrightarrow Y^*$. Each germ $f \in Y$ is represented by its Taylor expansion at a: $f(z) = \sum_{i=1}^{\infty} e'_{i,a}(f) e_{i,a}(z)$, which converges absolutely and uniformly on some neighborhood of a; the functionals $e'_{i,a} \in A(\{a\})^* \hookrightarrow Y^*$ are defined in (1.2). We introduce two directional Chebyshev constants characterizing approximative properties of this system of functionals with respect to the spaces Y or $X := Y^*$. The first one describes the asymptotic behavior of the least deviation of "monic polynomials with respect to the system of analytic functionals (1.2)" in the space X:

$$\tag{5.1} \begin{aligned} \tau_Y^*(a,\theta) &:= \limsup_{k(i)/|k(i)| \to \theta} (\Delta_{i,X})^{1/s(i)} \\ &:= \sup_{L \in \mathcal{L}_\theta} \limsup_{i \in L} (\Delta_{i,X})^{1/s(i)}, \qquad \theta \in \Sigma, \end{aligned}$$

where

$$\tag{5.2} \Delta_{i,X} = \Delta_{i,a,X} := \inf\Big\{ \Big\| e'_{i,a} + \sum_{j<i} c_j e'_{j,a} \Big\|_X : (c_j) \in \mathbb{C}^{i-1} \Big\}, \qquad i \in \mathbb{N},$$

and $\mathcal{L}_\theta$ is defined in the Preliminaries. One can see here an analogy with the one-dimensional case (see (3.2)) in that the continuous linear functionals (1.2) can be expressed via

$$\tag{5.3} e'_{i,a}(f) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}_r(a)} \frac{f(\zeta)\,d\zeta}{(\zeta - a)^{k(i)+I}}, \qquad f \in A(\{a\}),\ i \in \mathbb{N},$$

where $I = (1, \dots, 1)$, and $\mathbb{T}_r(a) := \{z = (z_\nu) \in \mathbb{C}^n : |z_\nu - a_\nu| = r\}$ with some sufficiently small $r = r(f) > 0$.

The characteristic (5.1) is dual, in a sense, to another one, defined by

$$\tag{5.4} \tau_Y(a,\theta) := \liminf_{k(i)/|k(i)| \to \theta} (\delta_{i,Y})^{1/s(i)} := \inf_{L \in \mathcal{L}_\theta} \liminf_{i \in L} (\delta_{i,Y})^{1/s(i)}, \qquad \theta \in \Sigma,$$

where

$$\tag{5.5} \begin{aligned} \delta_{i,Y} &= \delta_{i,a,Y} := \inf\{\|f\|_Y : f \in \mathcal{N}_i\}, \\ \mathcal{N}_i &= \mathcal{N}_{i,a,Y} := \{f \in Y : e'_{j,a}(f) = 0,\ j < i;\ e'_{i,a}(f) = 1\}. \end{aligned}$$

If the space Y is closely related to the given strictly pluriregular domain D, then (5.1) describes the size of the boundary ∂D viewed from a, while (5.4) coincides on Σ°, as will be shown below, with the characteristic $\tau(a, D; \theta)$ introduced in the previous section. In the next definition we deal with the special space $Y = AC(\overline{D})$, but it will be shown below that the space Y can vary in a quite wide range leaving the above characteristics unchanged.

DEFINITION 11. Let D be a strictly pluriregular domain in $\mathbb{C}^n$, $a \in D$, and $Y = AC(\overline{D})$. Then the number $\tau(a, \partial D; \theta) := \tau_Y^*(a, \theta)$ is called the *directional Chebyshev constant of ∂D viewed from a in the direction $\theta \in \Sigma$.* The *principal Chebyshev constant of ∂D viewed from a* is defined by

$$\tau(a, \partial D) := \exp\Big(\int_\Sigma \ln \tau(a, \partial D; \theta)\, d\sigma(\theta)\Big). \tag{5.6}$$

That the integral (5.6) exists follows from $\tau(a, \partial D, \theta) = \tau(a, D, \theta)^{-1}$, $\theta \in \Sigma^\circ$, which will be proved below (see Theorem 12).

Given a domain $D \subset \mathbb{C}^n$ and $a \in D$, consider the sublevel sets of the pluripotential Green function ($\lambda < 0$):

$$D_\lambda := \{z \in D : g_D(a, z) < \lambda\}, \qquad K_\lambda := \{z \in D : g_D(a, z) \le \lambda\}. \tag{5.7}$$

THEOREM 12. *Let $D \subset \mathbb{C}^n$ be a strictly pluriregular domain, $a \in D$, Y any Banach space with dense embeddings*

$$A(\overline{D}) \hookrightarrow Y \hookrightarrow A(D), \tag{5.8}$$

and set $X = Y^*$. Then for each $\theta \in \Sigma^\circ$ the usual limit exists in (5.1) *and* (5.4), *and*

$$\tau_Y(a, \theta) = \tau(a, D; \theta) = \tau_Y^*(a, \theta)^{-1} = \tau(a, \partial D; \theta)^{-1} \qquad \text{if } \theta \in \Sigma^\circ.$$

Moreover

$$\lim_{s\to\infty} \Big| \prod_{i=m_{s-1}+1}^{m_s} \delta_{i,Y} \Big|^{1/sN_s} = \lim_{s\to\infty} \Big| \prod_{i=m_{s-1}+1}^{m_s} \frac{1}{\Delta_{i,X}} \Big|^{1/sN_s} = \tau(a, D). \tag{5.9}$$

Furthermore,

$$\tau(a, D_\lambda; \theta) = \tau(a, D; \theta) \exp \lambda, \quad \tau(a, \partial D_\lambda; \theta) = \tau(a, \partial D; \theta) \exp(-\lambda). \tag{5.10}$$

This theorem will be proved in the next section after some preliminary considerations.

