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Sets with the Bernstein and generalized Markov properties

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Abstract. It is known that for C^∞ determining sets Markov's property is equivalent to Bernstein's property. We are interested in finding a generalization of this fact for sets which are not C^∞ determining. In this paper we give examples of sets which are not C^∞ determining, but have the Bernstein and generalized Markov properties.

1. Notation and definitions. Throughout this paper we use the following notation.

$\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{N}_k$ is the set of integers which are greater than or equal to k, $\mathbb{B}^N := \{x \in \mathbb{R}^N : |x| = \sqrt{x_1^2 + \cdots + x_N^2} \le 1\}$ is the Euclidean ball, $\mathbb{S}^{N-1} := \partial\mathbb{B}^N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1^2 + \cdots + x_N^2 = 1\}$ is the Euclidean sphere.

For $E \subset \mathbb{R}^N$ set $\mathcal{P}(E) = \{f : E \to \mathbb{R} : f = P|_E$ for some P in $\mathbb{R}[x_1, \ldots, x_N]\}$.

We shall see that for the Euclidean sphere $\mathbb{S}^{N-1}$ we have

$$\mathcal{P}(\mathbb{S}^{N-1}) = \mathbb{R}[x_1, \ldots, x_{N-1}] + \mathbb{R}[x_1, \ldots, x_{N-1}]x_N.$$

It is easy to check that if $F, G \in \mathbb{R}[x_1, \ldots, x_{N-1}] + \mathbb{R}[x_1, \ldots, x_{N-1}]x_N$ and $F|_{\mathbb{S}^{N-1}} = G|_{\mathbb{S}^{N-1}}$ then $F = G$.

We denote by $\mathcal{P}_k(E)$ the space of (the restrictions to E of) polynomials of degree at most k ($k \in \mathbb{Z}_+$): $\mathcal{P}_k(E) = \{f \in \mathcal{P}(E) : \deg_* f \le k\}$, where $\deg_* f = \inf\{\deg P : P \in \mathbb{R}[x_1, \ldots, x_N]$ and $f = P|_E\}$ (see [Sk]).

We shall use the notation $\widetilde{x}$ for $(x_1, \ldots, x_{N-1})$, where $x = (x_1, \ldots, x_N)$. Thus $\widetilde{x} = \pi_N(x)$, where $\pi_N : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the natural projection.

If $f : E \to \mathbb{R}$ is a bounded function, then

$$d_k(f) := \operatorname{dist}(f, \mathcal{P}_k(E)) := \inf_{g \in \mathcal{P}_k(E)} \{\|f - g\|_E\}.$$

Let $\Omega \subset \mathbb{R}^N$ and f be a real-valued function defined on a neighbourhood of the closure of Ω. We say that f *vanishes on* $\overline{\Omega}$ *to order at most* d

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if for any $x \in \overline{\Omega}$ there exists $\alpha \in \mathbb{Z}_+^N$ such that $|\alpha| \le d$ and $D^\alpha f(x) \ne 0$ (cf. [G]).

2. Introduction. Markov's inequality for derivatives of polynomials and its generalizations are still the object of many investigations. Let us recall that a compact set $E \subset \mathbb{R}^N$ has *Markov's property* if there exist constants $M, m > 0$ such that for each polynomial $P \in \mathcal{P}(E)$ and each $i = 1, \dots, N$ the following *Markov inequality* holds:

$$\|D_i P\|_E \le M(\deg P)^m \|P\|_E.$$

By writing $E \in \mathcal{M}_N(m, M)$ we mean that $E \subset \mathbb{R}^N$ has Markov's property with constants $M, m > 0$. There are many sets which have Markov's property (see [P2] and [P3]), but Zerner gave an example of a set for which Markov's inequality does not hold (see [Z]). Many authors have tried to determine when a set has Markov's property. It is known (see [P1, Remark 3.5]) that if a compact set E has Markov's property then it is C^∞ *determining*, which means that for each function $f \in C^\infty(\mathbb{R}^N)$ the following implication holds: $f|_E = 0 \Rightarrow \forall \alpha \in \mathbb{Z}_+^N \ \ D^\alpha f|_E = 0$.

In 1990, Pleśniak proved an important theorem (see [P1, Theorem 3.3]) that completes previous results obtained by Pawłucki and Pleśniak for a special class of UPC sets (see [PP]) and provides equivalents of Markov's property for C^∞ determining sets.

THEOREM 2.1 (cf. [P1, Theorem 3.3]). *If E is a C^∞ determining compact subset of $\mathbb{R}^n$, then the following statements are equivalent:*

(i) *E has Markov's property.*
(ii) *There exist positive constants M and r such that for every polynomial p of degree at most $k \in \mathbb{N}_1$, $|p(x)| \le M\|p\|_E$ if $x \in \mathbb{C}^n$ and* $\operatorname{dist}(x, E) \le 1/k^r$.
(ii′) *There exist positive constants M and r such that for every polynomial p of degree at most $k \in \mathbb{N}_1$, $|p(x)| \le M\|p\|_E$ if $x \in \mathbb{R}^n$ and* $\operatorname{dist}(x, E) \le 1/k^r$.
(iii) *E has Bernstein's property: for every function $f : E \to \mathbb{R}$, if for each $s > 0$, $\lim_{k\to\infty} k^s \operatorname{dist}(f, \mathcal{P}_k(E)) = 0$, then there is $\tilde{f} \in C^\infty(\mathbb{R}^N)$ such that $\tilde{f}|_E = f$.*

