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Existence and uniqueness of solutions of nonlinear infinite systems of parabolic differential-functional equations

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Abstract. We consider the Fourier first initial-boundary value problem for an infinite system of weakly coupled nonlinear differential-functional equations of parabolic type. The right-hand sides of the system are functionals of unknown functions. The existence and uniqueness of the solution are proved by the Banach fixed point theorem.

1. Introduction. We consider an infinite system of weakly coupled nonlinear differential-functional equations of the form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z), \qquad i \in S, \tag{1}$$

where

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i, \qquad \mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k},$$

$x = (x_1, \ldots, x_m)$, $(t,x) \in (0,T] \times G := D$, $T < \infty$, $G \subset \mathbb{R}^m$, G is an open bounded domain with boundary $\partial G \in C^{2+\alpha} \cap C^{2-0}$ $(0 < \alpha \le 1)$. S is a set of indices (finite or infinite) and z stands for the mapping

$$z : S \times \overline{D} \ni (i,t,x) \mapsto z^i(t,x) \in \mathbb{R},$$

composed of the unknown functions z^i.

Let $B(S)$ be the Banach space of mappings

$$v : S \ni i \mapsto v^i \in \mathbb{R},$$

with the finite norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\}.$$

The case of finite systems ($B(S) = \mathbb{R}^r$) was treated in [1]. For S infinite countable we have $B(S) = l^\infty$ and we now focus on such infinite systems.

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According to what has just been stated,

$$\|v\|_{B(S)} = \|v\|_{l^\infty}.$$

Denote by $C_S(\overline{D})$ the Banach space of mappings

$$w : \overline{D} \ni (t,x) \mapsto (w(t,x) : S \ni i \mapsto w^i(t,x) \in \mathbb{R}) \in l^\infty,$$

where the functions w^i are continuous in $\overline{D}$, with the finite norm

$$\|w\|_0 := \sup\{|w^i(t,x)| : (t,x) \in \overline{D},\ i \in S\}.$$

A mapping z will be called *regular in* $\overline{D}$ if the functions z^i $(i \in S)$ are continuous in $\overline{D}$ and have continuous derivatives $\partial z^i/\partial t, \partial^2 z^i/\partial x_j \partial x_k$ in D for $j,k = 1,\ldots,m$.

For system (1) we consider the Fourier first initial-boundary value problem:

Find a *regular solution* (or briefly: *solution*) z of system (1) in $\overline{D}$ satisfying the initial-boundary value condition

$$z(t,x) = g(t,x) \quad \text{for } (t,x) \in \Gamma, \tag{2}$$

where $g = (g^1, g^2, \ldots)$.

We define $\sigma = (0,T] \times \partial G$, $D_0 = \{(t,x) : t = 0,\ x \in \overline{G}\}$, $\Gamma = D_0 \cup \sigma$, $\overline{D} = D \cup \Gamma$. For any $\tau \in (0,T]$ we set $D^\tau = (0,\tau] \times G$, $\sigma^\tau = (0,\tau] \times \partial G$, $\Gamma^\tau = D_0 \cup \sigma^\tau$, $\overline{D}^\tau = D^\tau \cup \Gamma^\tau$. Obviously $D^T = D$.

In [2], to solve the above problem, we used the monotone iterative method (sometimes also called the method of lower and upper functions). However, this method requires assuming the monotonicity of the right-hand side functions f^i with respect to the function argument (cf. [1]). This is not a typical assumption in existence and uniqueness theorems. Yet an unquestionable advantage of the monotone method is the possibility of constructing sequences of successive approximations which tend monotonically—one from above and the other from below—to the desired exact solution. Moreover, the speed of the convergence is at least exponential.

In the present paper, to prove the existence and uniqueness of solution, we apply the Banach fixed point theorem. Considering mainly Banach spaces of bounded continuous functions, we give some natural sufficient conditions for the existence and uniqueness. We notice that finite systems were studied by H. Ugowski [5] and H. Leszczyński [4].

