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New fixed point theorems for mappings satisfying a generalized weakly contractive condition with weaker control functions

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Abstract. The purpose of this paper is to derive new common fixed point theorems for a pair of mappings satisfying a more general weakly contractive condition with weaker control functions in a complete metric space. Applications to new fixed point results with conditions of integral type are also given. We furnish an example to demonstrate that these results improve the previously existing ones.

1. Introduction and preliminaries. The literature on fixed point theory presents a lot of generalizations of the Banach contraction mapping principle [4]. One of the most interesting is the result of Khan et al. [9]. They addressed a new category of fixed point problems for a single self-map with the help of a control function. A function $\varphi : [0,\infty) \to [0,\infty)$ is called a *distance altering function* if φ is continuous, nondecreasing and $\varphi(0) = 0$.

Khan et al. [9] gave the following result.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete metric space, let φ be a distance altering function, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \le c\varphi(d(x,y)) \tag{1.1}$$

for all $x, y \in \mathcal{X}$ and for some $0 < c < 1$. Then $\mathcal{T}$ has exactly one fixed point.

In fact Khan et al. proved a more general theorem of which the above result is a particular case. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabrière [3] in Hilbert spaces by introducing the concept of weakly contractive mappings.

A self-mapping $\mathcal{T}$ of a metric space $\mathcal{X}$ is called *weakly contractive* if for each $x, y \in \mathcal{X}$,

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$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y) - \phi(d(x, y)) \tag{1.2}$$

where ϕ is a distance altering function.

Rhoades [12] showed that most of the results from [3] are still true for any Banach space. He also proved the following very interesting fixed point theorem which contains the contraction condition as a special case for $\phi(t) = (1-k)t$.

Theorem 1.2. *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a weakly contractive mapping, then $\mathcal{T}$ has exactly one fixed point.*

Alber and Guerre-Delabrière [3] assumed additionally that $\lim_{t\to\infty} \phi(t) = \infty$. But Rhoades [12] obtained the result of Theorem 1.2 without using this assumption. One of the main generalizations of the Banach principle is the following theorem established by Boyd and Wong [5]. In their theorem it is assumed that $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right (that is, $r_n \to r \geq 0$ implies $\limsup_{n\to\infty} \psi(r_n) \leq \psi(r)$).

Theorem 1.3. *Let $(\mathcal{X}, d)$ be a complete metric space and suppose that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y)) \quad \text{for each } x, y \in \mathcal{X}, \tag{1.3}$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then $\mathcal{T}$ has a unique fixed point x^*, and $\{\mathcal{T}^n(x)\}$ converges to x^* for each $x \in X$.

Similarly, Reich [11] presented the following:

Theorem 1.4. *Let $(\mathcal{X}, d)$ be a complete metric space and suppose that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \beta(d(x, y))d(x, y) \quad \text{for each } x, y \in \mathcal{X},\ x \neq y, \tag{1.4}$$

where $\beta : [0, \infty) \to [0, 1)$ and $\limsup_{t\to r+} \beta(t) < 1$ for all $0 < r < \infty$. Then $\mathcal{T}$ has a fixed point x^*.

Weak contractions are also closely related to maps of Boyd and Wong [5] and Reich [11] type. Namely, if ϕ is a lower semicontinuous function from the right then $\psi(t) = t - \phi(t)$ is upper semicontinuous from the right, and moreover, (1.2) turns into (1.3). Therefore the weak contraction is of Boyd and Wong type. And if we define $\beta(t) = 1 - \phi(t)/t$ for $t > 0$ and $\beta(0) = 0$, then (1.2) turns into (1.4). Therefore the weak contraction becomes a Reich type one.

Recently, the following generalized result has been given by Dutta and Choudhoury [8], combining Theorems 1.1 and 1.2.

THEOREM 1.5. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(x,y)) - \phi(d(x,y)) \tag{1.5}$$

for all $x, y \in \mathcal{X}$, where $\varphi, \phi : [0,\infty) \to [0,\infty)$ are both continuous and nondecreasing functions with $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then $\mathcal{T}$ has exactly one fixed point.

Dorić [7] gave the following generalized version of Theorems 1.5 and 1.2.