Note that the converse to (iii) follows from Jackson's inequality for a cube. The result of [PP] has been extended by Goetgheluck [G], who also established the following result:

THEOREM 2.2 (Goetgheluck's theorem, [G, Theorem 1]). *Let Ω be a bounded subset of $\mathbb{R}^N$ which has Markov's property with exponent m, and let $h \in C^\infty(\mathbb{R}^N)$ vanish on $\overline{\Omega}$ to order at most d. Then there exists a positive* *constant $C(h,\Omega)$ such that for every $k \in \mathbb{N}_1$ and $P \in \mathcal{P}_k(\Omega)$ we have*

$$\|P\|_\Omega \le C(h,\Omega)k^{md}\|Ph\|_\Omega.$$

This theorem is an important generalization of the classical *Schur inequality*

$$\forall P \in \mathbb{C}[t] \quad \|P\|_{[-1,1]} \le (\deg P + 1)\|P(t)\sqrt{1-t^2}\|_{[-1,1]}.$$

The following inequality is an easy generalization of the Schur inequality to the Euclidean ball:

$$\forall P \in \mathbb{R}[x] \quad \|P\|_{\mathbb{B}^{N-1}} \le (\deg P + 1)\|P(x)\sqrt{1-x^2}\|_{\mathbb{B}^{N-1}}, \tag{2.1}$$

where $x^2 = x_1^2 + \cdots + x_{N-1}^2$. The above inequality can be verified by taking $Q(t) = P(tu)$, where $x = tu \in \mathbb{B}^{N-1}$, $t \in [-1,1]$ and $\|u\| = 1$. Some versions of Markov's and Bernstein's inequalities for the Euclidean ball have also been proved in [Sa] and [B]:

$$\forall P \in \mathbb{R}[x_1,\ldots,x_{N-1}] \quad \|D_jP\|_{\mathbb{B}^{N-1}} \le (\deg P)^2\|P\|_{\mathbb{B}^{N-1}}, \tag{2.2}$$

$$\forall P \in \mathbb{R}[x]\ \forall x \in \mathbb{B}^{N-1} \quad |D_jP(x)| \le \frac{\deg P}{\sqrt{1-x^2}}\|P\|_{\mathbb{B}^{N-1}}. \tag{2.3}$$

3. Generalized Markov property. There are many algebraic subsets of $\mathbb{R}^N$ which are not C^∞ determining and have Bernstein's property. Such sets cannot have Markov's property, but some of them do have a generalized Markov property that is defined below.

Definition 3.1. For a compact set $E \subset \mathbb{R}^N$ and a polynomial $f \in \mathcal{P}(E)$, we set

$$\|f\|_j^* = \|f\|_E + \inf\{\|D_jF\|_E : F \in \mathbb{R}[x_1,\ldots,x_N],\ F|_E = f\},\ j = 1,\ldots,N.$$

Let us observe that $\|f\|_j^*$ is a norm on $\mathcal{P}(E)$. Moreover if E is a C^∞ determining compact subset of $\mathbb{R}^N$ then $\|f\|_j^* = \|f\|_E + \|D_jf\|_E$ for j in $\{1,\ldots,N\}$.

Definition 3.2. We say that a compact set $E \subset \mathbb{R}^N$ has the *generalized Markov property* if there exist constants $M, m > 0$ such that for each $k \in \mathbb{N}_1$, $f \in \mathcal{P}_k(E)$, $j = 1,\ldots,N$,

$$\|f\|_j^* \le Mk^m\|f\|_E.$$

We shall see that the Euclidean sphere has this property.

Proposition 3.3. *The set $E = \mathbb{S}^{N-1}$ has the generalized Markov property with $m = 2$.*

Proof. Let $f \in \mathcal{P}_k(E)$ with $k \in \mathbb{N}_1$. Then $f(\tilde{x}, x_N) = p(\tilde{x}) + q(\tilde{x})x_N$ for some $p, q \in \mathbb{R}[\tilde{x}]$ such that $\deg p \le k$ and $\deg q \le k-1$. We remark that this extension of f to all of $\mathbb{R}^N$ is not necessarily unique.

Since $\mathbb{S}^{N-1}$ is symmetric, for each $f \in \mathcal{P}(E)$ we have

$$\|f\|_E = \max_{\tilde{x}\in\mathbb{B}^{N-1}} \{|p(\tilde{x})| + |q(\tilde{x})|\sqrt{1-\tilde{x}^2}\}.$$

Set $A := \max\{\|p\|_{\mathbb{B}^{N-1}}, \|q(\tilde{x})\sqrt{1-\tilde{x}^2}\|_{\mathbb{B}^{N-1}}\}$. Hence

$$A \le \|f\|_E \le 2A.$$

For $(\tilde{x}, x_N) \in E$ and $1 \le j \le N-1$ we have

$$\begin{aligned}|D_j f(\tilde{x}, x_N)| &\le \|D_j p(\tilde{x}) + D_j q(\tilde{x}) x_N\|_E \\ &= \max_{\tilde{x}\in\mathbb{B}^{N-1}} \{|D_j p(\tilde{x})| + |D_j q(\tilde{x})\sqrt{1-\tilde{x}^2}|\}.\end{aligned}$$