6. Asymptotics of leading coefficients of internal orthonormal bases

LEMMA 13. *Let $a \in \mathbb{C}^n$ and H be any Hilbert space with dense embedding $H \hookrightarrow A(\{a\})$, $a \in \mathbb{C}^n$. Let $\varphi_i' = \sum_{j=1}^i a_{ji} e_j'$ be the orthonormal system in the dual space $H^* \hookleftarrow A(\{a\})^*$, obtained by the Gram–Schmidt procedure applied to the system $\{e_i'\}$, defined by* (1.2)*; and let $\{\varphi_i\} \subset H$ be the biorthogonal system to $\{\varphi_i'\}$. Then*

$$\delta_{i,H} = 1/\Delta_{i,H^*} = |a_{ii}|$$

and for each $\theta \in \Sigma$, *we have*

$$\tau_H^*(a;\theta) = \limsup_{k(i)/|k(i)|\to\theta} \frac{1}{|a_{ii}|^{1/s(i)}}, \qquad \tau_H(a,\theta) = \liminf_{k(i)/|k(i)|\to\theta} |a_{ii}|^{1/s(i)}.$$

Let H_λ *be the Hilbert space of all* $x = \sum_{i=1}^\infty \xi_i\varphi_i \in A(\{a\})$ *with*

$$\|x\|_{H_\lambda} := \Big(\sum_{i=1}^\infty |\xi_i|^2 \exp(2\lambda s(i))\Big)^{1/2} < \infty, \qquad \lambda \le 0. \tag{6.1}$$

Then

$$\tau_{H_\lambda}(a,\theta) = \tau_H(a;\theta)\exp\lambda, \quad \tau_{H_\lambda}^*(a,\theta) = \tau_H^*(a;\theta)\exp(-\lambda), \qquad \lambda < 0. \tag{6.2}$$

Proof. Consider also the dual Hilbert scale

$$G_\lambda := \Big\{x' = \sum_{i=1}^\infty \xi_i'\varphi_i' \in G : \|x'\|_{G_\lambda} := \Big(\sum_{i=1}^\infty |\xi_i'|^2 \exp(-2\lambda s(i))\Big)^{1/2} < \infty\Big\},$$

with $\lambda \le 0$, $G_0 = G = H^*$. The system $\{\varphi_i\}$ is an orthogonal basis in each space H_λ and has an expansion

$$\varphi_i(z) = \sum_{j\ge i} b_{j,i}e_j(z), \tag{6.3}$$

converging in some neighborhood of a, while $\{\varphi_i'\}$ is an orthogonal basis in any G_λ. Moreover,

$$b_{i,i} = \frac{1}{a_{i,i}}, \quad \|\varphi_i\|_{H_\lambda} = \exp(\lambda s(i)), \quad \|\varphi_i'\|_{G_\lambda} = \exp(-\lambda s(i)), \quad i\in\mathbb{N},\ \lambda\le 0.$$

By the extremal property of orthogonal systems, we have

$$\delta_{i,H_\lambda} = |a_{ii}|\exp\lambda s(i), \qquad \varDelta_{i,H_\lambda} = |a_{ii}|^{-1}\exp(-\lambda s(i)). \tag{6.4}$$

Taking the logarithm and passing to the lower (resp. upper) limit along subsequences $L\in\mathcal{L}_\theta$, $\theta\in\Sigma$, we complete the proof. ∎

Proof of Theorem 12. Since $A(\overline{D})$ is a nuclear locally convex space, by Pietsch [Pt, Section 4.4] there exists a Hilbert space H with dense embeddings

$$A(\overline{D}) \hookrightarrow H \hookrightarrow Y, \qquad A(\overline{D}) \hookrightarrow H \hookrightarrow AC(\overline{D}) \hookrightarrow A(D). \tag{6.5}$$

It is known (see, e.g., [Z2, Ze, Z3, Z4]) that, under these restrictions on H, the system $\{\varphi_i\}$ is a common basis in the spaces $A(D)$, $A(\{a\})$, $A(D_\lambda)$, $A(K_\lambda)$, $\lambda<0$, and the following embeddings hold:

$$A(K_\lambda) \hookrightarrow H_\lambda \hookrightarrow A(D_\lambda), \quad \lambda < 0, \tag{6.6}$$

where H_λ is the scale (6.1) and the sublevel sets K_λ, D_λ are defined in (5.7). Therefore $H\hookrightarrow Y\hookrightarrow A(D)\hookrightarrow H_\lambda$ for every $\lambda<0$. Due to (6.4) and (6.5), there are positive constants C and $c = c(\lambda)$ such that

$$c\delta_{i,H_\lambda} = c\delta_{i,H}\exp(\lambda s(i)) \le \delta_{i,Y} \le C\delta_{i,H}, \qquad i\in\mathbb{N}. \tag{6.7}$$

On the other hand, since $H \hookrightarrow H^\infty(D) \hookrightarrow A(D) \hookrightarrow H_\lambda$, $\lambda < 0$, we obtain, taking into account Lemma 10,

$$\limsup_{i\in L}\,(c\delta_{i,H})^{1/s(i)}\exp\lambda \le \tau(a,D;\theta) \le \liminf_{i\in L}\,(C\delta_{i,H})^{1/s(i)} \tag{6.8}$$

for any $L \in \mathcal{L}_\theta$, $\theta \in \Sigma^\circ$ and $\lambda < 0$. Hence, if $\theta \in \Sigma^\circ$, the usual limit exists in (5.4) with $Y = H$ and $\tau(a,D;\theta) = \tau_H(a,\theta)$. Applying now (6.7) with an arbitrary Y satisfying the conditions of the theorem, we conclude the same with $\tau_Y(a,\theta)$ instead of $\tau_H(a,\theta)$. Then, applying the embeddings dual to (6.6), we obtain

$$G_\lambda \hookrightarrow A(D)^* \hookrightarrow Y^* \hookrightarrow G, \qquad \lambda < 0.$$

In the same token, by Lemma 13, we conclude that

$$\tau_Y^*(a,\theta) = \tau(a,\partial D;\theta) = 1/\tau(a,D;\theta), \qquad \theta \in \Sigma^\circ, \tag{6.9}$$

and the usual limit exists in (5.1) for $\theta \in \Sigma^\circ$.