THEOREM 1.6. *Let $(\mathcal{X}, d)$ be a nonempty complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be such that for each $x, y \in \mathcal{X}$,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Phi(x,y)) - \phi(\Phi(x,y)), \tag{1.6}$$

where

(i) $\Phi(x,y) = \max\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{1}{2}[d(y,\mathcal{T}x) + d(x,\mathcal{T}y)]\}$.
(ii) $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous, nondecreasing function with* $\varphi(t) = 0$ *if and only if* $t = 0$,
(iii) $\phi : [0,\infty) \to [0,\infty)$ *is a lower semicontinuous function with* $\phi(0) = 0$ *if and only if* $t = 0$.

Then there exists a fixed point $z \in \mathcal{X}$ such that $z = \mathcal{T}z$.

Abbas and Khan [1] also gave an extension of Theorem 1.5 as follows:

THEOREM 1.7. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ satisfy*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(\mathcal{S}x, \mathcal{S}y)) - \phi(d(\mathcal{S}x, \mathcal{S}y)),$$

where $\phi, \varphi : [0,\infty) \to [0,\infty)$ are both continuous and decreasing functions with $\phi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Then $\mathcal{T}$ and $\mathcal{S}$ have exactly one common fixed point.

Abbas and Dorić [2] also extended Theorems 1.6 and 1.7 to pairs of maps.

Recently, Popescu [10, Theorem 4] proved Theorem 1.6 for some weaker conditions on control functions.

In this article, an attempt has been made to generalize Theorem 4 of [10] by taking into account two self-mappings instead of one. Our result also improves Theorem 1.7 by considering weaker conditions for control functions φ and ϕ. An example shows that these result improve those in the literature. At the end of the paper, an application to fixed point theorems with conditions of integral type is also given.

2. Main results

THEOREM 2.1. *Let $(\mathcal{X}, d)$ be a complete metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ be such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \tag{2.1}$$

for all $x, y \in \mathcal{X}$, where

(a)

$$\begin{aligned}\Theta(x, y) \\ &= \max\{d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{T}y), \tfrac{1}{2}[d(\mathcal{S}y, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{T}y)]\}.\end{aligned} \tag{2.2}$$

(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function with* $\varphi(t) = 0$ *if and only if* $t = 0$.
(c) $\phi : [0, \infty) \to [0, \infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(\alpha_n) > 0$ *if* $\lim_{n\to\infty} \alpha_n = \alpha > 0$,
(d) $\phi(\alpha) > \varphi(\alpha) - \varphi(\alpha-)$ *for any* $\alpha > 0$, *where* $\varphi(\alpha-)$ *is the left limit of* φ *at* α.

If $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X})$ is closed, then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point. Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then they have exactly one common fixed point.

Proof. Note that the left limit of φ at a exists by the monotonicity of φ.

Pick $x_0 \in X$. If $\mathcal{T}x_0 = \mathcal{S}x_0$, then we have a coincidence point. Suppose $\mathcal{T}x_0 \neq \mathcal{S}x_0$ for $x_0 \in \mathcal{X}$. Now since $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can choose $x_1 \in \mathcal{X}$ so that $\mathcal{S}(x_1) = \mathcal{T}(x_0)$. Again, from $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can find $x_2 \in \mathcal{X}$ so that $\mathcal{S}(x_2) = \mathcal{T}(x_1)$. Continuing, we find a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{S}x_{n+1} = \mathcal{T}x_n \quad \text{for all } n \geq 0. \tag{2.3}$$

If there exists $n_0 \in \{1, 2, \ldots\}$ such that $\Theta(x_{n_0}, x_{n_0-1}) = 0$ then it is clear that $\mathcal{S}(x_{n_0-1}) = \mathcal{T}(x_{n_0}) = \mathcal{T}x_{n_0-1}$ and so $\mathcal{T}$ and $\mathcal{S}$ have a coincidence at $x = x_{n_0-1}$; therefore the assertion is proved. Now we can suppose

$$\Theta(x_n, x_{n-1}) > 0 \tag{2.4}$$

for all $n \geq 1$.