From (2.2) for $f \in \mathcal{P}_k(E)$ we obtain

$$|D_j f(\tilde{x}, x_N)| \le k^2 \|p\|_{\mathbb{B}^{N-1}} + \|D_j q(\tilde{x})\sqrt{1-\tilde{x}^2}\|_{\mathbb{B}^{N-1}}.$$

Applying (2.3) and (2.1) with $\tilde{x} \in \mathbb{B}^{N-1}$ and $f \in \mathcal{P}_k(E)$ we get

$$\begin{aligned}|D_j f(\tilde{x}, x_N)| &\le k^2 \|p\|_{\mathbb{B}^{N-1}} + (k-1)\|q\|_{\mathbb{B}^{N-1}} \\ &\le k^2 (\|p\|_{\mathbb{B}^{N-1}} + \|q(\tilde{x})\sqrt{1-\tilde{x}^2}\|_{\mathbb{B}^{N-1}}).\end{aligned}$$

We conclude that

$$\|f\|_j^* \le 3k^2 \|f\|_E, \quad j = 1, \dots, N-1.$$

Similarly, by (2.1), we have $\|f\|_N^* \le 2k\|f\|_E$, and the assertion follows with $m = 2$ and $M = 3$. ∎

An inspection of the above proof permits one to establish a similar proposition for any subset E of the sphere which is symmetric in the following sense: for each $(\tilde{x}, x_N) \in E$, $(\tilde{x}, -x_N)$ is an element of E as well.

Proposition 3.4. *Let $E \subset \mathbb{S}^{N-1}$ be a symmetric compact set and let $\widetilde{E} = \pi_N(E)$. If $\widetilde{E} \in \mathcal{M}_{N-1}(m, M)$, then E has the generalized Markov property in $\mathbb{R}^N$. More precisely, it has the generalized Markov property in $\mathbb{R}^N$ with exponent m if $\widetilde{E} \subset \operatorname{int} \mathbb{B}^{N-1}$, and with exponent $2m$ otherwise.*

Proof. We have $\mathcal{P}(E) \subset \mathbb{R}[x_1, \dots, x_{N-1}] + \mathbb{R}[x_1, \dots, x_{N-1}]x_N$. Let $f \in \mathcal{P}_k(E)$ and $f(\tilde{x}, x_N) = p(\tilde{x}) + q(\tilde{x})x_N$. The symmetry of E implies $\|f\|_E = \max_{\tilde{x}\in\widetilde{E}} \{|p(\tilde{x})| + |q(\tilde{x})\sqrt{1-\tilde{x}^2}|\}$. Hence, by Goetgheluck's theorem, for $h(\tilde{x}) = 1 - \tilde{x}^2$ and $\Omega = \widetilde{E}$ there exists a constant C independent of f such that

$$\|D_N f\|_E = \sqrt{\|q^2\|_{\widetilde{E}}} \le \sqrt{C(2k)^{2m} \|q^2(\tilde{x})(1-\tilde{x}^2)\|_{\widetilde{E}}} \le C_1 k^m \|f\|_E$$

with $C_1 = 2^m\sqrt{C}$. Moreover, if $1 \le j \le N-1$, then

$$\begin{aligned}\|D_j f\|_E &\le \|D_j p\|_{\widetilde{E}} + \|D_j q\|_{\widetilde{E}} \le M k^m (\|p\|_{\widetilde{E}} + \|q\|_{\widetilde{E}}) \\ &\le M k^m (\|p\|_{\widetilde{E}} + C_1 k^m \|q(\tilde{x})\sqrt{1-\tilde{x}^2}\|_{\widetilde{E}}) \\ &\le 2M \max\{1, C_1\} k^{2m} \|f\|_E,\end{aligned}$$

which yields the generalized Markov property with $m' = 2m$ and $M' = 2\max\{1, 2M, 2MC_1, C_1\}$.

If $\widetilde{E} \subset \operatorname{int} \mathbb{B}^{N-1}$ then by the compactness of $\widetilde{E}$ we obtain

$$B \max\{\|p\|_{\widetilde{E}}, \|q\|_{\widetilde{E}}\} \le \|f\|_E \le 2\max\{\|p\|_{\widetilde{E}}, \|q\|_{\widetilde{E}}\}$$

with $0 < B := \min_{\tilde{x}\in\widetilde{E}} \sqrt{1-\tilde{x}^2} \le 1$ depending only on E. It follows that

$$\|D_N f\|_E = \|q\|_{\widetilde{E}} \le (1/B)\|f\|_E$$

and for $1 \le j \le N-1$,

$$\begin{aligned}\|D_j f\|_E &\le \|D_j p\|_{\widetilde{E}} + \|D_j q\|_{\widetilde{E}} \\ &\le 2Mk^m \max\{\|p\|_{\widetilde{E}}, \|q\|_{\widetilde{E}}\} \le (2M/B)k^m\|f\|_E.\end{aligned}$$

This means that E has the generalized Markov property with constants $M' = 2\max\{1, 2M\}/B$ and m. ■

Let $\Phi = (\Phi_1, \ldots, \Phi_N) : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial automorphism of degree r.