2. Notations, definitions and assumptions. The Hölder space $C^{l+\alpha}(\overline{D}) := C^{(l+\alpha)/2, l+\alpha}(\overline{D})$ $(l = 0,1,2,\ldots; 0 < \alpha < 1)$ is the space of continuous functions h in $\overline{D}$ whose derivatives $\partial^{r+s}h/\partial t^r \partial x^s := D_t^r D_x^s h(t,x)$ $(0 \le 2r + s \le l)$ all exist and are Hölder continuous with exponent α $(0 <$

$\alpha < 1$) in D, with the finite norm

$$|h|_{l+\alpha} := \sup_{\substack{P\in D\\ 0\le 2r+s\le l}} |D_t^r D_x^s h(P)| + \sup_{\substack{P,P'\in D\\ 2r+s=l\\ P\neq P'}} \frac{|D_t^r D_x^s h(P) - D_t^r D_x^s h(P')|}{[d(P,P')]^\alpha},$$

where $d(P,P')$ is the parabolic distance of points $P=(t,x), P'=(t',x')\in\mathbb{R}^{m+1}$,

$$d(P,P') = (|t-t'| + |x-x'|^2)^{1/2},$$

and $|x| = (\sum_{j=1}^m x_j^2)^{1/2}$.

In particular we have the following norms

$$|h|_0 = \sup_{P\in D} |h(P)|, \qquad |h|_{0+\alpha} = |h|_0 + \sup_{\substack{P,P'\in D\\ P\neq P'}} \frac{|h(P)-h(P')|}{[d(P,P')]^\alpha},$$

$$|h|_{1+\alpha} = |h|_{0+\alpha} + \sum_{j=1}^m |D_{x_j} h|_{0+\alpha},$$

$$|h|_{2+\alpha} = |h|_{0+\alpha} + \sum_{j=1}^m |D_{x_j} h|_{0+\alpha} + \sum_{j,k=1}^m |D^2_{x_j x_k} h|_{0+\alpha} + |D_t h|_{0+\alpha}.$$

By $C_S^{l+\alpha}(\overline{D})$ we denote the Banach space of mappings w such that $w^i\in C^{l+\alpha}(\overline{D})$ for all $i\in S$ with the finite norm

$$\|w\|_{l+\alpha} := \sup\{|w^i|_{l+\alpha} : i\in S\}.$$

The boundary norm $\|\cdot\|^{\Gamma}_{l+\alpha}$ for a function $\phi\in C_S^{l+\alpha}(\Gamma)$ is defined as

$$\|\phi\|^{\Gamma}_{l+\alpha} := \inf_{\Phi} \|\Phi\|_{l+\alpha},$$

where the infimum is taken over all extensions Φ of ϕ onto $\overline{D}$.

Finally, by $|\cdot|^{D^\tau}_{l+\alpha}$ and $\|\cdot\|^{D^\tau}_{l+\alpha}$ we denote the relevant norms in the spaces $C^{l+\alpha}(\overline{D}^\tau)$ and $C_S^{l+\alpha}(\overline{D}^\tau)$, respectively.

We denote by $C^{k-0}(D)$ ($k=1,2$) the space all functions h for which the following norms are finite (see [3], p. 190):

$$|h|_{1-0} = |h|_0 + \sup_{\substack{P,P'\in D\\ P\neq P'}} \frac{|h(P)-h(P')|}{|t-t'|+|x-x'|}, \qquad |h|_{2-0} = |h|_{1-0} + \sum_{j=1}^m |D_{x_j} h|_{1-0}.$$

We assume that the functions

$$f^i : \overline{D}\times C_S(\overline{D}) \ni (t,x,s) \mapsto f^i(t,x,s)\in\mathbb{R}, \qquad i\in S,$$

are continuous and satisfy the following assumptions:

(H_f) The functions f^i ($i\in S$) are *uniformly Hölder continuous* (with exponent α) with respect to t and x in $\overline{D}$, i.e., $f(\cdot,\cdot,s)\in C_S^{0+\alpha}(\overline{D})$.

(L) f^i $(i \in S)$ satisfy the *uniform Lipschitz condition* with respect to s, i.e., for all $s, \widetilde{s} \in C_S(\overline{D})$ we have

$$|f^i(t,x,s) - f^i(t,x,\widetilde{s})| \le L\|s - \widetilde{s}\|_0 \quad \text{for } (t,x) \in D,$$

where $L > 0$ is a constant.