First we will prove that $\lim_{n\to\infty} d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) = 0$. By (2.2), we have, for $n \geq 1$,

$$\begin{aligned}\Theta(x_n, x_{n-1}) &= \max\{d(\mathcal{S}x_n, \mathcal{S}x_{n-1}), d(\mathcal{S}x_n, \mathcal{T}x_n), d(\mathcal{S}x_{n-1}, \mathcal{T}x_{n-1}), \\ &\qquad \tfrac{1}{2}[d(\mathcal{S}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}x_{n-1})]\} \\ &= \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), \tfrac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\} \\ &\leq \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)\}.\end{aligned}$$

Now we claim that

$$d(\mathcal{T}x_{n+1},\mathcal{T}x_n)\leq d(\mathcal{T}x_n,\mathcal{T}x_{n-1}) \tag{2.5}$$

for all $n\geq 1$. Suppose this is not true, that is, there exists $n_0\geq 1$ such that $d(\mathcal{T}(x_{n_0+1}),\mathcal{T}(x_{n_0}))>d(\mathcal{T}(x_{n_0}),\mathcal{T}(x_{n_0-1}))$.

Substituting $x=x_{n_0+1}$ and $y=x_{n_0}$ into (2.1), we have

$$\begin{aligned}\varphi(d(\mathcal{T}x_{n_0+1},\mathcal{T}x_{n_0}))&\leq\varphi(\Theta(x_{n_0+1},x_{n_0}))-\phi(\Theta(x_{n_0+1},x_{n_0}))\\&\leq\varphi(\max\{d(\mathcal{T}x_{n_0},\mathcal{T}x_{n_0-1}),d(\mathcal{T}x_{n_0},\mathcal{T}x_{n_0+1})\})\\&\quad-\phi(\Theta(x_{n_0+1},x_{n_0}))\\&=\varphi(d(\mathcal{T}x_{n_0},\mathcal{T}x_{n_0+1}))-\phi(\Theta(x_{n_0+1},x_{n_0})).\end{aligned}$$

This implies that $\phi(\Theta(x_{n_0+1},x_{n_0}))=0$. By (c), we have $\Theta(x_{n_0+1},x_{n_0})=0$, which contradicts (2.4). Therefore, (2.5) is true and so the sequence $\{d(\mathcal{T}(x_{n+1}),\mathcal{T}(x_n))\}$ is nonincreasing and bounded. Thus there exists $\rho\geq 0$ such that $\lim_{n\to\infty}d(\mathcal{T}(x_{n+1}),\mathcal{T}(x_n))=\rho$. Therefore by (2.2),

$$\begin{aligned}&\lim_{n\to\infty}d(\mathcal{T}(x_n),\mathcal{T}(x_{n-1}))\\&\qquad\leq\lim_{n\to\infty}\Theta(x_n,x_{n-1})\\&\qquad=\lim_{n\to\infty}\max\{d(\mathcal{S}x_n,\mathcal{S}x_{n-1}),d(\mathcal{S}x_n,\mathcal{T}x_n),d(\mathcal{S}x_{n-1},\mathcal{T}x_{n-1}),\\&\qquad\quad\tfrac{1}{2}[d(\mathcal{S}x_{n-1},\mathcal{T}x_n)+d(\mathcal{S}x_n,\mathcal{T}x_{n-1})]\}\\&\qquad=\lim_{n\to\infty}\max\{d(\mathcal{T}x_{n-1},\mathcal{T}x_{n-2}),d(\mathcal{T}x_{n-1},\mathcal{T}x_n),\tfrac{1}{2}d(\mathcal{T}x_{n-2},\mathcal{T}x_n)\}.\end{aligned}$$

This implies $\rho\leq\lim_{n\to\infty}\Theta(x_n,x_{n-1})\leq\rho$ and so $\lim_{n\to\infty}\Theta(x_n,x_{n-1})=\rho$. Now we claim that $\rho=0$. By (2.1), we have

$$\varphi(d(\mathcal{T}x_n,\mathcal{T}x_{n-1}))\leq\varphi(\Theta(x_n,x_{n-1}))-\phi(\Theta(x_n,x_{n-1}))$$

and taking the limit as $n\to\infty$, we have

$$\varphi(\rho+)\leq\varphi(\rho+)-\liminf_{n\to\infty}\phi(\Theta(x_n,x_{n+1})),$$

which is contradictory unless $\rho=0$. Hence

$$\rho=0=\lim_{n\to\infty}d(\mathcal{T}x_{n+1},\mathcal{T}x_n). \tag{2.6}$$

Next we show that $\{\mathcal{T}x_n\}$ is Cauchy. Suppose it is not. Then there is an $\varepsilon>0$ such that for an integer k there exist integers $m(k)>n(k)>k$ such that

$$d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)})>\varepsilon. \tag{2.7}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.7) and such that

$$d(\mathcal{T}x_{n(k)},\mathcal{T}x_{m(k)-1})<\varepsilon. \tag{2.8}$$

Now

$$\varepsilon \le d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) \le d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) + d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).$$