Proposition 3.5. *If $E \subset \mathbb{R}^N$ has the generalized Markov property, then so does $\Phi(E)$.*

Proof. Let $f \in \mathcal{P}_k(\Phi(E))\setminus\{0\}$. Then there exists $G \in \mathbb{R}[x_1, \ldots, x_N]$ such that $\deg G = k\cdot r$ and $G|_E = f\circ\Phi$, so $f\circ\Phi \in \mathcal{P}_{kr}(E)$. Moreover, since E has the generalized Markov property, there exist constants $M, m > 0$ such that for each $x \in E$ and $j = 1, \ldots, N$ we have

$$|D_j F(x)| \le Mk^m r^m \|f\circ\Phi\|_E = Mk^m r^m \|f\|_{\Phi(E)}$$

for some polynomial F such that $F|_E = f\circ\Phi$. Let $y \in \Phi(E)$ and $x = \Phi^{-1}(y)$. Then

$$D_j(F\circ\Phi^{-1})(y) = \sum_{l=1}^N D_l F(x) D_j \Psi_l(y),$$

where $\Phi^{-1} = (\Psi_1, \ldots, \Psi_N)$. Hence

$$|D_j(F\circ\Phi^{-1})(y)| \le Mk^m r^m \|f\|_{\Phi(E)} \sum_{l=1}^N |D_j\Psi_l(y)| \le M_1 k^m \|f\|_{\Phi(E)},$$

where $M_1 = N\|D_j\Phi^{-1}\|_{\Phi(E)} Mr^m$. This completes the proof. ■

Proposition 3.6. *If $E \subset \mathbb{R}^N$ has Bernstein's property, then so does $Z := \Phi(E)$.*

Proof. Let $g : Z \to \mathbb{R}$ be such that $\lim_{l\to\infty} l^s d_l(g) = 0$ for each $s > 0$. Then there exist constants $C_s(l)$ such that $\lim_{l\to\infty} C_s(l) = 0$ and $\|g - p_l\|_Z \le C_s(l) l^{-s}$ for some $p_l \in \mathcal{P}_l(Z)$. Hence $d_{lr}(g\circ\Phi) \le C_s(l)l^{-s}$, which yields

$\lim_{l\to\infty} l^s d_{lr}(g) = 0$. For each $k \in \mathbb{N}_r$ there exists $l \in \mathbb{Z}_+$ such that $rl \le k < r(l+1)$ and

$$d_k(g \circ \Phi) \le d_{lr}(g \circ \Phi).$$

From this we have $\lim_{k\to\infty} k^s d_k(g \circ \Phi) = 0$. As E has Bernstein's property, there exists $G \in C^\infty(\mathbb{R}^N)$ such that $G|_E = g \circ \Phi$. Hence $G \circ \Phi^{-1} \in C^\infty(\mathbb{R}^N)$ and $G \circ \Phi^{-1}|_Z = g$, and this completes the proof. ∎

We shall need a generalization of Pleśniak's condition $(\mathcal{P})$ (see [P1, Theorem 3.3(ii)]), which plays an important role in problems of the existence of a continuous linear operator extending the traces of C^∞ functions on a compact set $E \subset \mathbb{R}^N$.

For $f \in \mathcal{P}(E)$, $a \ge 1$ and $\epsilon > 0$ we define

$$|f|_{a,\epsilon} = \inf\{\|F\|_{E_\epsilon} : F \in \mathbb{R}[x_1, \ldots, x_N],\ F|_E = f,\ \deg F \le a \deg_* f\},$$

where $E_\epsilon = \{z \in \mathbb{R}^N : \operatorname{dist}(z, E) \le \epsilon\}$.

Definition 3.7. We say that a compact set $E \subset \mathbb{R}^N$ satisfies *Pleśniak's condition* $(\mathcal{P}_\sigma)$, $\sigma \ge 0$, if there exist positive constants m, M_1, M_2 and a constant $a \ge 1$ such that

$$|f|_{a,\epsilon} \le M_2 k^\sigma \|f\|_E \quad \text{if } f \in \mathcal{P}_k(E),\ \epsilon \le M_1/k^m.$$

If $\sigma = 0$, we denote this condition by $(\mathcal{P})$.

Definition 3.8. For a compact set $E \subset \mathbb{R}^N$, $f \in \mathcal{P}(E)$ and $n \in \mathbb{Z}_+$, we put

$$\|f\|_n = \inf\Big\{\max_{|\alpha|\le n} \|D^\alpha F\|_E : F \in \mathbb{R}[x_1, \ldots, x_N],\ F|_E = f\Big\}.$$

We say that a set E has the *strong generalized Markov property* if there exist constants $M, m > 0$ such that for each $n \in \mathbb{Z}_+$ and $f \in \mathcal{P}(E)$,

$$\|f\|_n \le M(\deg_* f)^{nm} \|f\|_E.$$

We have a relation between condition $(\mathcal{P}_\sigma)$ and the strong generalized Markov property, which is similar to the implication (ii)⇒(i) of [P1].