(V) The f_i satisfy the *Volterra condition*: for all $(t,x) \in \overline{D}$ and $s, \widetilde{s} \in C_S(\overline{D})$, if $s^j(\overline{t},x) = \widetilde{s}^j(\overline{t},x)$ for $0 \le \overline{t} \le t,\ j \in S$, then $f^i(t,x,s) = f^i(t,x,\widetilde{s})$ $(i \in S)$.

(H_a) The coefficients $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ $(j,k = 1,\ldots,m,\ i \in S)$ in (1) are uniformly Hölder continuous (with exponent α) in $\overline{D}$, i.e., $a^i_{jk} = a^i_{jk}(\cdot,\cdot) \in C^{0+\alpha}(\overline{D})$ and a^i_{jk} belong to $C^{1-0}(\sigma)$.

This implies the existence of constants $K_1, K_2 > 0$ such that

$$\sum_{j,k=1}^{m} |a^i_{jk}|_{0+\alpha} \le K_1, \qquad \sum_{j,k=1}^{m} |a^i_{jk}|^{\varGamma}_{1-0} \le K_2, \qquad i \in S.$$

Moreover, we assume that

(H_g) $g \in C^{2+\alpha}_S(\varGamma) \cap C^{1+\beta}_S(\varGamma)$, where $0 < \alpha < \beta < 1$.

We also assume that the operators $\mathcal{F}^i$ $(i \in S)$ are uniformly parabolic in $\overline{D}$ (the operators $\mathcal{A}^i$ are uniformly elliptic in $\overline{D}$), i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a^i_{jk}(t,x)\xi_j\xi_k \ge \mu \sum_{j=1}^{m} \xi_j^2$$

for all $\xi = (\xi_1,\ldots,\xi_m) \in \mathbb{R}^m$, $(t,x) \in \overline{D}$, $i \in S$.

Remark 1. If the function f^i $(i \in S)$ satisfies the Lipschitz (L) and the Volterra (V) conditions then f^i satisfies the following condition:

(L^*) for all $s, \widetilde{s} \in C_S(\overline{D})$ we have

$$|f^i(t,x,s) - f^i(t,x,\widetilde{s})| \le L\|s - \widetilde{s}\|^{D^t}_0 \quad \text{for } (t,x) \in D^t,\ i \in S,$$

where $L > 0$ is a constant,

and vice versa.

The fact that in condition (L^*) we have $\|s - \widetilde{s}\|^{D^t}_0$ means that for functions $s, \widetilde{s} \in C_S(\overline{D})$ such that $s^i(\widetilde{t},\widetilde{x}) = \widetilde{s}^i(\widetilde{t},\widetilde{x})$ for $(\widetilde{t},\widetilde{x}) \in \overline{D}^t$, the function f as a functional of s takes the same values. Therefore f satisfies the Volterra condition (V). Moreover, if f satisfies (L^*), then (L) holds because $\|s - \widetilde{s}\|^{D^t}_0 \le \|s - \widetilde{s}\|_0$. The reverse implication is obvious.

Remark 2. We remark that if $g \in C^{2+\alpha}_S(\varGamma)$ and $\partial G \in C^{2+\alpha}$ then, without loss of generality, we can consider the homogeneous initial-boundary condition

$$z(t,x) = 0 \qquad \text{for } (t,x) \in \varGamma. \tag{3}$$

Accordingly, in what follows we confine ourselves to considering the homogeneous problem (1), (3) in $\overline{D}$.

3. Existence and uniqueness theorems

THEOREM 1. *Under the above assumptions, $\tau^* \in (0,T]$ be sufficiently small. Then there exists a unique solution z of the problem* (1), (3) *in the domain $\overline{D}^\tau$, where $0 < \tau < \tau^* \le T$, and $z \in C_S^{2+\alpha}(\overline{D}^\tau) \cap C_S^{1+\beta}(\overline{D}^\tau)$, $0 < \alpha < \beta < 1$.*

Before going into the proof we introduce the Nemytskiĭ operator and prove some lemmas.