Then by (2.7) and (2.8) it follows that

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) = \varepsilon. \tag{2.9}$$

Also, by the triangle inequality, we have

$$|d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) - d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})| < d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).$$

By using (2.9) we get

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) = \varepsilon. \tag{2.10}$$

Now by (2.2) we get

$$\begin{aligned} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) &\le \Theta(x_{n(k)}, x_{m(k)-1}) \\ &= \max\{d(\mathcal{S}x_{n(k)}, \mathcal{S}x_{m(k)-1}), d(\mathcal{S}x_{n(k)}, \mathcal{T}x_{n(k)}), \\ &\quad d(\mathcal{S}x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), \\ &\quad \tfrac{1}{2}[d(\mathcal{S}x_{m(k)-1}, \mathcal{T}x_{n(k)}) + d(\mathcal{S}x_{n(k)}, \mathcal{T}x_{m(k)-1})]\} \\ &\le \max\{d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2}), d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}), \\ &\quad d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{m(k)-1}), \\ &\quad \tfrac{1}{2}[d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{n(k)}) + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-1})]\} \\ &\le \max\{d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2}), d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}), \\ &\quad d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{m(k)-1}), \tfrac{1}{2}[d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{n(k)-1}) \\ &\quad\ + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}) + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-1})]\}, \end{aligned}$$

and letting $k \to \infty$ and using (2.9) and (2.10), we have

$$\varepsilon \le \lim_{k\to\infty} \Theta(x_{n(k)}, x_{m(k)-1}) \le \varepsilon$$

and so

$$\lim_{k\to\infty} \Theta(x_{n(k)}, x_{m(k)-1}) = \varepsilon.$$

If there is a subsequence $\{k(p)\}$ of $\{k\}$ with $\varepsilon < d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))})$ for any p, then by (2.2) we get

$$\begin{aligned} \varphi(\varepsilon+) &= \limsup_{k\to\infty} \varphi(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})) \\ &\le \limsup_{k\to\infty} \varphi(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)+1}) + d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1})) \\ &= \limsup k \to \infty \varphi(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1})) \\ &\le \limsup_{k\to\infty} [\varphi(\Theta(x_{n(k)}, x_{m(k)-1})) - \phi(\Theta(x_{n(k)}, x_{m(k)-1}))] \\ &= \varphi(\varepsilon+) - \liminf_{k\to\infty} \phi(\Theta(x_{n(k)}, x_{m(k)-1})), \end{aligned}$$

which is a contradiction. We repeat the procedure if there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\varepsilon < d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p)+1)})$ for any p or $\varepsilon < d(\mathcal{T}x_{n(k(p)+1)}, \mathcal{T}x_{m(k(p))})$ for any p. Therefore, we can suppose that $d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))}) = \varepsilon$, $d(\mathcal{T}x_{n(k(p)+1)}, \mathcal{T}x_{m(k(p))}) \leq \varepsilon$ and $d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p)+1)}) \leq \varepsilon$ for any $k \geq k_1$. Then $\Theta(x_{n(k)}, x_{m(k)}) = \varepsilon$ for $k \geq k_3 = \max\{k_1, k_2\}$, where k_2 is such that $d(\mathcal{T}x_k, \mathcal{T}x_{k+1}) < \varepsilon$ for all $k \geq k_2$. Substituting $x = x_{n(k)}$, $x = x_{m(k)}$ in (2.1), we have

$$\varphi(d(\mathcal{T}x_{n(k)+1}, \mathcal{T}x_{m(k)+1})) \leq \varphi(\varepsilon) - \phi(\varepsilon)$$

for any $k \geq k_2$. Obviously $d(\mathcal{T}x_{n(k)+1}, \mathcal{T}x_{m(k)+1}) < \varepsilon$, otherwise we have $\phi(\varepsilon) = 0$. Letting $k \to \infty$ we obtain

$$\varphi(\varepsilon-) \leq \varphi(\varepsilon) - \phi(\varepsilon),$$

which contradicts hypothesis (c). Thus $\{\mathcal{T}x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$ there exists $z \in \mathcal{X}$ such that $\mathcal{T}x_n \to z$ as $n \to \infty$. Next, we show that z is a fixed point for both $\mathcal{T}$ and $\mathcal{S}$.