Proposition 3.9. *If a compact set $E \subset \mathbb{R}^N$ satisfies Pleśniak's condition $(\mathcal{P}_\sigma)$ then it has the strong generalized Markov property.*

Proof. The proof is a slight modification of the proof of (ii′)⇒(i) in [P1]. For $x \in E$, we set

$$I_k(x) := \{z \in \mathbb{R}^N : |z_j - x_j| \le M_1/(N^{1/2} k^m),\ j = 1, \ldots, N\} \subset E_{\epsilon_k},$$

where $\epsilon_k = M_1/k^m$.

Let $f \in \mathcal{P}_k(E)$ and let $F \in \mathbb{R}[x_1, \ldots, x_N]$ be such that $\deg F \le a \deg_* f$ and $F|_E = f$. By the classical Markov inequality for a cube we get

$$|D^\alpha F(x)| \le [(a \deg_* f)^2/(M_1/(N^{1/2}(\deg_* f)^m)]^{|\alpha|}\|F\|_{I_k(x)}$$
$$\le M_3^{|\alpha|}(\deg_* f)^{(m+2)|\alpha|}\|F\|_{E_{\epsilon_k}},$$

where $M_3 = a^2 N^{1/2}/M_1$. Taking the supremum over all α with $|\alpha| \le n$ and then the infimum over F, from the assumption that E satisfies Pleśniak's condition we derive

$$\|f\|_n \le M_2 M_3^n (\deg_* f)^{n(m+2)+\sigma}\|f\|_E.$$

Finally, we obtain

$$\|f\|_n \le M_4 (\deg_* f)^{nm_1}\|f\|_E,$$

where the constant M_4 is determined by the equivalence of the norms on the space $\mathcal{P}_1(E)$, and $m_1 = m + 2 + \sigma + s$ with s such that $M_3 \le 2^s$. ∎

The same results can be obtained by taking the following generalizations of Markov's property and condition $(\mathcal{P}_\sigma)$.

Definition 3.10. Let E be a compact subset of $\mathbb{R}^N$. The set E has the *generalized Markov property* $(\mathcal{M}^*)$ if:

(a) there exist a linear map $\Lambda : \mathcal{P}(E) \to \mathbb{R}[x_1, \ldots, x_N]$ and a constant $a \ge 1$ such that $\Lambda(f)|_E = f$ and $\deg \Lambda(f) \le a \deg_* f$;

(b) there exist constants M, m such that for each $f \in \mathcal{P}(E)$,

$$\|D_j \Lambda(f)\|_E \le M(\deg \Lambda(f))^m \|f\|_E, \quad j = 1, \ldots, N.$$

The set E has the *strong generalized Markov property* $(\mathcal{M}_s^*)$ if it fulfils both condition (a) and

(c) there exist constants M_1, m_1 such that for each $f \in \mathcal{P}(E)$,

$$\|D^\alpha \Lambda(f)\|_E \le M_1(\deg \Lambda(f))^{|\alpha| m_1}\|f\|_E, \quad \alpha \in \mathbb{Z}_+^N.$$

The set E satisfies *condition* $(\mathcal{P}_\sigma^*)$ (for some $\sigma \ge 0$) if it fulfils (a) and

(d) there exist constants $m_2, M_2, M_3 > 0$ such that

$$\|\Lambda(f)\|_{E_\epsilon} \le M_2 k^\sigma \|f\|_E \quad \text{for } f \in \mathcal{P}_k(E),\ \epsilon \le M_3 k^{-m_2},\ k \in \mathbb{N}_2.$$

Proposition 3.11. *We have* $(\mathcal{P}_\sigma^*) \Leftrightarrow (\mathcal{M}_s^*) \Rightarrow (\mathcal{M}^*)$.

Proof. The implication $(\mathcal{M}_s^*) \Rightarrow (\mathcal{M}^*)$ is obvious. The equivalence $(\mathcal{P}_\sigma^*) \Leftrightarrow (\mathcal{M}_s^*)$ can be proved as in [P1]. Assume $(\mathcal{P}_\sigma^*)$. Let $x \in E$. For $k \in \mathbb{N}_1$, we define

$$I_k(x) := \{z \in \mathbb{R}^N : |z_j - x_j| \le M_3/(N^{1/2}k^{m_2})\} \subset E_{\epsilon_k},$$

where M_3, m_2 are the constants from condition $(\mathcal{P}_\sigma^*)$ and $\epsilon_k = M_3/k^{m_2}$. Let $f \in \mathcal{P}(E)$. By the classical Markov inequality for a cube we have

$$|D^\alpha \Lambda(f)(x)| \le \big[(\deg \Lambda(f))^2/\big(M_3/(N^{1/2}(\deg \Lambda(f))^{m_2})\big)\big]^{|\alpha|}\|\Lambda(f)\|_{I_{\deg \Lambda(f)}(x)}.$$

By a similar argument to that of the previous proof, we get

$$|D^\alpha \Lambda(f)(x)| \le M_2 N^{|\alpha|/2} M_3^{-|\alpha|} (\deg \Lambda(f))^{(m_2+2)|\alpha|+\sigma} \|f\|_E \\ \le M_4 (\deg \Lambda(f))^{(m_2+2+s)|\alpha|+\sigma} \|f\|_E,$$

where the constant M_4 is determined by the equivalence of the norms on $\mathcal{P}_1(E)$, and s is a constant such that $N^{1/2} M_3^{-1} \le 2^s$. We can take $M_1 = M_4$ and $m_1 = m_2 + 2 + s + \sigma$.