An examination of the proofs of Lemmas 5 and 6 in [Z1] shows that, since the function $\tau(a,D;\theta)$ is continuous on Σ° (see Corollary 7 above) and the usual limits exist in (5.1), (5.4), we can establish, in the same way as in [Z1], the following relations:

$$\lim_{s\to\infty}\frac{1}{N_s}\sum_{|k(i)|=s}\ln\tau_{i,Y} = \int_\Sigma \ln\tau(a,D;\theta)\,d\sigma(\theta) = \ln\tau(a,D),$$
$$\lim_{s\to\infty}\frac{1}{N_s}\sum_{|k(i)|=s}\ln\tau_{i,Y}^* = \int_\Sigma \ln\tau(a,\partial D;\theta)\,d\sigma(\theta) = \ln\tau(a,\partial D),$$

where σ is the normalized Lebesgue measure on Σ. Thus (5.9) is proved.

Applying (6.6) once more, we obtain

$$H_{\lambda+\varepsilon} \hookrightarrow H^\infty(\overline{D}_\lambda) \hookrightarrow H_{\lambda-\varepsilon}, \qquad \lambda < 0,\ 0 < \varepsilon < -\lambda.$$

Therefore there exist constants $C = C(\lambda,\varepsilon)$ and $c = c(\lambda,\varepsilon)$ such that

$$\begin{aligned} c\delta_{i,H_{\lambda-\varepsilon}} &= c\delta_{i,H}\exp(\lambda-\varepsilon) \le \delta_i(a,D_\lambda)\\ &\le C\delta_{i,H_{\lambda+\varepsilon}} = C\delta_{i,H}\exp(\lambda+\varepsilon), \qquad i\in\mathbb{N}, \end{aligned}$$

where $\delta_i(a,D_\lambda)$ is defined in (4.1) with D_λ instead D. Passing to the limit along any sequence $L \in \mathcal{L}_\theta$, $\theta \in \Sigma^\circ$, and taking into account (6.2), we obtain

$$\tau(a,D;\theta)\exp(\lambda-\varepsilon) \le \tau(a,D_\lambda;\theta) \le \tau(a,D;\theta)\exp(\lambda+\varepsilon), \qquad \theta\in\Sigma^\circ.$$

The first relation in (5.10) follows by letting $\varepsilon \to 0$. The remaining statements of the theorem can be derived easily from the proved ones by applying Lemma 13. ■

Summarizing the above considerations we obtain the main result of this section.

THEOREM 14. *Let $Y = H$ be a Hilbert space satisfying the conditions of Theorem* 12, *and let*

$$\varphi_i' = \sum_{j=1}^{i} a_{j,i} e_j', \qquad \varphi_i = \sum_{j\geq i} b_{j,i} e_j$$

be the orthonormal systems constructed for the spaces H^* and H as in Lemma 13. *Then*

$$\lim_{i\in L} |b_{i,i}|^{1/s(i)} = 1/\tau(a, D; \theta), \qquad L \in \mathcal{L}_\theta,\ \theta \in \Sigma^\circ.$$

The geometric mean of the leading coefficients $a_{i,i} = 1/b_{i,i}$ of degree s satisfies an asymptotic relation, determined by the principal Chebyshev constants:

$$\lim_{s\to\infty} \Big(\Big(\prod_{|k(i)|=s} |a_{i,i}| \Big)^{1/N_s} \Big)^{1/s} = \tau(a, \partial D) = \frac{1}{\tau(a, D)}. \tag{6.10}$$

Indeed, by Lemma 13, $\delta_{i,H} = |a_{i,i}| = 1/|b_{i,i}|$, so it suffices to apply (5.9).

PROPOSITION 15. *Let D be a bounded complete logarithmically convex n-circular domain in $\mathbb{C}^n$ and*

$$h(\theta) = h_D(\theta) := \sup\Big\{ \sum_{\nu=1}^{n} \theta_\nu \ln|z_\nu| : z = (z_\nu) \in D \Big\}, \qquad \theta = (\theta_\nu) \in \Sigma,$$

its characteristic function. Then $\tau(0, D; \theta) = \tau(\overline{D}, \theta) = \exp(h(\theta))$, $\theta \in \Sigma$, and

$$\tau(0, D) = \tau(\overline{D}) = \exp\Big(\int_\Sigma h(\theta)\, d\sigma(\theta) \Big), \tag{6.11}$$

where σ is the normalized Lebesgue measure on Σ (here $\tau(\overline{D}, \theta)$ and $\tau(\overline{D})$ are, respectively, the directional and principal Chebyshev constants of the compact set $K = \overline{D}$; see [Z1, Z5]).