By (2.3) we have $\{\mathcal{T}(x_n)\} = \{\mathcal{S}(x_{n+1})\} \subseteq \mathcal{S}(\mathcal{X})$, therefore by the closedness of $\mathcal{S}(\mathcal{X})$, there exists $u \in \mathcal{X}$ such that $\mathcal{S}u = z$. We claim that $\mathcal{T}u = z$.

For each n, we can use the inequality (2.2) for x_n and u. Since

$$\begin{aligned}\Theta(u, x_n) &= \max\{d(\mathcal{S}u, \mathcal{S}x_n), d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}x_n, \mathcal{T}x_n),\\ &\qquad \tfrac{1}{2}[d(\mathcal{S}x_n, \mathcal{T}u) + d(\mathcal{S}u, \mathcal{T}x_n)]\}\\ &= \max\{d(z, \mathcal{T}x_{n-1}), d(z, \mathcal{T}u), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n),\\ &\qquad \tfrac{1}{2}[d(\mathcal{T}x_{n-1}, \mathcal{T}u) + d(z, \mathcal{T}x_n)]\},\end{aligned}$$

it follows that $\lim_{n\to\infty} \Theta(u, x_n) = d(z, \mathcal{T}u)$. Therefore,

$$\begin{aligned}\varphi(d(\mathcal{T}u, z)-) &= \limsup_{n\to\infty} \varphi(d(\mathcal{T}u, \mathcal{S}x_{n+1})) = \limsup_{n\to\infty} \varphi(d(\mathcal{T}u, \mathcal{T}x_n))\\ &\leq \limsup_{n\to\infty} [\varphi(\Theta(u, x_n)) - \phi(\Theta(u, x_n))]\\ &\leq \varphi(d(\mathcal{T}u, z)) - \phi(d(\mathcal{T}u, z)).\end{aligned}$$

which contradicts hypothesis (c) unless $\mathcal{T}u = z$. Therefore, $\mathcal{T}u = \mathcal{S}u = z$. Thus we have proved that $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point. If $\mathcal{T}$ and $\mathcal{S}$ commute, we have

$$\mathcal{T}(z) = \mathcal{T}(\mathcal{S}(u)) = \mathcal{S}(\mathcal{T}(u)) = \mathcal{S}(z). \tag{2.11}$$

Next we claim that $\mathcal{T}z = z$. From (2.1), we get

$$\varphi(d(\mathcal{T}u, \mathcal{T}z)) \leq \varphi(\Theta(u, z)) - \phi(\Theta(u, z))$$

where

$$\begin{aligned}\Theta(u, z) &= \max\{d(\mathcal{S}u, \mathcal{S}z), d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}z, \mathcal{T}z), \tfrac{1}{2}[d(\mathcal{S}u, \mathcal{T}z) + d(\mathcal{S}z, \mathcal{T}u)]\}\\ &= \max\{d(z, \mathcal{S}z), d(z, z), d(\mathcal{S}z, \mathcal{T}z), \tfrac{1}{2}[d(z, \mathcal{T}z) + d(\mathcal{S}z, z)]\}\\ &= d(z, \mathcal{T}z).\end{aligned}$$