Assume now that $(\mathcal{M}_s^*)$ holds. Let $f \in \mathcal{P}_k(E)$. For $z \in \mathbb{R}^N$, there exists $x \in E$ such that $\operatorname{dist}(z, E) = \operatorname{dist}(z, x)$. By Taylor's formula we get

$$\Lambda(f)(z) = \sum_{|\alpha| \le k} (D^\alpha \Lambda(f)(x)/\alpha!)(z - x)^\alpha.$$

By the assumption on $\delta := \operatorname{dist}(z, E)$, we obtain

$$|\Lambda(f)(z)| \le M_1 \|f\|_E \sum_{|\alpha| \le k} (a \deg_* f)^{|\alpha| m_1} \delta^{|\alpha|}/\alpha! \le M_1 \|f\|_E \sum_{l=0}^{k} [N(ak)^{m_1} \delta]^l / l!.$$

Hence for $\delta \le 1/(ak)^{m_1}$ we have

$$|\Lambda(f)(z)| \le M_1 \|f\|_E \sum_{l=0}^{k} N^l / l! \le M_1 e^N \|f\|_E.$$

We can take $m_2 = m_1$, $M_3 = 1/a^{m_1}$ and $M_2 = M_1 e^N$. ∎

4. Markov's and Bernstein's properties for subsets of algebraic sets. In this section we consider the sets of the form

$$\mathbb{V} = \{(\tilde{x}, x_N) \in \mathbb{R}^N : x_N^2 = Q(\tilde{x})\},$$

where $Q \in \mathbb{R}[x_1, \ldots, x_{N-1}]$ is such that $Q^{-1}([0, +\infty)) \ne \emptyset$ and $\deg Q \le d$. For symmetric subsets of this kind, we shall prove some theorems which correspond to the propositions of the previous section.

THEOREM 4.1. *Let $E \subset \mathbb{V}$ be a compact symmetric set and let $\widetilde{E} = \pi_N(E)$. If $\widetilde{E} \in \mathcal{M}(m, M)$, then E has the generalized Markov property with respect to $\mathbb{R}^N$.*

Proof. Observe that

$$\mathcal{P}_l(E) \subset \mathbb{R}_{d_1 l}[x_1, \ldots, x_{N-1}] + \mathbb{R}_{d_1 l - 1}[x_1, \ldots, x_{N-1}] x_N,$$

where $d_1 = [d/2] + 1$. Indeed, if $f \in \mathcal{P}(E)$ and $\deg_* f = l$, then there exists $F \in \mathbb{R}[x_1, \ldots, x_N]$ such that $F|_E = f$ and $\deg F = l$. Since $E \subset \mathbb{V}$, we have $x_N^2 = Q(\tilde{x})$. Hence $F(\tilde{x}, x_N) = p(\tilde{x}) + q(\tilde{x}) x_N$ for $(\tilde{x}, x_N) \in E$, where $\deg p \le dl/2$ and $\deg q \le dl/2 - 1$. We define $F_*(\tilde{x}, x_N) := p(\tilde{x}) + q(\tilde{x}) x_N$ for $(\tilde{x}, x_N) \in E$.

Since E is symmetric, we get $\|F_*\|_E = \max_{\tilde{x} \in \widetilde{E}} \{|p(\tilde{x})| + |Q(\tilde{x})|^{1/2} |q(\tilde{x})|\}$.

Applying Goetgheluck's theorem to Q and $\widetilde{E}$ we find that there exists a constant $C>0$ such that, for each $R\in\mathbb{R}[\tilde{x}]$ with $\deg R>0$,

$$\|R\|_{\widetilde{E}}\le C(\deg R)^{md}\|Q\cdot R\|_{\widetilde{E}}.$$

Setting $R(\tilde{x})=q^2(\tilde{x})$ for $\deg q>0$ gives

$$\|q\|_{\widetilde{E}}\le C_1(\deg q)^{md/2}\|\,|Q|^{1/2}q\|_{\widetilde{E}} \tag{4.1}$$

with $C_1=2^{md/2}\sqrt{C}$.

It remains to estimate the derivative D_NF_*. We have

$$\|D_NF_*\|_E\le C_2l^{md/2}\|\,|Q|^{1/2}q\|_{\widetilde{E}}\le C_2l^{md/2}\|F_*\|_E,$$

where $C_2=C_1d_1^{md/2}$. Moreover, for $1\le j\le N-1$ we have

$$D_jF_*(\tilde{x},x_N)=D_jp(\tilde{x})+D_jq(\tilde{x})x_N.$$

Hence, since E is symmetric and $\widetilde{E}$ has Markov's property, there exists a constant C_3 such that

$$\|D_jF_*\|_E\le C_3l^m\max\{\|p\|_{\widetilde{E}},\|q\|_{\widetilde{E}}\}.$$

Now, by condition (4.1),

$$\|D_jF_*\|_E\le C_4l^{m+md/2}\max\{\|p\|_{\widetilde{E}},\|\,|Q|^{1/2}q\|_{\widetilde{E}}\}\le C_4l^{m(1+d/2)}\|F_*\|_E$$

with $C_4=C_3\max\{C_2,1\}$. Therefore

$$\|f\|_j^*\le C_5(\deg_* f)^{m(1+d/2)}\|f\|_E\quad \text{for } j=1,\dots,N,$$

where $C_5=1+\max\{C_2,C_4\}$. This completes the proof. ∎

THEOREM 4.2. *Let $\mathbb{V}$ be as above, and let E be a compact symmetric subset of $\mathbb{V}$. Let $\widetilde{E}$ be the projection of E onto $\mathbb{R}^{N-1}$. If $\widetilde{E}$ has Bernstein's property, then so does E.*