Proof. Take any Hilbert space H with embeddings

$$A(\overline{D}) \hookrightarrow H \hookrightarrow A(D) \tag{6.12}$$

and such that the monomials $e_i = z^{k(i)}$ are pairwise orthogonal; for instance, one can take the Bergman space $AL^2(D)$ of all functions analytic and square integrable in D. Then $\{p_i = e_i/\|e_i\|_H\}$ is an orthonormal polynomial basis with $a_{i,i} = 1/\|e_i\|_H$ in the framework of Theorem 6.1 from [Z5]; on the other hand, in the context of Theorem 14 it is an orthonormal basis φ_i with $b_{i,i} = 1/\|e_i\|_H$. Therefore, by Theorem 14 above and Theorem 6.1 from [Z5],

$$\tau(\overline{D}, \theta) = \lim_{k(i)/s(i)\to\theta} \|e_i\|_H^{1/s(i)} = \tau(0, D; \theta), \qquad \theta \in \Sigma^\circ, \tag{6.13}$$

where $\tau(\overline{D}, \theta)$ is the directional Chebyshev constant of the compact set $K = \overline{D}$ in the direction θ (see [Z5]).

By (6.12), given $\varepsilon > 0$ there exist positive constants $c = c(\varepsilon)$ and $C = C(\varepsilon)$ such that

$$\begin{aligned} c\exp\big((1-\varepsilon)h_D(\theta(i))s(i)\big) &= c|e_i|_{(1-\varepsilon)D} \le \|e_i\|_H \le C|e_i|_{(1+\varepsilon)D} \\ &= C\exp\big((1+\varepsilon)h_D(\theta(i))s(i)\big), \end{aligned}$$

where $\theta(i) = k(i)/s(i)$, $i \in \mathbb{N}$. Hence, as h_D is continuous and $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{k(i)/s(i)\to\theta} \|e_i\|_H^{1/s(i)} = \exp(h_D(\theta)), \qquad \theta \in \Sigma.$$

Combining this with (6.13), we obtain

$$\tau(\overline{D},\theta) = \tau(0,D;\theta) = \exp(h_D(\theta)), \qquad \theta \in \Sigma^\circ,$$

and then, by integration, (6.11). ∎

Problem 16. *Characterize all domains $D \Subset \mathbb{C}^n$ with $0 \in D$ such that $\tau(\overline{D}) = \tau(0,D)$.*

7. Internal transfinite diameters. Let D be a domain in $\mathbb{C}^n$, $a \in D$, and let $e_i' = e_{i,a}' \in A(D)^*$, $i \in \mathbb{N}$, be the system of analytic functionals determined by (1.2). Since, in contrast to the one-dimensional case, evaluation at a point makes no sense for analytic functionals, there is no direct analog of Leja's Vandermondians. The general considerations of Section 4 in [Z5] turned out to be useful for an alternative equivalent definition of the transfinite diameter for compact sets (see [Z5, Theorem 5.1]). This approach provides a way out in the present situation as well.

Definition 17. The *transfinite diameter of ∂D viewed from the point* a is the number

$$d(a,\partial D) := \limsup_{i\to\infty} \tilde{\mathcal{V}}_i^{1/l_{s(i)}} \tag{7.1}$$

where $s(i) = |k(i)|$ (see Preliminaries), l_s is defined in (2.2), and

$$\tilde{\mathcal{V}}_i = \sup\{|\det(e_{i,\alpha}'(f_\beta))_{\alpha,\beta=1}^i| : f_\beta \in \mathbb{B}_{H^\infty(D)},\ \beta = 1,\dots,i\} \tag{7.2}$$

is the sequence of extremal Vandermondians. The *internal transfinite diameter of D with respect to a* is defined by

$$d(a;D) := 1/d(a;\partial D).$$

Let D be a strictly pluriregular domain in $\mathbb{C}^n$, Y be a Banach space with dense embeddings (5.8), and $X := Y^*$. Then, by Lemma 3,

$$e_{i,a}' \in A(\{a\})^* \hookrightarrow A(D)^* \hookrightarrow X, \qquad i \in \mathbb{N}.$$

Set

$$\tilde{\mathcal{V}}_i^Y := \sup\{|\det(e'_\mu(f_\nu))_{\mu,\nu=1}^i| : f_\nu \in \mathbb{B}_Y,\ \nu = 1,\ldots,i\},$$
$$d^Y := \limsup_{i\to\infty} (\tilde{\mathcal{V}}_i^Y)^{1/l_{s(i)}}. \tag{7.3}$$

THEOREM 18. *Under the above assumptions the usual limit exists in* (7.3), *it does not depend on the choice of the space* Y, *namely* $d^Y = d(a;\partial D)$, *and*

$$d(a,D) = \frac{1}{d(a;\partial D)} = \tau(a,D) = \exp\Big(\int_\Sigma \ln\tau(a,D;\theta)\,d\sigma(\theta)\Big), \tag{7.4}$$

where σ *is the normalized Lebesgue measure on* Σ.

Proof. By Lemma 4.2 in [Z5], we have the estimates

$$\Delta_{i,X} \le \tilde{\mathcal{V}}_i^Y/\tilde{\mathcal{V}}_{i-1}^Y \le i\Delta_{i,X}, \quad i \in \mathbb{N},$$

where $\Delta_{i,X}$ is defined in (5.2). Therefore

$$\prod_{i=m_{s-1}+1}^{m_s} \Delta_{i,X} \le \tilde{\mathcal{V}}_{m_s}^Y/\tilde{\mathcal{V}}_{m_{s-1}}^Y \le (m_s)^{N_s} \prod_{i=m_{s-1}+1}^{m_s} \Delta_{i,X}.$$

Since $\frac{\ln m_s}{s} \to 0$, due to (5.9) we have the asymptotic formula

$$\ln\tilde{\mathcal{V}}_{m_s}^Y - \ln\tilde{\mathcal{V}}_{m_{s-1}}^Y \sim sN_s\ln\tau(a,\partial D), \quad s\to\infty. \tag{7.5}$$