Hence

$$\varphi(d(z, \mathcal{T}z)-) = \varphi(d(\mathcal{T}u, \mathcal{T}z)) \leq \varphi(d(z, \mathcal{T}z)) - \phi(d(z, \mathcal{T}z)),$$

which contradicts hypothesis (c) unless $\mathcal{T}z = z$ and so $z = \mathcal{T}z = \mathcal{S}z$. Hence z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Finally, we prove the uniqueness of the common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Suppose there exists another common fixed point $\nu \in \mathcal{X}$ such that $\mathcal{S}\nu = \nu = \mathcal{T}\nu$. Then by (2.1),

$$\begin{aligned}\varphi(d(z, \nu)) &= \varphi(d(\mathcal{T}z, \mathcal{S}\nu)) \\ &\leq \varphi(\Theta(z, \nu)) - \phi(\Theta(z, \nu)) \leq \varphi(d(z, \nu)) - \phi(d(z, \nu)),\end{aligned}$$

a contradiction unless $d(z, \nu) = 0$, that is, $z = \nu$. ∎

An immediate consequence of Theorem 2.1 is as follows.

COROLLARY 2.2. *Let $(\mathcal{X}, d)$ be a complete metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ be such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(\mathcal{S}x, \mathcal{S}y)) - \phi(d(\mathcal{S}x, \mathcal{S}y)) \tag{2.12}$$

for all $x, y \in \mathcal{X}$, where

(a) *$\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,*
(b) *$\phi : [0, \infty) \to [0, \infty)$ is a function with $\phi(t) = 0$ if and only if $t = 0$, and $\liminf_{n\to\infty} \phi(\alpha_n) > 0$ if $\lim_{n\to\infty} \alpha_n = \alpha > 0$,*
(c) *$\phi(\alpha) > \varphi(\alpha) - \varphi(\alpha-)$ for any $\alpha > 0$.*

If $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X})$ is closed, then $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

Further, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then they have a common fixed point.

If $\mathcal{S} = I$, an identity mapping, in Theorem 2.1, then we obtain Theorem 4 of [10] as corollary:

COROLLARY 2.3. *Let $(\mathcal{X}, d)$ be a complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be such that*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \tag{2.13}$$

for all $x, y \in \mathcal{X}$, where

(a) $\Theta(x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)]\}$,
(b) *$\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,*
(c) *$\phi : [0, \infty) \to [0, \infty)$ is a function with $\phi(t) = 0$ if and only if $t = 0$, and $\liminf_{n\to\infty} \phi(\alpha_n) > 0$ if $\lim_{n\to\infty} \alpha_n = \alpha > 0$,*
(d) *$\phi(\alpha) > \varphi(\alpha) - \varphi(\alpha-)$ for any $\alpha > 0$.*

Then $\mathcal{T}$ has a fixed point.

The following example shows that Theorem 2.1 can be used in the situations when Theorem 1.7 and [10, Theorem 4] cannot.

Example 2.4. Let $\mathcal{X} = [0,1]$ be equipped with the standard metric and consider the following mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ and functions $\varphi, \phi : [0,\infty) \to [0,\infty)$:

$$\mathcal{S}x = \begin{cases} 1, & 0 \le x < 1/2, \\ 1/2, & x = 1/2, \\ 1/10, & 1/2 < x \le 2/3, \\ 0, & 2/3 < x \le 1, \end{cases} \qquad \mathcal{T}x = \begin{cases} 1/2, & 0 \le x \le 1/2, \\ 1, & 1/2 < x \le 1, \end{cases}$$

$$\varphi(t) = \begin{cases} (7/5)t, & 0 \le t < 1/2, \\ (2-\sqrt{2})t + (\sqrt{2}-1), & 1/2 \le t < \infty, \end{cases} \qquad \phi(t) = (1/10)t^2.$$

Then a careful computation shows that all the conditions of Theorem 2.1 are fulfilled. We just note the main points:

The only point of discontinuity of φ is $1/2$ and $\phi(1/2) = 0.025 > \sqrt{2}/2 - 0.7 = \varphi(1/2) - \varphi(1/2-)$, hence condition (d) is satisfied.

Since $\phi(t) \le \varphi(t)$ for all $t \in [0,1]$, the only nontrivial cases when the contractive condition (2.1) has to be checked are when $x \in [0,1/2)$, $y \in (1/2, 2/3]$ and $x \in [0,1/2)$, $y \in (2/3, 1]$ (or vice versa). In the first case, (2.1) becomes $\varphi(1/2) \le \varphi(9/10) - \phi(9/10)$ and in the second, $\varphi(1/2) \le \varphi(1) - \phi(1)$, and both of these inequalities are easily verified.