Let $\eta \in C_S(\overline{D})$. We define the *Nemytskiĭ operator* $\mathbf{F} = (\mathbf{F}^1, \mathbf{F}^2, \ldots)$ by setting

$$\mathbf{F}^i[\eta](t,x) := f^i(t,x,\eta) = f^i(t,x,\eta^1(\cdot,\cdot),\eta^2(\cdot,\cdot),\ldots), \qquad i \in S.$$

LEMMA 1. *If the function $f = (f^1, f^2, \ldots)$ generating the Nemytskiĭ operator $\mathbf{F}$ satisfies conditions (H_f) and (L), then*

(a) $\mathbf{F} : C_S^{0+\alpha}(\overline{D}) \to C_S^{0+\alpha}(\overline{D})$;

(b) $\mathbf{F}$ *maps every bounded subset of $C_S^{0+\alpha}(\overline{D})$ into a bounded set of $C_S^{0+\alpha}(\overline{D})$.*

Proof. (a) is obvious. If now $\eta \in C_S^{0+\alpha}(\overline{D})$ and $\|\eta\|_{0+\alpha} \le M$ then

$$\begin{aligned}
|\mathbf{F}^i[\eta](t,x)| &\le |\mathbf{F}^i[\eta](t,x) - \mathbf{F}^i[0](t,x)| + |\mathbf{F}^i[0](t,x)| \\
&= |f^i(t,x,\eta) - f^i(t,x,0)| + |f^i(t,x,0)| \\
&\le L\|\eta\|_0 + |f^i(t,x,0)| \le L\|\eta\|_{0+\alpha} + |f^i(t,x,0)|
\end{aligned}$$

and therefore

$$\|\mathbf{F}^i[\eta]\|_0 \le L\|\eta\|_{0+\alpha} + M_0,$$

where $M_0 := \|\mathbf{F}[0]\|_0$.

From (H_f) we have

$$\begin{aligned}
|\mathbf{F}^i[\eta](t,x) - \mathbf{F}^i[\eta](t',x')| &= |f^i(t,x,\eta) - f^i(t',x',\eta)| \\
&\le H(|t-t'| + |x-x'|^2)^{\alpha/2},
\end{aligned}$$

where $H = \text{const} > 0$ is the Hölder coefficient of f^i $(i \in S)$.

By the definition of the norm in $C_S^{0+\alpha}(\overline{D})$ we obtain

$$\|\mathbf{F}[\eta]\|_{0+\alpha} \le LM + M_0 + H. \ \blacksquare$$

We consider the linear initial-boundary value problem

$$
\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & \text{in } D,\ i \in S, \\ \gamma(t,x) = g(t,x) & \text{on } \varGamma, \end{cases} \tag{4}
$$

where $\gamma = (\gamma^1, \gamma^2, \ldots)$ and $\delta = (\delta^1, \delta^2, \ldots)$.

Lemma 2. *If $\delta \in C_S^{0+\alpha}(\overline{D})$, the assumptions (H_a), (H_g) hold and $\mathcal{F}^i[g^i](t,x) = \delta^i(t,x)$ on $\partial G \in C^{2+\alpha}$ $(i \in S)$ then problem* (4) *has a unique solution γ, and furthermore, $\gamma \in C_S^{2+\alpha}(\overline{D})$. Moreover, the following Schauder type $(2+\alpha)$-estimate holds*:

$$
\|\gamma\|_{2+\alpha} \le c(\|\delta\|_{0+\alpha} + \|g\|_{2+\alpha}^{\varGamma}), \tag{5}
$$

where $c > 0$ is a constant depending only on the constants μ, K_1, α and the geometry of the domain D.

Proof. Observe that (4) has the following property: the ith equation depends on the ith unknown function only. Therefore, the theorem on the existence and uniqueness of solution of the Fourier first initial-boundary value problem for linear parabolic equations ([3], Theorems 6 and 7, p. 65) and the definition of the norm in $C_S^{k+\alpha}(\overline{D})$ yield the statement of the lemma immediately.

Now we consider the linear homogeneous initial-boundary value problem

$$
\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & \text{in } D,\ i \in S, \\ \gamma(t,x) = 0 & \text{on } \varGamma. \end{cases} \tag{6}
$$

Using the same arguments as previously, from A. Friedman's theorem on the a priori estimates of the $(1+\beta)$-type for solutions of linear parabolic equations ([3], Theorem 4 and its proof, pp. 191–201) we directly get the following lemma.