Proof. We have

$$\mathcal{P}_k(E)\subset\mathbb{R}_{d_1k}[x_1,\dots,x_{N-1}]+\mathbb{R}_{d_1k-1}[x_1,\dots,x_{N-1}]x_N,$$

where $d_1=[d/2]+1$.

Let $g:E\to\mathbb{R}$ be such that for each $s>0$ one has

$$\lim_{k\to\infty}k^sd_k(g)=0.$$

Then $g\in C(E)$. Fix $k\in\mathbb{N}$. There exist $p_k\in\mathbb{R}_{dk}[\tilde{x}]$ and $q_k\in\mathbb{R}_{dk-1}[\tilde{x}]$ such that

$$d_k(g)=\sup_{(\tilde{x},x_N)\in E}|g(\tilde{x},x_N)-p_k(\tilde{x})-q_k(\tilde{x})x_N|.$$

We are going to show that $(p_k(\tilde{x}))_{k\in\mathbb{N}}$ and $(q_k(\tilde{x}))_{k\in\mathbb{N}}$ are Cauchy sequences in $C(\widetilde{E})$. Let $n,l\in\mathbb{N}$. We have

$$\|p_n(\tilde{x})+q_n(\tilde{x})x_N-p_l(\tilde{x})-q_l(\tilde{x})x_N\|_E\le 2\max\{d_n(g),d_l(g)\}.$$

Since E is symmetric,

$$\|p_n - p_l\|_{\widetilde{E}} \le 2\max\{d_n(g), d_l(g)\}.$$

On the other hand, by Goetgheluck's theorem,

$$\begin{aligned}\|q_n - q_l\|_{\widetilde{E}} &\le C_6 \max\{n,l\}^{md/2}\|(q_n - q_l)|Q|^{1/2}\|_{\widetilde{E}}\\ &\le 2C_6 \max\{n,l\}^{md/2}\max\{d_n(g), d_l(g)\}\end{aligned}$$

with $C_6 = C_1 d_1^{md/2}$. Hence for $l = n+1$ we have

$$\|q_n - q_{n+1}\|_{\widetilde{E}} \le C_7 n^{md/2} d_n(g) = O(1/n^2),$$

where $C_7 = C_6 2^{md/2+1}$. For $n < l$ we get

$$\|q_n - q_l\|_{\widetilde{E}} \le \sum_{k=n}^{l-1} \|q_k - q_{k+1}\|_{\widetilde{E}} = O(1/n).$$

In a similar way we show that

$$\|p_n - p_l\|_{\widetilde{E}} = O(1/n).$$

By the completeness of $C(\widetilde{E})$, there exist $g_1, g_2 \in C(\widetilde{E})$ such that

$$g(\tilde{x}, x_N) = g_1(\tilde{x}) + g_2(\tilde{x})x_N,$$

where

$$g_1(\tilde{x}) = \lim_{n\to\infty} p_n(\tilde{x}), \qquad g_2(\tilde{x}) = \lim_{n\to\infty} q_n(\tilde{x}).$$

Letting $l \to \infty$, for each $s > 0$ we obtain

$$\|g_1 - p_n\|_{\widetilde{E}} \le 2d_n(g) = o(n^{-s}) \quad \text{and} \quad \|g_2 - q_n\|_{\widetilde{E}} = o(n^{-s}).$$

Then for $n = [k/d]$ we have

$$d_k(g_1) \le \|g_1 - p_{[k/d]}\|_{\tilde{E}} = o(k^{-s}) \quad \text{and} \quad d_k(g_2) \le \|g_2 - q_{[k/d]}\|_{\tilde{E}} = o(k^{-s}).$$

Since $\widetilde{E}$ has Bernstein's property, there exist $G_1, G_2 \in C^\infty(\mathbb{R}^{N-1})$ such that $G_1|_{\widetilde{E}} = g_1$ and $G_2|_{\widetilde{E}} = g_2$. Define $G(\tilde{x}, x_N) := G_1(\tilde{x}) + G_2(\tilde{x})x_N$. Then $G \in C^\infty(\mathbb{R}^N)$ and $G|_E = g$, and this completes the proof. ∎

THEOREM 4.3. *Let $\mathbb{V}$, E and $\widetilde{E}$ satisfy the assumptions of Theorem* 4.1. *If $\widetilde{E} \in \mathcal{M}(m, M)$, then E has the strong generalized Markov property.*

Proof. Let as above

$$f \in \mathcal{P}_k(E) \subset \mathbb{R}_{d_1k}[x_1, \ldots, x_{N-1}] + \mathbb{R}_{d_1k-1}[x_1, \ldots, x_{N-1}]x_N$$

and $F(\tilde{x}, x_N) = p(\tilde{x}) + q(\tilde{x})x_N$. It is sufficient to estimate $\|D^\alpha F\|_E$ for $\alpha = (\beta, 0)$ and $\alpha = (\beta, 1)$ with $|\beta| \le dk$.