Thus, the mappings $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point (which is $1/2$).

3. Application to integral type problems. We now present applications of the results of the previous section. We obtain a fixed point theorem for a pair of mappings satisfying a general contractive condition of integral type (Branciari [6]) in a complete metric space.

Let Ψ be a nonnegative Lebesgue integrable function (with finite integral) on $\mathbb{R}^+$ such that $\int_0^\epsilon \Psi(t)\,dt > 0$ for each $\epsilon > 0$.

Theorem 3.1. *Let $\mathcal{S}$ and $\mathcal{T}$, as well as φ, ϕ, $\Theta(x,y)$ satisfy the conditions of Theorem* 2.1, *except that condition* (2.1) *is replaced by*

$$\int_0^{\varphi(d(\mathcal{T}x,\mathcal{T}y))} \Psi(t)\,dt \le \int_0^{\varphi(\Theta(x,y))} \Psi(t)\,dt - \int_0^{\phi(\Theta(x,y))} \Psi(t)\,dt \tag{3.1}$$

for all $x, y \in \mathcal{X}$. Then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. Define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)\,dt$. Then Λ is continuous and nondecreasing with $\Lambda(0) = 0$. Condition (3.1) becomes

$$\Lambda\left(\varphi(d(\mathcal{T}x,\mathcal{T}y))\right) \le \Lambda\left(\varphi(\Theta(x,y))\right) - \Lambda\left(\phi(\Theta(x,y))\right), \tag{3.2}$$

which can be further written as

$$\varphi_1(d(\mathcal{T}x,\mathcal{T}y)) \le \varphi_1(\Theta(x,y) - \phi_1(\Theta(x,y)),$$

where $\phi_1 = \Lambda \circ \phi$ and $\varphi_1 = \Lambda \circ \varphi$. Clearly, ϕ_1, φ_1 are control functions with $\phi_1(0) = 0 = \varphi_1(0)$. Hence by Theorem 2.1, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. ∎

THEOREM 3.2. *Let $\mathcal{S}$ and $\mathcal{T}$, as well as φ, ϕ, $\Theta(x,y)$ satisfy the conditions of Theorem* 2.1, *except that condition* (2.1) *is replaced by*

$$\int_0^{d(\mathcal{T}x,\mathcal{T}y)} \Psi(t)\,dt \le h \int_0^{\Theta(x,y)} \Psi(t)\,dt$$

for all $x, y \in \mathcal{X}$ and $h \in [0,1)$. Then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. Following the lines of Theorem 3.1, we can define $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Lambda(x) = \int_0^x \Psi(t)\,dt$ and check the other properties. If we set $\phi = (1-h)\varphi$, then by Theorem 2.1, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. ∎

REMARK 3.3. We can also establish similar integral results as applications of Corollaries 2.2 and 2.3.

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References

[1] M. Abbas and M. Ali Khan, *Common fixed point theorem of two mappings satisfying a generalized weak contractive condition*, Int. J. Math. Math. Sci. 2009, art. ID 131068, 9 pp.

[2] M. Abbas and D. Dorić, *Common fixed point theorem for four mappings satisfying generalized weak contractive condition*, Filomat 24 (2010), no. 2, 1–10.

[3] Ya. I. Alber and S. Guerre-Delabrière, *Principles of weakly contractive maps in Hilbert spaces*, in: New Results in Operator Theory and its Applications, I. Gohberg and Yu. Lyubich (eds.), Oper. Theory Adv. Appl. 98, Birkhäuser, Basel, 1997, 7–22.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922), 133–181.

[5] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969), 458–464.

[6] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. 29 (2002), 531–536.

[7] D. Dorić, *Common fixed point for generalized (ψ,ϕ)-weak contractions*, Appl. Math. Lett. 22 (2009), 1896–1900.

[8] P. N. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl. 2008, art. ID 406368, 8 pp.

[9] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc. 30 (1984), 1–9.

[10] O. Popescu, *Fixed points for (ψ, ϕ)-weak contractions*, Appl. Math. Lett. 24 (2011), 1–4.

[11] S. Reich, *Some fixed point problems*, Atti Acad. Naz. Lincei Rend. 57 (1974), 194–198.

[12] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. 47 (2001), 2683–2693.

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