Lemma 3. *Assume that $\delta \in C_S(\overline{D})$, $\partial G \in C^{2+\alpha} \cap C^{2-0}$ and (H_a) holds. Let $\delta(t,x)$ vanish on ∂G and let γ be a solution of problem* (6). *Then, for any $0 < \beta < 1$, there exists a constant $K > 0$, depending only on β, μ, K_1, K_2 and the geometry of the domain D, such that*

$$
\|\gamma\|_{1+\beta} \le K\|\delta\|_0. \tag{7}
$$

Moreover, there exists a constant $\overline{K} > 0$ depending on the same parameters as K such that

$$
\|\gamma\|_{1+\beta}^{D^\tau} \le \overline{K}\tau^{(1-\beta)/2}\|\delta\|_0^{D^\tau} \tag{8}
$$

for $0 < \tau \le T$.

Proof of Theorem 1. Define

$$
A = \{u \in C_S^{1+\alpha}(\overline{D}^\tau) : u(t,x) = 0 \text{ on } \varGamma^\tau,\ 0 < \tau \le T,\ 0 < \alpha < 1\}.
$$

The set A is closed in $C_S^{1+\alpha}(\overline{D}^\tau)$.

For $u \in A$ we define a mapping $\mathbf{T}$ setting $z = \mathbf{T}[u]$, where z is the (supposedly unique) solution of the linear initial-boundary value problem

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = \mathbf{F}^i[u](t,x) & \text{in } D^\tau,\ i \in S, \\ z(t,x) = 0 & \text{on } \Gamma^\tau. \end{cases} \tag{9}$$

For $u \in A$ by Lemma 1(a) we have $\mathbf{F}[u] \in C_S^{0+\alpha}(\overline{D}^\tau)$. From Lemma 2 it follows that problem (9) has a unique solution z and $z \in C_S^{2+\alpha}(\overline{D}^\tau)$. Moreover, by Lemma 3, $z \in C_S^{1+\beta}(\overline{D}^\tau)$ for any β, $0 < \beta < 1$. Therefore $\mathbf{T}$ maps the set A into itself.

We now prove that $\mathbf{T}$ is a contraction in $C_S^{1+\beta}(\overline{D}^\tau)$ for every β, $0 < \alpha < \beta < 1$.

Let $u, \widetilde{u} \in A$ and $z = \mathbf{T}[u]$, $\widetilde{z} = \mathbf{T}[\widetilde{u}]$. According to the definition of $\mathbf{T}$ we have

$$\begin{cases} \mathcal{F}^i[z^i - \widetilde{z}^i](t,x) = \mathbf{F}^i[u](t,x) - \mathbf{F}^i[\widetilde{u}](t,x) & \text{in } D^\tau,\ i \in S, \\ z(t,x) - \widetilde{z}(t,x) = 0 & \text{on } \Gamma^\tau. \end{cases} \tag{10}$$

Applying Lemma 3 and condition (L) to the solution $z - \widetilde{z}$ of problem (10) we obtain

$$\begin{aligned} \|z - \widetilde{z}\|_{1+\beta}^{D^\tau} &\le \overline{K}\tau^{(1-\beta)/2} \|\mathbf{F}[u] - \mathbf{F}[\widetilde{u}]\|_0^{D^\tau} \\ &= \overline{K}\tau^{(1-\beta)/2} \sup_{\substack{(t,x)\in D^\tau \\ i\in S}} |f^i(t,x,u) - f^i(t,x,\widetilde{u})| \\ &\le \overline{K}\tau^{(1-\beta)/2} L \|u - \widetilde{u}\|_0^{D^\tau} \le \overline{K} L \tau^{(1-\beta)/2} \|u - \widetilde{u}\|_{1+\beta}^{D^\tau}. \end{aligned}$$

If we now assume that

$$\theta := \overline{K} L \tau^{(1-\beta)/2} < 1,$$

or

$$0 < \tau < \tau^* := \min\{(\overline{K}L)^{2/(\beta-1)}, T\}, \tag{11}$$

then we get

$$\|z - \widetilde{z}\|_{1+\beta}^{D^\tau} \le \theta \|u - \widetilde{u}\|_{1+\beta}^{D^\tau}.$$

Hence the mapping $\mathbf{T}$ is a contraction in $C_S^{1+\beta}(\overline{D}^\tau)$, where $0 < \tau < \tau^*$ and τ^* defined by (11) is sufficiently small. Therefore by the Banach fixed point theorem the mapping $\mathbf{T}$ has a unique fixed point $z \in A$. It is obvious that $z \in C_S^{1+\beta}(\overline{D}^\tau) \cap C_S^{2+\alpha}(\overline{D}^\tau)$ and z is the unique solution of problem (1), (3) in $\overline{D}^\tau$.