By summing from 1 to s (see, e.g., [dB]), we derive the asymptotic formula

$$\ln\tilde{\mathcal{V}}_{m_s}^Y \sim \ln\tilde{\mathcal{V}}_{m_s}^Y - \ln\tilde{\mathcal{V}}_{m_0}^Y \sim \sum_{q=1}^s qN_q\cdot\ln\tau(a,\partial D) = l_s\ln\tau(a,\partial D), \quad s\to\infty.$$

Let $m_{s-1} < i \le m_s$, that is, $s(i) = s$. Take positive numbers r, R so that $\mathbb{U}_r(a) \Subset D \Subset \mathbb{U}_R(a)$, $2R > 1 > r/2$. Then, due to Proposition 15, there is i_0 such that

$$\left(\frac{1}{2R}\right)^{s(i)} \le \Delta_{i,X} \le \left(\frac{2}{r}\right)^{s(i)}, \quad i \ge i_0.$$

Therefore

$$\tilde{\mathcal{V}}_i^Y\left(\frac{1}{2R}\right)^{sN_s} \le \tilde{\mathcal{V}}_i^Y\prod_{j=i+1}^{m_s}\Delta_{j,X} \le \tilde{\mathcal{V}}_{m_s}^Y \le \tilde{\mathcal{V}}_i^Y(m_s)^{N_s}\prod_{j=i+1}^{m_s}\Delta_{j,X}$$
$$\le \tilde{\mathcal{V}}_i^Y(m_s)^{N_s}\left(\frac{2}{r}\right)^{sN_s}$$

for $i \ge i_0$ and $s = s(i)$. Since $sN_s/l_s \to 0$ as $s\to\infty$, by (7.5) we have

$$\ln\tilde{\mathcal{V}}_i^Y \sim l_{s(i)}\ln\tau(a,\partial D) \quad \text{as } i\to\infty,$$

which implies that the usual limit exists in (7.3), and it does not depend on Y, namely $d^Y = \tau(a,\partial D)$. Since $Y_1 := AC(\overline{D}) \hookrightarrow H^\infty(D) \hookrightarrow AL^2(D) =: Y_2$

and both Y_1, Y_2 satisfy the conditions of the theorem, we find that $d^Y = d(a;\partial D)$. Then, by Theorem 12 and (4.3), we have

$$d(a;\partial D) = \tau(a,\partial D) = \exp\Big(\int_{\Sigma} \ln \tau(a,\partial D;\theta)\, d\sigma(\theta)\Big),$$

so (7.4) is proved. ∎

Notice that

$$l_s \sim \lambda_s := \frac{s^{n+1}}{(n-1)!(n+1)} \quad \text{as } s \to \infty. \tag{7.6}$$

The following statement can be proved similarly to Theorem 5.2 in [Z5].

THEOREM 19. *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then the Chebyshev constant $\tau(a;\partial D)$ is expressed by the formula*

$$\tau(a;\partial D) = d(a,\partial D) = \bigg(\exp\sum_{\nu=1}^{n+1}\frac{1}{\nu}\bigg)\lim_{i\to\infty}\frac{\mathcal{W}_{i,a}^{1/\lambda_{s(i)}}}{s(i)}, \tag{7.7}$$

where $\lambda_{s(i)}$ is defined in (7.6),

$$\mathcal{W}_{i,a} = \sup\big\{|\mathcal{W}_a((f_\nu)_{\nu=1}^i)| : |f_\nu|_D \le 1,\ \nu = 1,\dots,i\big\},$$

and

$$\mathcal{W}_a((f_\nu)_{\nu=1}^i) = \det\,(f_\nu^{(k(\mu))}(a))_{\mu,\nu=1}^i$$

is the multivariate Wronskian of the system $\{f_\nu\}_{\nu=1}^i$, evaluated at a.

In particular, we get

COROLLARY 20. *Let D be a strictly regular domain in $\mathbb{C}$, and $a \in D$. Then*

$$c(a,D) = \frac{1}{c(a,\partial D)} = \exp\bigg(-\frac{3}{2}\bigg)\lim_{s\to\infty}\frac{s}{\mathcal{W}_{s,a}^{2/s^2}},$$

where

$$\mathcal{W}_{s,a} = \max\big\{|\det\,(f_\alpha^{(\beta)}(a))_{a,\beta=0}^{s-1}| : |f_\alpha|_D \le 1,\ \alpha = 0,\dots,s-1\big\}.$$

In particular, if D is simply connected and $\omega : D \to \mathbb{B}$ is an analytic bijection such that $\omega(a) = 0$, then

$$|\omega'(a)| = \exp\bigg(\frac{3}{2}\bigg)\lim_{s\to\infty}\frac{\mathcal{W}_{s,a}^{2/s^2}}{s}.$$

8. Internal Robin function and capacities in $\mathbb{C}^n$. For $n \ge 2$, the function $g_D(a,z) - \ln|z-a|$, in general, has infinitely many partial limits as $z \to a$. So, in contrast to the case $n = 1$, there are many ways to define capacities of D relative to a. By analogy with [Z1, Z2, Z5], one can define a natural capacity by

$$C(a,D) := \exp\Big(-\limsup_{z\to a}(g_D(a,z) - \ln|z-a|)\Big). \tag{8.1}$$

Similarly to the compact set case (for a survey of related results see [Z5]), in order to get an analog of the Szegö equality, one can modify the definition of Chebyshev constants, by normalizing the leading homogeneous polynomial parts (relative to the variable $\zeta = z - a$), instead of normalizing the leading coefficients. Namely, let

$$\mathcal{M}_s := \{f \in A(D) : e'_{i,a}(f) = 0,\ s(i) < s\}. \tag{8.2}$$

Given $f \in \mathcal{M}_s$, let

$$\hat{f}_s(z) = \sum_{s(i)=s} e'_{i,a}(f)(z-a)^{k(i)} = \lim_{|w|\to\infty} w^s f\left(a + \frac{z-a}{w}\right). \tag{8.3}$$

be its homogeneous part of degree s (it may be identically zero). Consider a Chebyshev-type characteristic (cf. [Z1, Z2, Si2]), given by $T(a,D) := \liminf_{s\to\infty} T_s(a,D)$, where

$$T_s(a,D) := (\inf\{|f|_D : f \in \mathcal{M}_s,\ |\hat{f}_s|_{\mathbb{B}^n} \geq 1\})^{1/s}.$$