Let us first examine $\|D^{(\beta,1)}F\|_E$. From (4.1), since $\widetilde{E} \in \mathcal{M}(m, M)$, for $k \geq 2$ we have

$$\|D^{(\beta,1)}F\|_E \leq M^{|\beta|}(d_1 k)^{|\beta|m}\|q\|_{\widetilde{E}} \leq C_1 M^{|\beta|}(d_1 k)^{|\beta|m+dm/2}\|q|Q|^{1/2}\|_{\widetilde{E}}$$
$$\leq C_1 M^{|\beta|} d_1^{|\beta|m_1} k^{|\beta|m_1}\|f\|_E \leq C_1 k^{|\beta|m_2}\|f\|_E,$$

where $m_1 = m + dm/2 > m$ and $m_2 = m_1 + s > m_1$ with s such that $Md_1^{m_1} < 2^s$. Since E is compact and symmetric, there exists a constant A such that $\|D^{(\beta,0)}F\|_E \leq A \max\{\|D^\beta p\|_{\widetilde{E}}, \|D^\beta q\|_{\widetilde{E}}\}$. In a similar way, for $k \geq 2$ we get

$$\|D^{(\beta,0)}F\|_E \leq AM^{|\beta|}(d_1 k)^{|\beta|m} \max\{\|p\|_{\widetilde{E}}, \|q\|_{\widetilde{E}}\}$$
$$\leq B_1 M^{|\beta|}(d_1 k)^{|\beta|m+dm/2} \max\{\|p\|_{\widetilde{E}}, \|q|Q|^{1/2}\|_{\widetilde{E}}\}$$
$$\leq B_1 k^{|\beta|m_2}\|f\|_E,$$

where $B_1 = A\max\{1, C_1\}$ and m_1, m_2 are defined above.

On the other hand, if $k = 1$, then for $|\beta| > 1$ we have $D^\beta p = D^\beta q = 0$, so there exists a constant B_2 such that E has the strong generalized Markov property with exponent m_2. ∎

Theorem 4.4. *Let* $\mathbb{V}$, E *and* $\widetilde{E}$ *satisfy the assumptions of Theorem* 4.1. *If* $\widetilde{E}$ *has Markov's property in* $\mathbb{R}^{N-1}$, *then* E *satisfies condition* $(\mathcal{P}_\sigma)$ *with* $\sigma = md/2$, *where* m *is the constant of Markov's inequality for* $\widetilde{E}$.

Proof. Since $\widetilde{E}$ has Markov's property, it satisfies condition $(\mathcal{P})$ with some constants m, M_1, M_2 (see [P1]). With the notation of Definition 3.7, letting M_1 decrease and M_2 increase we obtain

$$|f|_{d,\epsilon_k} \leq C_1 \max\{\|p\|_{\widetilde{E}_{\epsilon_k}}, \|q\|_{\widetilde{E}_{\epsilon_k}}\} \leq C_1 M_2 \max\{\|p\|_{\widetilde{E}}, \|q\|_{\widetilde{E}}\}$$
$$\leq C_2 k^{md/2}\|f\|_E,$$

and the theorem follows. ∎

Remark 4.5. From the proofs of Theorems 4.3 and 4.4 we get even more: the sets under consideration satisfy conditions $(\mathcal{M}^*)$, $(\mathcal{M}_s^*)$ and $(\mathcal{P}_\sigma^*)$ of Definition 3.10.

Example 4.6. One can provide other examples of sets having the (strong) generalized Markov property, Bernstein's property or satisfying condition $(\mathcal{P})$ by considering algebraic sets of the form

$$\mathbb{V} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j^2 = Q_j(x_{m+1}, \ldots, x_N) \text{ for } j = 1, \ldots, m\},$$

where Q_j $(j = 1, \ldots, m)$ are polynomials such that $Q_j^{-1}([0, +\infty)) \neq \emptyset$ and $m \leq N$.

Example 4.7. We can also take images of symmetric subsets of $\mathbb{V}$ from Example 4.6 under polynomial automorphisms $\Phi = (\Phi_1, \ldots, \Phi_N) : \mathbb{R}^N \to \mathbb{R}^N$.

Open problems

1) Do the sets $x^3 + y^3 = 1$ and $x^4 + y^4 = 1$ have the generalized Markov property?

2) Are the generalized Markov property of Definition 3.2 and the generalized Markov property $(\mathcal{M}^*)$ equivalent?

3) It is obvious that a set with the strong generalized Markov property also has the generalized Markov property. Does the converse hold? An answer may bring a solution to the following problem.

4) By Theorem 2.1, Markov's property is equivalent to Bernstein's property for C^∞ determining sets. One can ask whether there is equivalence between the strong generalized Markov property and Bernstein's property for subsets of semialgebraic sets.

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