Remark 3. Applying (7) to the solution $z - \widetilde{z}$ of (10) in $\overline{D}$ we obtain the following inequality in the whole set $\overline{D}$:

$$\|z - \widetilde{z}\|_{1+\beta} \le K \|\mathbf{F}[u] - \mathbf{F}[\widetilde{u}]\|_0 \le KL \|u - \widetilde{u}\|_0 \le KL \|u - \widetilde{u}\|_{1+\beta}.$$

Hence the mapping $\mathbf{T}$ is a contraction in $C_S^{1+\beta}(\overline{D})$ if $KL < 1$. ■

COROLLARY 1. *The solution of problem* (1), (3) *can be obtained by the method of successive approximations.*

Proof. Set

$$z_0 = \Phi, \qquad z_{n+1} = \mathbf{T}[z_n], \qquad n = 1, 2, \ldots,$$

where $\Phi \in C_S^{2+\alpha}(\overline{D})$ is an extension of $g = 0$ from Γ onto $\overline{D}$.

It is easily seen that $z_n \in C_S^{2+\alpha}(\overline{D}^\tau) \cap C_S^{1+\beta}(\overline{D}^\tau)$ for $n = 1, 2, \ldots$

From the Banach theorem the sequence $\{z_n\}$ is convergent in $C_S^{1+\beta}(\overline{D}^\tau)$ to the fixed point z of the mapping $\mathbf{T}$. Therefore

$$\lim_{n\to\infty} z_n^i(t,x) = z^i(t,x),$$

$$\lim_{n\to\infty} \frac{\partial z_n^i(t,x)}{\partial x_j} = \frac{\partial z^i(t,x)}{\partial x_j}, \qquad j = 1, \ldots, m,\ i \in S,$$

uniformly in $\overline{D}^\tau$.

By Lemmas 2 and 1(b) the sequence $\{\|z_n\|_{2+\alpha}^{D^\tau}\}$ is bounded. In fact, if $\|z_n\|_{2+\alpha}^{D^\tau} \le M$ (where the constant $M > 0$ will be specified later), then

$$\|z_{n+1}\|_{2+\alpha}^{D^\tau} \le c\|\mathbf{F}[z_n]\|_{0+\alpha}^{D^\tau} \le c(LM + M_0 + H).$$

If we now choose

$$M \ge c(M_0 + H)(1 - cL)^{-1}$$

and if $cL < 1$, we obtain

$$\|z_{n+1}\|_{2+\alpha}^{D^\tau} \le M.$$

Therefore, if we additionally assume that

$$cL < 1 \tag{12}$$

then the sequence $\{z_n\}$ is bounded in $C_S^{2+\alpha}(\overline{D}^\tau)$.

Hence, in view of the Ascoli–Arzelà theorem, there is a subsequence $\{z_{n'}\}$ such that

$$\lim_{n'\to\infty} \frac{\partial^2 z_{n'}^i(t,x)}{\partial x_j \partial x_k} = \frac{\partial^2 z^i(t,x)}{\partial x_j \partial x_k},$$

$$\lim_{n'\to\infty} \frac{\partial z_{n'}^i(t,x)}{\partial t} = \frac{\partial z^i(t,x)}{\partial t}, \qquad j,k = 1, \ldots, m,\ \ i \in S,$$

uniformly in $\overline{D}^\tau$. It follows that $z = z(t,x)$ is the desired solution of problem (1), (3) in $\overline{D}^\tau$.

We remark that the constant c (from the estimate (5)) depends on the geometry of the domain D. Therefore, due to assumptions (11) and (12), the "height" of D, i.e., τ^*, has to be an appropriately chosen, sufficiently small number. ∎

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