Theorem 21. *Let D be a strictly pluriregular domain in $\mathbb{C}^n$ and $a \in D$. Then $T(a,D) = C(a,D)$.*

This theorem will be proved below after some preliminary considerations. Without restrictions on D it may not be true, as is seen from

Example 22. Let $D = \mathbb{B}_R \setminus K \subset \mathbb{C}$, where K is the standard Cantor set on the real line, $R > 1$, and $a \in D$. Then, since K is regular, but negligible for bounded analytic functions, we have $T(a,D) = T(a,\mathbb{B}_R) = C(a,\mathbb{B}_R) \neq C(a,D)$.

The following notion was introduced in [BT] (cf. [Lel, Az]).

Definition 23. The *Robin function of a Stein manifold D relative to a point $a \in D$* is defined by

$$\rho_D(a,\zeta) := \limsup_{|\lambda|\to 0}(g_D(a,a+\lambda\zeta) - \ln|\lambda|), \qquad \zeta \in \mathbb{C}^n.$$

Let D be a bounded domain in $\mathbb{C}^n$. Then the Robin function $\rho(\zeta) = \rho_D(a,\zeta)$ is continuous, plurisubharmonic in $\mathbb{C}^n$, and logarithmically homogeneous, that is,

$$\rho(t\zeta) = \rho(\zeta) + \ln|t|, \qquad \zeta \in \mathbb{C}^n,\ t \in \mathbb{C}.$$

Therefore the open set

$$\breve{D} = \breve{D}_a := \{\zeta \in \mathbb{C}^n : \rho_D(a,\zeta) < 0\}$$

is a complete circular domain, that is, $\lambda z \in \check{D}$ if $z \in \check{D}$ and $|\lambda| \le 1$. It is clear that

$$g_{\check{D}_a}(0,\zeta) = \rho_D(a,\zeta), \qquad \rho_{\check{D}_a}(0,\zeta) \equiv \rho_D(a,\zeta).$$

DEFINITION 24 (cf. Jarnicki–Pflug [JP1, JP2, JP3], Nivoche [Ni1, Ni2]). The *ζ-directional analytic capacity of order s for a domain $D \subset \mathbb{C}^n$ relative to a point a* is the number

$$\gamma_s(a,D;\zeta) := \left(\sup\left\{\frac{1}{s!}|d_\zeta^{(s)} f(a)| : f \in \mathcal{M}_s,\ |f|_D \le 1\right\}\right)^{-1/s} = (\sup\{|\hat{f}_s(a+\zeta)| : f \in \mathcal{M}_s,\ |f|_D \le 1\})^{-1/s}, \quad \zeta \in \mathbb{C}^n, \tag{8.4}$$

where $d_\zeta^{(s)}$ stands for the derivative of order s in the direction of the tangent vector $\zeta \in \mathbb{C}^n$ at the point a, and $\hat{f}$ is defined in (8.3). One can consider the reciprocal

$$\gamma_s(a,\partial D;\zeta) := (\gamma_s(a,D;\zeta))^{-1}$$

as the *analytic capacity of order s of ∂D viewed from a in the direction ζ.*

For every $\zeta \in \mathbb{C}^n$ the following limit exists (see, e.g., [Ni2]):

$$\gamma_\infty(a,D;\zeta) := \lim_{s\to\infty} \gamma_s(a,D;\zeta) = \inf\{\gamma_s(a,D;\zeta) : s \in \mathbb{N}\}. \tag{8.5}$$

This characteristic can be considered as the *analytic capacity of infinite order* (*transfinite analytic capacity*) of D relative to a in the direction ζ.

PROPOSITION 25 (Nivoche [Ni2]). *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then*

$$-\ln \gamma_\infty(a,D;\zeta) \le \rho_D(a,\zeta), \quad \zeta \in \mathbb{S}. \tag{8.6}$$

Equality holds in (8.6) *quasi-everywhere on $\mathbb{S}$ (i.e., except a set $A \subset \mathbb{S}$ with $[A] = \{[z] \in \mathbb{CP}^{n-1} : z \in A\}$ polar in $\mathbb{CP}^{n-1}$).*

We introduce related directional Chebyshev constants:

$$T_s(a,D;\zeta) := \inf\{|f|_D : f \in \mathcal{M}_s, |\hat{f}_s(a+\zeta)| \ge 1\}^{1/s}, \quad s \in \mathbb{N},$$
$$T(a,D;\zeta) := \liminf_{s\to\infty} T_s(a,D;\zeta),$$

with $\zeta \in \mathbb{S}$. It is easily seen that they coincide with their capacity counterparts (8.4) and (8.5): $T_s(a,D;\zeta) = \gamma_s(a,D;\zeta)$, $T(a,D;\zeta) = \gamma_\infty(a,D;\zeta)$, $\zeta \in \mathbb{S}$.

Proof of Theorem 21. It is obvious that

$$T_s(a,D)^{-1} = (\sup\{|\hat{f}_s|_{\mathbb{B}^n} : f \in \mathcal{M}_s,\ |f|_D \le 1\})^{1/s},$$

so, taking into account (8.4), we have

$$T_s(a,D)^{-1} = \sup_{\zeta\in\mathbb{S}}\{1/\gamma_s(D,a;\zeta) : \zeta \in \mathbb{S}\}.$$

It follows from the definition (8.1) that $C(a,D)=\exp(-\lambda(a,D))$ with

$$\lambda(a,D):=\max\{\rho_D(a,\zeta):\zeta\in\mathbb{S}\}. \tag{8.7}$$

Then, by (8.6), we obtain

$$-\ln T(a,D)=\limsup_{s\to\infty}(-\ln T_s(a,D))\le\lambda(a,D).$$

For contradiction, suppose that $-\ln T(a,D)<r<\lambda(a,D)$. Then, by (8.5), $-\ln\gamma_\infty(D,\zeta)\le r$, $\zeta\in\mathbb{S}$, hence, since equality holds in (8.6) quasi-everywhere on $\mathbb{S}$, we obtain $\rho_D(a,\zeta)=\limsup_{\xi\to\zeta}(-\ln\gamma_\infty(D,\xi))\le r<\lambda(a,D)$ for every $\zeta\in\mathbb{S}$. This contradicts (8.7) and hence yields $C(a,D)=T(a,D)$. ∎

Let D be a strictly pluriregular domain in $\mathbb{C}^n$, and Δ and u be as in Definition 1. Then there is $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ the connected component Δ_ε of the set $\{z\in\Delta:u(z)<\varepsilon\}$ containing $\overline{D}$ is relatively compact in Δ. The following stability properties can be found, e.g., in [BT, Dem, Ni3].

Lemma 26. *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then*

$$g_D(a,z)=\lim_{\lambda\nearrow 0}g_{D_\lambda}(a,z),\qquad g_D(a,z)=\lim_{\varepsilon\searrow 0}g_{\Delta_\varepsilon}(a,z),\qquad z\in D\setminus\{a\},$$
$$\rho_D(a,\zeta)=\lim_{\lambda\nearrow 0}\rho_{D_\lambda}(a,\zeta),\qquad \rho_D(a,\zeta)=\lim_{\varepsilon\searrow 0}\rho_{\Delta_\varepsilon}(a,\zeta),\qquad \zeta\in\mathbb{C}^n\setminus\{0\},$$

where D_λ are the sublevel domains defined in (5.7) *and Δ_ε are defined above. Moreover, the convergence in all these relations is locally uniform.*

The following statement shows how the Robin function can be expressed in terms of orthonormal bases (cf. [Ze, Theorem 2]).

Theorem 27. *Let D be a strictly pluriregular domain in $\mathbb{C}^n$, H any Hilbert space with dense embeddings $A(\overline{D})\hookrightarrow H\hookrightarrow A(D)$, $\{\varphi_i\}$ the orthonormal basis from Lemma* 13, *and $g_i(a+\zeta):=\sum_{s(j)=s(i)}e'_{j,a}(\varphi_i)\zeta^{k(j)}$ the homogeneous part of φ_i of degree $s(i)$, $i\in\mathbb{N}$. Then*

$$\rho_D(a,\zeta)=\limsup_{\xi\to\zeta}\limsup_{i\to\infty}\frac{\ln|g_i(a+\zeta)|}{s(i)},\qquad \zeta\in\mathbb{C}^n. \tag{8.8}$$

Proof. Take $\lambda<0$. Then there exists a positive constant $c=c(\lambda)$ such that

$$c|f|_{D_\lambda}\le\|f\|_H,\qquad f\in H, \tag{8.9}$$

where D_λ is defined in (5.7). Set

$$V(\xi):=\limsup_{i\to\infty}\frac{\ln|g_i(a+\xi)|}{s(i)}=\limsup_{s\to\infty}V_s(\xi),$$

where $V_s(\xi)=\sup\{\ln|g_i(a+\xi)|/s:s(i)=s\}$. Since, by (8.9),

$$\{c\varphi_i:s(i)=s\}\subset\mathcal{M}_s\cap\mathbb{B}_{H^\infty(D_\lambda)},$$

we have

$$V_s(\xi) + \frac{\ln c}{s} \le -\ln \gamma_s(a, D_\lambda; \xi), \qquad \xi \in \mathbb{C}^n,$$

and hence, by Proposition 25,

$$\rho_{D_\lambda}(a,\zeta) = \limsup_{\xi\to\zeta}(-\ln\gamma(a, D_\lambda;\xi)) \ge \limsup_{\xi\to\zeta} V(\xi), \qquad \zeta\in\mathbb{C}^n. \tag{8.10}$$

Let Δ and u be as in Definition 1. Then there exists ε_0 such that for $0<\varepsilon<\varepsilon_0$ the connected component Δ_ε of the set $\{z\in\Delta : u(z)<\varepsilon\}$ containing $\overline{D}$ is relatively compact in Δ. Consider $f\in\mathcal{M}_s\cap\mathbb{B}_{H^\infty(\overline{\Delta_\varepsilon})}$. Then there exists $C=C(\varepsilon)$ such that $\|f\|_H\le C|f|_{\Delta_\varepsilon}$. Since

$$\hat{f}_s(a+\xi) = \sum_{s(i)=s} c_i g_i(a+\xi),$$

where $c_i=(f,\varphi_i)_H$, we have $|c_i|\le\|f\|_H\le C$. Therefore

$$|\hat{f}_s(a+\xi)| \le \sum_{s(i)=s} |c_i|\,|g_i(a+\xi)| \le CN_s \sup_{s(i)=s}|g_i(a+\xi)|.$$

Hence,

$$-\ln\gamma_s(a,\Delta_\varepsilon;\xi) \le V_s(\xi) + \frac{\ln CN_s}{s}, \qquad \xi\in\mathbb{C}^n,$$

and, after passing to the limit, by Proposition 25 we obtain

$$-\ln\gamma(a,\Delta_\varepsilon;\xi)\le V(\xi), \qquad \xi\in\mathbb{C}^n.$$

Thus

$$\rho_{\Delta_\varepsilon}(a,\zeta) = \limsup_{\xi\to\zeta}(-\ln\gamma(a,\Delta_\varepsilon;\xi)) \le \limsup_{\xi\to\zeta} V(\xi), \qquad \zeta\in\mathbb{C}^n. \tag{8.11}$$

Combining (8.10) and (8.11) and applying Lemma 26, we complete the proof. ∎

Introduce an average capacity by

$$\mathfrak{C}(a,D) := \exp\Big(-\int_{\mathbb{S}} \ln\rho_D(a,\zeta)\,d\omega(\zeta)\Big).$$

Corollary 28. *Under the assumptions of Theorem 27 we have*

$$\lim_{s\to\infty}\bigg(\exp\bigg(\frac{1}{s}\sup_{s(i)=s}\Big\{\int_{\mathbb{S}}\ln|\hat{f}_i(a+\zeta)|\,d\omega(\zeta)\Big\}\bigg)\bigg) = \frac{1}{\mathfrak{C}(a,D)}.$$

9. Conclusion

9.1. Theorem 27 could be useful for confirming the following conjecture (cf. [BC, Theorem 2]).

Conjecture 29. *For strictly pluriregular domains,*

$$\tau(0,\check{D}_a;\theta) = \tau(a,D;\theta), \qquad \theta\in\Sigma,$$

so that the directional Chebyshev constants and hence the transfinite diameter $d(a, D)$ are uniquely determined by the Robin function $\rho(a, D; \zeta)$.

The estimate $\tau(0, \breve{D}_a; \theta) \le \tau(a, D; \theta)$, $\theta \in \Sigma$, can be easily proved similarly to [BC]. In order to get $\tau(a, D; \theta) \le \tau(0, \breve{D}_a; \theta)$, $\theta \in \Sigma$, *one needs to prove an internal analogue of Bloom's Theorem* [Bl, Theorem 3.2].

9.2. Rumely [Ru] (see also [DR]) discovered a formula expressing the transfinite diameter of a compact set K via its Robin function.

Problem 30. *Let D be a strictly pluriregular domain in $\mathbb{C}^n$ and $a \in D$. Give an analogue of the Rumely formula, expressing the internal transfinite diameter $d(a, \partial D)$ via the Robin function $\rho_D(a, z)$.*

9.3. The characteristics from Definitions 5, 9, and 17 can be extended to Stein manifolds (they will depend on the choice of local coordinates at a!). Let D be a Stein manifold, $a \in D$, and suppose the analytic mapping $\varphi = (\varphi_\nu) : D \to \mathbb{C}^n$ gives local coordinates at a with $\varphi(a) = 0$. For example, the directional Chebyshev constant $\tau_\varphi(a, D; \theta)$ can be defined as in Definition 5 with the functionals (1.2) expressed in terms of the chosen local coordinates at a:

$$e'_i(f) = e'_{i,a}(\varphi; f) := \frac{1}{k(i)!} \left. \frac{\partial^{|k(i)|} f(\varphi^{-1}(\zeta))}{\partial \zeta^{k(i)}} \right|_{\zeta=0} . \tag{9.1}$$

For concrete Stein manifolds one can use some preferable local coordinates at a. If, for example, D is an unbranched Riemann domain over $\mathbb{C}^n$, $\pi : D \to \mathbb{C}^n$ the projection, and $a \in D$, then it is natural to define Chebyshev constants by applying the local coordinates $\varphi(z) = \pi(z) - \pi(a)$.

9.4. There is another way of defining the transfinite diameter and Chebyshev constants for an arbitrary Stein manifold D and given local coordinates φ at $a \in D$. Consider a continuous plurisubharmonic function u in D such that $\{u(z) < s\}$ is relatively compact in D for every $s \in \mathbb{N}$ and $u(a) < 1$. Let G_s be the connected component of $\{u(z) < s\}$ which contains a.

Definition 31. The *exhausting directional Chebyshev constants of the domain D relative to the point a and to the local coordinates φ* are defined by

$$\tilde{\tau}_\varphi(a, D; \theta) := \lim_{s \to +\infty} \tau_\varphi(a, G_s; \theta) = \sup_{s \in \mathbb{N}} \tau_\varphi(a, G_s; \theta), \tag{9.2}$$

and the corresponding *principal Chebyshev constant* and *transfinite diameter* are defined by

$$\tilde{\tau}_\varphi(a, D) := \lim_{s \to +\infty} \tau_\varphi(a, G_s), \qquad \tilde{d}_\varphi(a, D) := \lim_{s \to +\infty} d_\varphi(a, G_s). \tag{9.3}$$

If $D \subset \mathbb{C}^n$ and $\varphi(z) = z - a$, we use the notation $\tilde{\tau}(a, D; \theta)$, $\tilde{\tau}(a, D)$, $\tilde{d}(a, D)$, respectively.

For strictly pluriregular domains on Stein manifolds, these new characteristics coincide with $\tau_\varphi(a, D; \theta)$, $\tau_\varphi(a, D)$, $d_\varphi(a, D)$, respectively, but in general they do not (see, e.g., Example 22). It is easily seen that the equality (7.4) holds with $\tilde{\tau}_\varphi$, $\tilde{d}_\varphi$ instead of τ, d.

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